ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2017. Випуск 83. С. 5–8
Visnyk of the Lviv Univ. Series Mech. Math. 2017. Issue 83. P. 5–8

УДК 512.582

EXTENDING MONOMORPHIC FUNCTORS WITH FINITE SUPPORTS

Taras BANAKH, Viktoriia BRYDUN

Ivan Franko National University of Lviv,
Universitetska str., 1, 79000 Lviv, Ukraine
e-mail: tbanakh@yahoo.com, v_frider@yahoo.com

We prove that each monomorphic functor with finite supports $F\colon \mathbf{Comp} \to \mathbf{Comp}$ has a unique extension $\bar{F}\colon \mathbf{TYCH} \to \mathbf{TYCH}$ to the category **TYCH** of Tychonoff spaces and their arbitrary maps such that $\bar{F}|\mathbf{Tych} = F_\beta$ where $F_\beta : \mathbf{Tych} \to \mathbf{Tych}$ is the extension of the functor F to the category **Tych** of Tychonoff spaces and their continuous maps, constructed by Chigogidze.

Key words: monomorphism, monomorphic functor, finite support, extension of functor

In this article we describe a general construction of an extension of a monomorphic functor $F : \mathbf{Comp} \to \mathbf{Comp}$ with finite supports in the category **Comp** of compact Hausdorff spaces and their continuous maps to a functor $\bar{F} : \mathbf{TYCH} \to \mathbf{TYCH}$ in the category **TYCH** whose objects are Tychonov spaces and morphisms are arbitrary (not necessarily continuous) maps between Tychonoff spaces. More information on functors in the category **Comp** can be found in the book [4].

We shall say that a functor $F : \mathbf{Comp} \to \mathbf{TYCH}$

- is *monomorphic* if F preserves monomorphisms, which means that for any injective continuous map $f : X \to Y$ between compact Hausdorff spaces the map $Ff : FX \to FY$ is injective;
- has *finite supports* if for each compact Hausdorff space X and each $a \in FX$ there is a finite subset $A \subset X$ such that $a \in Fi_{A,X}(FA)$ where $i_{A,X} : A \to X$ is the identity inclusion.

More information on monomorphic functors with finite supports can be found in the paper [1].

Given a functor $F : \mathbf{Comp} \to \mathbf{TYCH}$ we first extend F to a functor $F_\beta : \mathbf{Tych} \to \mathbf{TYCH}$ defined on the category **Tych** of Tychonoff spaces and their continuous maps. Given a Tychonoff space X let βX be the Stone-Čech compactification of X and $\mathcal{K}(X)$ be the family of all compact subsets of X. For each compact subset $K \in \mathcal{K}(X)$ let

---

2010 Mathematics Subject Classification: 18A22

$i_{K,\beta X} : K \to X \subset \beta X$ be the identity inclusion of K into the Stone-Čech compactification of X. Applying the functor F to the inclusion $i_{K,\beta X} : K \to \beta X$, we get a map $Fi_{K,\beta X} : FK \to F(\beta X)$.

Now let

$$F_\beta X := \bigcup_{K\in\mathcal{K}(X)} Fi_{K,\beta X}(FK).$$

For any continuous function $f : X \to Y$ between Tychonoff spaces let $\beta f : \beta X \to \beta Y$ be the Stone-Čech extension of f and $F_\beta f : F_\beta X \to F_\beta Y$ be the restriction of the map $F\beta f$ to $F_\beta X$. In such way we define the extension F_β : **Tych** $\to$ **TYCH** of the functor F to the category **Tych**. For functors F : **Comp** $\to$ **Comp** the extension F_β was introduced and studied by Chigogidze in [2].

Now assuming that the functor F : **Comp** $\to$ **TYCH** is monomorphic and has finite supports, we shall further extend the functor F_β to a functor $\bar{F}$: **TYCH** $\to$ **TYCH** defined on the category **TYCH** of Tychonoff spaces and their arbitrary (not necessarily continuous) maps. For a Tychonoff space X let $[X]^{<\omega}$ be the family of all finite subspaces of X.

Proposition 1. *If a functor F :* **Comp** $\to$ **TYCH** *has finite supports, then*

$$F_\beta X = \bigcup_{A\in[X]^{<\omega}} Fi_{A,\beta X}(FA).$$

Proof. The inclusion $\bigcup_{A\in[X]^{<\omega}} Fi_{A,\beta X}(FA) \subset F_\beta X$ follows from the inclusion $[X]^{<\omega} \subset \mathcal{K}(X)$. To prove the reverse inclusion, fix any element $a \in F_\beta X$ and find a compact subset $K \subset X$ such that $a \in Fi_{K,\beta X}(FK)$. Find an element $b \in FK$ such that $a = Fi_{K,\beta X}(b)$. Since F has finite supports, there exists a finite subset $A \subset K$ such that $b \in Fi_{A,K}(FA)$ and hence $b = Fi_{A,K}(c)$ for some $c \in FA$. Since $i_{A,\beta X} = i_{K,\beta X} \circ i_{A,K}$, we get

$$\begin{aligned} a &= Fi_{K,\beta X}(b) = \\ &= Fi_{K,\beta X}(Fi_{A,K}(c)) = \\ &= F(i_{K,\beta X} \circ i_{A,K})(c) = \\ &= Fi_{A,\beta X}(c) \in Fi_{A,\beta X}(FA). \end{aligned}$$

□

Now we are able to prove the main result of this note.

Theorem 1. *Each monomorphic functor F :* **Comp** $\to$ **TYCH** *with finite supports has a unique extension $\bar{F}$:* **TYCH** $\to$ **TYCH** *such that $\bar{F}|$***Tych** $= F_\beta$.

Proof. For any Tychonoff space X put $\bar{F}X = F_\beta X$. Given any function $f : X \to Y$ between Tychonoff spaces and any $a \in \bar{F}X = F_\beta X$, find a finite subspace $A_1 \subset X$ such that $a \in Fi_{A_1,\beta X}(FA_1)$. Such subspace exists by Proposition 1. Find an element $a_1 \in FA_1$ such that $a = Fi_{A_1,\beta X}(a_1)$. Applying the functor F_β to the continuous map $f_1 = f|A_1 : A_1 \to Y$, we get a map $F_\beta f_1 : FA_1 \to F_\beta Y$. Finally, put

$$\bar{F}f(a) := F_\beta f_1(a_1) \in F_\beta Y = \bar{F}Y.$$

Let us show that the value $\bar{F}f(a) = F_\beta f_1(a_1)$ depends only on a (but not on A_1 or a_1).

Let $A_2 \subset X$ be a finite set such that $a \in Fi_{A_2,\beta X}(FA_2)$ and $a_2 \in FA_2$ be an element such that $a = Fi_{A_2,\beta X}(a_2)$. Consider the finite set $A = A_1 \cup A_2$, and for $i \in \{1,2\}$ let $i_{A_i,A} : A : A_i \to A$ denote the identity inclusion. Let $\tilde{a}_i = Fi_{A_i,A}(a_i) \in FA$ and observe that

$$a = Fi_{A_i,\beta X}(a_i) = Fi_{A,\beta X} \circ Fi_{A_i,A}(a_i) = Fi_{A,\beta X}(\tilde{a}_i).$$

Since the functor F is monomorphic, the map $Fi_{A,\beta X}$ is injective and hence $\tilde{a}_1 = \tilde{a}_2$. Then

$$F_\beta f_1(a_1) = F_\beta(f|A) \circ Fi_{A_1,A}(a_1) = F_\beta(f|A)(\tilde{a}_1) = F_\beta(f|A)(\tilde{a}_2) = F_\beta(f|A_2)(a_2),$$

so the map $\bar{F} : \bar{F}X \to \bar{F}Y$ is well-defined.

Thus, we obtain an extension of the functor $F : \mathbf{Comp} \to \mathbf{TYCH}$ to the functor $\bar{F} : \mathbf{TYCH} \to \mathbf{TYCH}$ such that $\bar{F}|\mathbf{Tych} = F_\beta$. To see that this extension is unique, observe that for any function $f : X \to Y$ between Tychonoff spaces and any $a \in \bar{F}X = F_\beta X$, we can apply Proposition 1 and find a finite set $A \subset X$ with $a \in Fi_{A,\beta X}FA$ and an element $a_1 \in FA$ such that $a = Fi_{A,\beta X}(a_1)$. Let $B = f(A) \subset Y$. Applying the functor $\bar{F}$ to the equality $f|A = f \circ i_{A,X}$ we obtain the equality $F_\beta(f|A) = \bar{F}(f|A) = \bar{F}f \circ \bar{F}i_{A,X} = \bar{F}f \circ F_\beta i_{A,X}$, which implies that the value $\bar{F}f(a) = \bar{F}f \circ F_\beta i_{A,X}(a_1) = F_\beta(f|A)(a_1)$ is uniquely determined. □

Theorem 1 allows us to aks the following problem which will be considered in subsequent publications.

Problem 1. *Detect monomorphic functors* $F : \mathbf{Comp} \to \mathbf{TYCH}$ *with finite support whose extension* $\bar{F} : \mathbf{TYCH} \to \mathbf{TYCH}$ *preserves certain property* $\mathcal{P}$ *of functions between Tychonoff spaces.*

In the role of the property $\mathcal{P}$ we can consider one of properties of generalized continuity, listed in the survey [3].

References

1. T. Banakh, M. Martynenko, and M. Zarichnyi. *On monomorhic topological functors with finite supports*, Carpathian Math. Publ. **4** (2012), no. 1, 4–12.
2. A. Chigogidze, *Extension of normal functors*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **6** (1984), 23–26.
3. O. Karlova, *Classification of analogues of continuous functions*, Chernivtsi Nat. Univ., Chernivtsi, 2015, Preprint.
4. A. Teleiko and M. Zarichnyi. *Categorical topology of compact Hausdorff spaces*, VNTL Publ., Lviv, 1999.

Стаття: надійшла до редколегії 02.06.2017
прийнята до друку ??.??.2017

ПРОДОВЖЕННЯ МОНОМОРФНОГО ФУНКТОРА ЗІ СКІНЧЕННИМИ НОСІЯМИ

Тарас БАНАХ, Вікторія БРИДУН

Львівський національний університет ім. І. Франка,
м. Львів, 79000, вул. Університетська, 1
e-mail: tbanakh@yahoo.com, v_frider@yahoo.com

Доведено, що кожен мономорфний функтор зі скінченними носіями $F\colon \mathbf{Comp} \to \mathbf{Comp}$ має продовження $\bar{F}\colon \mathbf{TYCH} \to \mathbf{TYCH}$ на категорію **TYCH** тихонівських просторів і довільних відображень (не обов'язково неперервних). Причому $\bar{F}|\mathbf{Tych} = F_\beta$, де $F_\beta : \mathbf{Tych} \to \mathbf{Tych}$ – побудоване Чігогідзе продовження функтора F на категорію **Tych** тихонівських просторів і неперервних відображень.

Ключові слова: мономорфізм, мономорфний функтор, скінченний носій, продовження функтора