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ON GROUPS WITH NILPOTENT QUOTIENTS WITH RESPECT TO INFINITE NORMAL SUBGROUPS

O. V. Turash

Turash O.V. On groups with nilpotent quotients with respect to infinite normal subgroups. We characterize Černikov groups whose quotients by infinite normal subgroups are nilpotent of the class $\leqslant c$.

Last time, one can notice an increasing interest to studying the structure of groups via investigating the properties of their quotients. In many papers groups with some restrictions on the system of all quotients by infinite normal subgroups are considered. In particular, in [5] Kalashnikova describes the locally soluble groups with abelian quotients by infinite normal subgroups.

In this paper we characterize Černikov groups whose quotients by infinite normal subgroups are nilpotent of the class $\leqslant c$.

All the notations are standard and can be found in [6]. Let us remark only that c is a non-negative integer, $\zeta(G)$ the center of the group G, $\mathbb{C}_{p^\infty}$ the quasicyclic p-group, $C_G(H)$ the centralizer of a subgroup H in G.

An infinite group G is defined to be an N_cQI-group if for every infinite normal subgroup H the quotient-group G/H is nilpotent of the class $\leqslant c$. For an N_cQI-group G by $I(G)$ denote the intersection of all infinite normal subgroups of G. If $I(G) = \langle 1 \rangle$, then, by Remak's Theorem, G is nilpotent of the class $\leqslant c$. So, further we assume that $I(G) \neq \langle 1 \rangle$. Let H be a normal in G subgroup of $I(G)$, $H \neq I(G)$. Clearly, H is finite.

An infinite normal subgroup L of a group A is said to be A-quasifinite if L satisfies the following two conditions:

1) every proper normal in A subgroup of L is finite;
2) L coincides with the union of all its proper finite normal in A subgroups.

Therefore, either all normal subgroups of G are finite or G is quasifinite, or $I(G)$ contains a finite normal in G subgroup B such that $I(G)/B$ is an infinite G-principal quotient.

Lemma 1. *Let G be an N_cQI-group, H its infinite normal subgroup. If H has an infinite family $\mathcal{G}$ of infinite normal in G subgroups such that $\cap\mathcal{G} = \langle 1 \rangle$, then G is nilpotent of the nilpotency class $\leqslant c$.*

Proof. If $L \in \mathcal{G}$, then the quotient G/L is nilpotent of the class $\leqslant c$. By Remak's Theorem, G is nilpotent of the class $\leqslant c$.

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Corollary 1.1. *Let G be an N_cQI-group, H its infinite abelian normal subgroup of finite rank. If $\cap_{n\in\mathbb{N}}H^n = \langle 1\rangle$, then G is nilpotent of the class $\leqslant c$.*

Corollary 1.2. *Let G be an N_cQI-group. If G is either non-nilpotent or nilpotent of the class $> c$, then its center $\zeta(G)$ is a periodic subgroup.*

Lemma 2. *Let G be either non-nilpotent or nilpotent of the class $> c$. Then its center $\zeta(G)$ contains a quasicyclic p-subgroup of an finite index, whenever $\zeta(G)$ is infinite.*

Proof. By Corollary 1.2 the center $\zeta(G)$ of G is periodic. Suppose that $\zeta(G)$ contains a subgroup $A = \prod_{\lambda\in\Lambda} A_\lambda$, $|\Lambda| = \infty$, with $A_\lambda \neq \langle 1\rangle$ for all $\lambda \in \Lambda$. Then for some infinite subsets Λ_1 and Λ_2 in Λ with $\Lambda_1 \sqcup \Lambda_2 = \Lambda$ the subgroups $B_1 = \prod_{\lambda\in\Lambda_1} A_\lambda$, $B_2 = \prod_{\lambda\in\Lambda_2} A_\lambda$ are infinite with $B_1 \cap B_2 = \langle 1\rangle$. Therefore, the quotients G/B_1 and G/B_2 are nilpotent of the class $\leqslant c$. Thus, by Remak's theorem the group G is nilpotent of the class $\leqslant c$.

Hence, the center $\zeta(G)$ does not contain subgroups which can be decomposed into an infinite direct product of nontrivial subgroups. Then $|\pi(\zeta(G))| < \infty$. Since the center $\zeta(G)$ is infinite, there exists a prime number p such that a Sylov p-subgroup P of $\zeta(G)$ is infinite. The subgroup $\Omega_1(P)$ is elementary abelian, thus, it is finite. Hence, P is a Černikov subgroup [2]. Let D be the divisible subgroup of P. The subgroup D is quasicyclic. Indeed, otherwise D contains an isomorphic to $\mathbb{C}_{p^\infty} \times \mathbb{C}_{p^\infty}$ subgroup $C_1 \times C_2$. Since C_i is infinite, the quotient G/C_i is nilpotent of the class $\leqslant c$ $(i = 1, 2)$. By Remak's Theorem, the group G is nilpotent of the class $\leqslant c$. Contradiction.

Let $q \in \pi(\zeta(G))$, $q \neq p$, and let Q be a Sylov q-subgroup of $\zeta(G)$. If $|Q| < \infty$, then applying Remak's Theorem, we obtain that G is nilpotent of the class $\leqslant c$. Therefore,the Sylov p'-subgroup of $\zeta(G)$ is finite. Hence $|\zeta(G) : D|$ is finite. Lemma is proved.

Theorem 1. *Let G be a nilpotent group of the class $> c$. If G is a Černikov group, then G is N_cQI-group iff it satisfies the following conditions:*

1) $\zeta(G) = D \times F$ for the quasicyclic p-group D (p is a prime number) and some finite group F;
2) $\gamma_{c+1}(G)$ is a finite subgroup of D;
3) the quotient $G/\zeta(G)$ is finite.

Proof. $(\Rightarrow)$ Since G is a nilpotent Černikov group, the quotient $G/\zeta(G)$ is finite.

Therefore, $|\zeta(G)| = \infty$, by Lemma 2, the center $\zeta(G)$ contains a subgroup $D \cong \mathbb{C}_{p^\infty}$ (p is a prime number) for which $|\zeta(G) : D| < \infty$. Then $\zeta(G) = D \times F$ for some finite subgroup F. Since $|G/\zeta(G)| < \infty$, by Shur's Theorem, the derived subgroup $[G, G]$ is finite. Hence, $\gamma_{c+1}(G)$ is also finite. Hence, G/D is nilpotent of the class $\leqslant c$, because D is infinite. Thus, $\gamma_{c+1}(G) \leqslant D$.

$(\Leftarrow)$ Suppose G satisfies conditions (1)–(3), and H is an infinite normal subgroup of G. Since $|G : D| < \infty$, the intersection $H \cap D$ is infinite. Therefore, $D \leqslant H$. Then G/H is nilpotent of the class $\leqslant c$, because $\gamma_{c+1}(G) \leqslant D$. Theorem is proved.

Theorem 2. *Suppose group G is not nilpotent. If G is a Černikov group, then G is an N_cQI-group, iff it satisfies the following conditions:*

1) for any divisible subgroup D of G the quotient $G/C_G(D)$ is a finite cyclic group;
2) every infinite normal in G subgroup L of D coincides with D (in particular, D is a p-group for some prime p);
3) the quotient G/D is nilpotent of the class $\leqslant c$;

4) $\gamma_{c+1}(G) = D$.

Proof. ($\Rightarrow$) Consider a minimal normal in G divisible subgroup D_1 of D. Suppose that $D \neq D_1$. Then there exists a normal in G divisible subgroup D_2 such that $D = D_1 \cdot D_2$ and $|D_1 \cap D_2| < \infty$ [1]. In particular, D_2 is an infinite normal subgroup of G. Therefore, the quotient G/D_2 is nilpotent of the class $\leqslant c$. By Remak's Theorem, $G/(D_1 \cap D_2)$ is a nilpotent group of the class $\leqslant c$. Thus, the group γ_{c+1} is finite. In particular, γ_{c+1} is an FC-group.

Let $D(\gamma_{c+1}(G))$ be a normal in G divisible part of $\gamma_{c+1}(G)$. Then $D(\gamma_{c+1}(G)) \leqslant \zeta(\gamma_{c+1}(G))$ [4]. Since G is an N_cQI-group, the quotient $\gamma_{c+1}(G)/D(\gamma_{c+1}(G))$ is nilpotent of the class $\leqslant c$. Hence, $\gamma_{c+1}(G)$ is nilpotent, and therefore, G is also nilpotent, a contradiction. Hence, we proved the equality $D = D_1$ and condition (2). Since G is an N_cQI-group, the quotient G/D is finite nilpotent of the class $\leqslant c$. We obtain condition (3). Condition (2) and Lemma 3.1 of [3] imply condition (1). Since G/D is nilpotent of the class $\leqslant c$, we have $\gamma_{c+1} \leqslant D$. Supposing $\gamma_{c+1}(G) \neq D$, by (2) we obtain that $|\gamma_{c+1}(G)| < \infty$. But, as above, is this case G is nilpotent. Hence, $\gamma_{c+1}(G) = d$ and condition (4) holds.

($\Leftarrow$) Let G satisfy (1)–(4). Arguing as in the proof of Theorem 1, we easy by obtain that every infinite normal subgroup of G contains D. Since $D = \gamma_{c+1}(G)$, the group G is an N_cQI-group. Theorem is proved.

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