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ON LIFTING OF CONTRAVARIANT FUNCTORS ONTO THE EILENBERG-MOORE CATEGORY.

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Levyts'ka V.S. On lifting of contravariant functors onto the Eilenberg-Moore category. We consider the problem of lifting contravariant functors onto the category Eilenberg-Moore of a monad. The results are applied to the monad in the category of Tychonov spaces generated by the second iteration of the functor C_p (the space of functions in the topology of pointwise convergence).

1°. A monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$, where $T : \mathcal{C} \to \mathcal{C}$ is a covariant functor and $\eta : 1_{\mathcal{C}} \to T$, $\mu : T^2 \to T$ are natural transformations satisfying the conditions: $\mu \circ \eta T = \mu \circ T\eta = 1_T$ and $\mu \circ \mu T = \mu \circ T\mu$.

A couple (X, ξ), where $\xi : TX \to X$ is a morphism, is called a $\mathbb{T}$-algebra iff $\xi \circ \eta X = 1_X$ and $\xi \circ T\xi = \xi \circ \mu X$. A morphism $f : X \to X'$ is called a morphism of a $\mathbb{T}$-algebra (X, ξ) into a $\mathbb{T}$-algebra (X', ξ') if $f \circ \xi = \xi' \circ Tf$. $\mathbb{T}$-algebras and their morphisms form a category which is usually denoted by $\mathcal{C}^{\mathbb{T}}$ (the Eilenberg-Moore category). We can define the forgetful functor $U^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ by $U^{\mathbb{T}}(X, \xi) = X$, $U^{\mathbb{T}}(f) = f$. (For details see [1].)

A lifting of functor $F : \mathcal{C} \to \mathcal{C}$ on the category $\mathcal{C}^{\mathbb{T}}$ is a functor $\overline{F} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}$ such that $U^{\mathbb{T}}\overline{F} = FU^{\mathbb{T}}$

It is easy to see that the couple $(TX, \mu X)$ is a $\mathbb{T}$-algebra (the free $\mathbb{T}$-algebra).

In [5] M. Zarichnyi considered the following problem. Suppose $F : \mathcal{C} \to \mathcal{C}$ is a covariant functor; is there a covariant functor $\overline{F} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}$ such that $U^{\mathbb{T}}\overline{F} = FU^{\mathbb{T}}$ (the problem of lifting of functor onto the category of $\mathbb{T}$-algebras)?

In this paper we consider the corresponding problem for a contravariant functor F.

2°. In what follows we fix a monad $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$.

The following result is a counterpart of a result of Zarichnyi [5].

Proposition 1. *There exists a bijective correspondence between the lifting of a contravariant functor F onto the category $\mathcal{C}^{\mathbb{T}}$ and the natural transformations $\delta : TFT \to F$ satisfying the conditions: (i) $\delta \circ \eta FT = F\eta$; (ii) $\delta \circ \mu FT = \delta \circ T\delta T \circ T^2 F\mu$.*

Proof. Suppose there is a natural transformations $\delta : TFT \to F$ such that conditions (i) and (ii) are satisfied. For every $(X, \xi) \in |\mathcal{C}^{\mathbb{T}}|$ put $\overline{F}(X, \xi) = (FX, \overline{\xi})$, where $\overline{\xi} = \delta X \circ TF\xi$ and for every $f \in \mathcal{C}^{\mathbb{T}}(X, Y)$ put $\overline{F}f = Ff$.

It is easy to see that $FU^{\mathbb{T}} = U^{\mathbb{T}}\overline{F}$. We have to check that $(FX, \overline{\xi})$ is a $\mathbb{T}$-algebra:

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$\overline{\xi} \circ \eta FX = \delta X \circ TF\xi \circ \eta FX = \delta X \circ \eta FTX \circ F\xi = F\eta X \circ F\xi = F(\xi \circ \eta X) = F(1_X) = 1_{FX}$.

Besides,

$\overline{\xi} \circ \mu FX = \delta X \circ TF\xi \circ \mu FX = \delta X \circ \mu FTX \circ T^2F\xi = \delta X \circ T\delta TX \circ T^2F\mu X \circ T^2F\xi =$
$= \delta X \circ T\delta TX \circ T^2F(\xi \circ \mu X) = \delta X \circ T\delta TX \circ T^2F(\xi \circ T\xi) = \delta X \circ T\delta TX \circ T^2FT\xi \circ T^2F\xi =$
$= \delta X \circ TF\xi \circ T\delta X \circ T^2F\xi = \overline{\xi} \circ T\overline{\xi}$.

Denote by f a morphism of a $\mathbb{T}$-algebra (X, ξ) into a $\mathbb{T}$-algebra (X', ξ'). Show that $\overline{F}f$ is a morphism of the $\mathbb{T}$-algebra $(FX', \overline{\xi}')$ into the $\mathbb{T}$-algebra $(FX, \overline{\xi})$:

$\overline{\xi} \circ TFf = \delta X \circ TF\xi \circ TFf = \delta X \circ TF(f \circ \xi) = \delta X \circ TF(\xi' \circ Tf) = \delta X \circ TFTf \circ TF\xi' =$
$= Ff \circ \delta X' \circ TF\xi' = Ff \circ \overline{\xi}'$.

It is easy to see that $\overline{F}(g \circ f) = \overline{F}f \circ \overline{F}g$.

Summing up we see that $\overline{F}$ is a lifting F contravariant endofunctor on the category $\mathcal{C}^{\mathbb{T}}$.

On the other hand, suppose $\overline{F} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}$ is a lifting of F onto $\mathcal{C}^{\mathbb{T}}$. Since $(TX, \mu X)$ is a free $\mathbb{T}$-algebra, we see that $\overline{F}(TX, \mu X) = (FTX, \overline{\mu}X)$ is a $\mathbb{T}$-algebra.

Put $\delta = F\eta \circ \overline{\mu} : TFT \to F$.

Show that $\delta = (\delta X)$ is a natural transformation from TFT to F. Given $f \in \mathcal{C}(X, Y)$, we obtain

$\delta X \circ TFTf = F\eta X \circ \overline{\mu}X \circ TFTf = F\eta X \circ FTf \circ \overline{\mu}Y = F(Tf \circ \eta X) \circ \overline{\mu}Y = F(\eta Y \circ$
$\circ f) \circ \overline{\mu}Y = Ff \circ F\eta Y \circ \overline{\mu}Y = Ff \circ \delta Y$.

Show that (i) holds. We have

$\delta X \circ \eta FTX = F\eta X \circ \overline{\mu}X \circ \eta FTX = F\eta X$.

Finally, we have to check (ii):

$\delta X \circ T\delta TX \circ T^2F\mu X = F\eta X \circ \overline{\mu}X \circ TF\eta TX \circ T\overline{\mu}TX \circ T^2F\mu X = F\eta X \circ \overline{\mu}X \circ TF\eta TX \circ$
$\circ TF\mu X \circ T\overline{\mu}X = F\eta X \circ \overline{\mu}X \circ TF(\mu X \circ \eta TX) \circ T\overline{\mu}X = F\eta X \circ \overline{\mu}X \circ T\overline{\mu}X = F\eta X \circ \overline{\mu}X \circ$
$\circ \mu FTX = \delta X \circ \mu FTX$.

Show that the above correspondence is a bijection. Given a natural transformation $\delta = (\delta X)$ satisfying (i) and (ii) consider the lifting $\overline{F}$ defined by $\overline{F}(X, \xi) = (FX, \delta X \circ TF\xi)$, $\overline{F}f = Ff$. Then $\overline{F}$ determines the natural transformation $\hat{\delta} = (\hat{\delta}X)$, $\hat{\delta}X = F\eta X \circ \overline{\mu}X$ and we have $\hat{\delta}X = F\eta X \circ \overline{\mu}X = F\eta X \circ \delta TX \circ TF\mu X = \delta X \circ TFT\eta X \circ TF\mu X = = \delta X \circ TF(\mu X \circ T\eta X) = \delta X$.

Conversely, given a lifting $\overline{F}$ of F onto the category $\mathcal{C}^{\mathbb{T}}$, consider the natural transformation $\delta = (\delta X)$ defined by $\delta X = F\eta X \circ \overline{\mu}X$, $X \in |\mathcal{C}|$. The natural transformation δ determines the lifting $\hat{F}$ of F onto $\mathcal{C}^{\mathbb{T}}$ by the formula $\hat{F}(TX, \mu X) = (FTX, \delta TX \circ TF\mu X)$.

We have

$\hat{F}(TX, \mu X) = (FTX, \delta TX \circ TF\mu X) = (FTX, F\eta TX \circ \overline{\mu}TX \circ TF\mu X) = (FTX, F\eta TX \circ$
$\circ F\mu X \circ \overline{\mu}X) = (FTX, \overline{\mu}X) = \overline{F}(TX, \mu X)$.

Let $(X, \xi) \in |\mathcal{C}^{\mathbb{T}}|$. Since ξ is a morphism of a $\mathbb{T}$-algebra $(TX, \mu X)$ into the $\mathbb{T}$-algebra (X, ξ), we see that $\hat{F}\xi = F\xi$ is a morphism of the $\mathbb{T}$-algebra $\hat{F}(X, \xi) = (FX, u)$ into the $\mathbb{T}$-algebra $\hat{F}(TX, \mu X) = \overline{F}(TX, \mu X) = (FTX, \overline{\mu}X)$. Thus, $F\xi \circ u = \overline{\mu}X \circ TF\xi$,

$F\eta X \circ F\xi \circ u = F\eta X \circ \overline{\mu}X \circ TF\xi$,

and we obtain $u = F\eta X \circ \overline{\mu}X \circ TF\xi$.

Thus, $\hat{F}(X, \xi) = (FX, F\eta X \circ \overline{\mu}X \circ TF\xi) = (FX, \delta X \circ TF\xi) = \overline{F}(X, \xi)$.

We see that any lifting of a contravariant functor onto $\mathcal{C}^{\mathbb{T}}$ is completely determined by its values onto the free algebras.

Remark. From the proof of Proposition 1 we see that a bijective correspondence between lifting $\overline{F}$ of F onto $\mathcal{C}^{\mathbb{T}}$ and natural transformations δ satisfying (i) and (ii) can be given by:

given δ, we set $\overline{F}(X, \xi) = (FX, \delta X \circ TF\xi)$ for $(X, \xi) \in |\mathcal{C}^{\mathbb{T}}|$;

given $\overline{F}$ we set $\delta = F\eta \circ \overline{\mu}$.

Recall that $\mathbb{T}$ is said to be projective [4] provided there exists a natural transformation $\pi : T \to 1$ (projection) such that $\pi \circ \eta = 1$ and $\pi \circ \mu = \pi \circ \pi T = \pi \circ T\pi$. The following is a counterpart of a result of Zarichnyi.

Proposition 2. *For any contravariant functor F and any projective monad $\mathbb{T}$ there exists a lifting of F onto the category $\mathcal{C}^{\mathbb{T}}$.*

Proof. Put $\delta = F\eta \circ \pi FT$ (here π denotes the projection), then
$\delta \circ \eta FT = F\eta \circ \pi FT \circ \eta FT = F\eta$.
Besides,
$\delta \circ T\delta T \circ T^2F\mu = F\eta \circ \pi FT \circ TF\eta T \circ T\pi FT^2 \circ T^2F\mu = F\eta \circ \pi FT \circ TF\eta T \circ T(\pi FT^2 \circ$ $\circ TF\mu) = F\eta \circ \pi FT \circ TF\eta T \circ TF\mu \circ T\pi FT = F\eta \circ \pi FT \circ T\pi FT = F\eta \circ \pi FT \circ \mu FT = \delta \circ \mu FT$.

3°. Suppose $C : \mathcal{C} \to \mathcal{C}$ is a contravariant functor such that there exists a natural transformation $\eta : 1 \to C^2$ satisfying the property: $C\eta \circ \eta C = 1_C$. Put $T = C^2$ and define the natural transformation $\mu : T^2 = C^4 \to C^2 = T$ by the formula: $\mu = C\eta C$.
Remark that the triple $\mathbb{T} = (T, \eta, \mu)$ is a monad on the category $\mathcal{C}$ (see [2]).

Proposition 3. *The natural transformation $\delta = C\eta \circ C^3\eta : TCT = C^5 \to C$ satisfies conditions (i) and (ii) from Proposition 1.*

Proof. We have
$C\eta \circ C^3\eta \circ \eta C^3 = C\eta \circ \eta C \circ C\eta = C\eta$.
To prove (ii), we see that
$C\eta \circ C^3\eta \circ C^3\eta C^2 \circ C^5\eta C^2 \circ C^6\eta C = C\eta \circ C^3\eta \circ C^3(C^2\eta C^2 \circ \eta C^2) \circ C^6\eta C = C\eta \circ C^3\eta \circ$ $\circ C^3(\eta C^4 \circ \eta C^2) \circ C^6\eta C = C\eta \circ C^3\eta \circ C^3\eta C^2 \circ C^3\eta C^4 \circ C^6\eta C = C\eta \circ C^3\eta \circ C^3\eta C^2 \circ C^4\eta C \circ$ $\circ C^3\eta C^2 = C\eta \circ C^3\eta \circ C^3(C\eta C \circ \eta C^2) \circ C^3\eta C^2 = C\eta \circ C^3\eta \circ C^3\eta C^2 = C(C^2\eta \circ \eta) \circ C^3\eta C^2 =$ $= C(\eta C^2 \circ \eta) \circ C^3\eta C^2 = C\eta \circ C\eta C^2 \circ C^3\eta C^2 = C\eta \circ C(C^2\eta C^2 \circ \eta C^2) = C\eta \circ C(\eta C^4 \circ \eta C^2) =$ $= C\eta \circ C\eta C^2 \circ C\eta C^4 = C(\eta C^2 \circ \eta) \circ C\eta C^4 = C(C^2\eta \circ \eta) \circ C\eta C^4 = C\eta \circ C^3\eta \circ C\eta C^4$.

Let $Tych$ denote the category of Tychonov spaces and their continuous maps. For a Tychonov space X we denote by C_pX the space of real-valued functions on X endowed by the topology of pointwise convergence. This construction determines a contravariant functor in $Tych$: for a map $f : X \to Y$ we have $C_pf(\varphi) = \varphi \circ f$, $\varphi \in C_pY$.
It is well-known that there exists a natural transformation $\eta : 1_{Tych} \to C_pC_p = C_p^2$.
It is defined by the condition:
$\eta X(x)(\varphi) = \varphi(x)$, where $x \in X$, $\varphi \in C_pX$.
It is known that $C_p\eta \circ \eta C_p = 1_{C_p}$ (see [2]). We see that the functor $T_p = C_p^2$ determines a monad on the category $Tych$ (see [3]).

Corollary. *The contravariant functor C_p has a lifting onto the category $Tych^{\mathbb{T}}$.*

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