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ON HEREDITARY RADICALS OF TORSION-FREE LOCALLY NILPOTENT GROUPS

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Artemovych O.D. On hereditary radicals of torsion-free locally nilpotent groups. We studied the hereditary radicals of torsion-free locally nilpotent groups. It was proved that any hyperabelian torsion-free locally nilpotent (respectively torsion-free Baer) group G has only the trivial hereditary proper radical (in the sence of Kurosh). Any Baer (in particular, a Fitting) torsion-free group G has only the trivial hereditary proper radical (in the sence of Plotkin). Moreover, the Baer radical $\mathcal{B}G$ (respectively the Fitting radical $\mathcal{F}G$) of G is isolated in a torsion-free locally nilpotent group G.

0. Let $\mathfrak{X}$ be an abstractive group theoretical property. A group G (respectively subgroup H of G) is said to be an $\mathfrak{X}$-group (respectively $\mathfrak{X}$-subgroup) if it has the property $\mathfrak{X}$.

Throughout this paper $\mathfrak{N}$ be a class of groups closed under homomorphic images and normal subgroups. A mapping $\mathfrak{X} : \mathfrak{N} \longrightarrow \mathfrak{N}$ is called a radical (in the sense of Kurosh) if it assing to each group G of $\mathfrak{N}$ the subgroup $\mathfrak{X}(G)$ which contains all normal $\mathfrak{X}$-subgroups of G and satisfies the following properties:

$(R1)$ $\mathfrak{X}(\mathfrak{X}(G)) = \mathfrak{X}(G)$ for each group $G \in \mathfrak{N}$;

$(R2)$ if $\phi : G \longrightarrow F$ is an epimorphism of groups G, F of $\mathfrak{N}$ then $\phi(\mathfrak{X}(G)) \leqslant \mathfrak{X}(F)$;

$(R3)$ $\mathfrak{X}(G/\mathfrak{X}(G)) = E$ is the identity group for each $G \in \mathfrak{N}$.

The subgroup $\mathfrak{X}(G)$ of G is a radical of G (in the sense of Kurosh) if it satisfies the properties $(R1) - (R3)$. Moreover, say that the radical $\mathfrak{X}(G)$ of G is hereditary if

$(R4)$ $H \leqslant \mathfrak{X}(G)$ implies $\mathfrak{X}(H) = H$ for each normal subgroup H of G, $G \in \mathfrak{N}$.

Recall that a subgroup H of torsion-free group G which coincides with their isolator

$$I_G(H) = \{x \in G \mid \exists n \in \mathbb{N} : \ x^n \in H\}$$

in G is said to be isolated in G.

Recall the following theorem, which we'll use later (see A.G. Kurosh [1], Yu.M. Ryabuhin [2], S.E. Dickson [3]; and Proposition 2.17 [4]).

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Proposition 0.1. *A radical $\mathfrak{X}$ is hereditary (in the sense of Kurosh) in the class of all Abelian groups $\mathfrak{A}$ if and only if it is one of the following types:*

(1) *R, i.e. $R(G) = G$ for each $G \in \mathfrak{A}$;*
(2) *R_π, i.e. for each $G \in \mathfrak{A}$ $R_\pi(G)$ is the maximal π-subgroup of G where π is some set of primes;*
(3) *R_0, i.e. $R_0(G) = E$ for each $G \in \mathfrak{A}$.*

Most of our notation is standard; in particular we refer to [4–7]. Throughout this paper we shall use the following notation: $\mathcal{B}(G)$ denotes the Baer radical of G , $\mathcal{F}(G)$ the Fitting radical of G.

1. In this section we shall study the hereditary radicals (in the sense of Kurosh) of torsion-free locally nilpotent groups.

The following theorem is an extension of Proposition 0.1 to the class of all hypercentral groups $\mathfrak{H}$.

Theorem 1.1. *A radical $\mathfrak{X}$ is hereditary (in the sense of Kurosh) in the class of all hypercentral groups $\mathfrak{H}$ if and only if it is of one of the following types:*

(1) *R_0, i.e. $R_0(G) = E$ for each $G \in \mathfrak{H}$;*
(2) *R_π, i.e. $R_\pi(G)$ is the maximal π-subgroup of G for each $G \in \mathfrak{H}$ where π is some set of primes;*
(3) *R, i.e. $R(G) = G$ for each $G \in \mathfrak{H}$.*

Proof. Suppose that $\mathfrak{X}(G)$ is any nontrivial hereditary radical of hypercentral group G, $Z = Z(G)$ is the centre of G. Then $Z \cap \mathfrak{X}(G) \neq E$ [5, Proposition 1.6] and therefore $Z \cap \mathfrak{X}(G) = \mathfrak{X}(Z)$. Without restricting generality, we can assume that $\mathfrak{X}(Z) \neq Z$. Then in view of Proposition 0.1 $\mathfrak{X}(Z)$ is the maximal π-subgroup of Z for some set π of primes. It is easy to see that $\mathfrak{X}(G)$ is a periodic group. If further $\mathfrak{X}(G) = \mathfrak{X}_1 \times \mathfrak{X}_2$ is a decomposition in a direct product of Sylow π-subgroup $\mathfrak{X}_1$ and π'-subgroup $\mathfrak{X}_2$ then from $\mathfrak{X}_2 \cap Z = E$ in view of Proposition 1.6 [5] we obtain that $\mathfrak{X}_2 = E$. Finally, since $\mathfrak{X}(Z)$ is the maximal π-subgroup of Z, we conclude that $\mathfrak{X}(G)$ is the maximal π-subgroup of G; this completes the proof.

A sketch of proof allows us to obtain, for any radical (in the sense of Kurosh) in the class of all Abelian groups $\mathfrak{A}$, its extension in the class of hypercentral groups. For example, an extension of Theorem 12 [1] is the following

Proposition 1.2. *In the class of all hypercentral p-groups there exists only one nontrivial proper radical $\mathfrak{D}$, i.e. for each hypercentral p-group G the subgroup $\mathfrak{D}(G)$ is the divisible part of G.*

Now we consider hereditary radicals in torsion-free groups. From Theorem 1.1 it follows

Corollary 1.3. *For any radical (in the sense of Kurosh) $\mathfrak{X}$ all infinite cyclic groups are either $\mathfrak{X}$-radical or $\mathfrak{X}$-semiprime.*

Lemma 1.4. *Let $\mathfrak{X}(G)$ be a nontrivial hereditary radical (in the sense of Kurosh) of torsion-free group G. Then for all subnormal cyclic subgroups A of G either $A \leqslant \mathfrak{X}(G)$ or $A \cap \mathfrak{X}(G) = E$.*

Proof. In consequence of subnormality of A in G we have $\mathfrak{X}(A) = A \cap \mathfrak{X}(G)$ whence by Corollary 1.3 either $A \leqslant \mathfrak{X}(G)$ or $A \cap \mathfrak{X}(G) = E$, as desired.

Corollary 1.5. *Let $\mathfrak{X}(G)$ be a nontrivial hereditary radical (in the sense of Kurosh) of torsion-free group G. Then either $\mathcal{B}(G) \leqslant \mathfrak{X}(G)$ or $\mathcal{B}(G) \cap \mathfrak{X}(G) = E$.*

In fact, if b is an arbitrary element of $\mathcal{B}(G) \cap \mathfrak{X}(G)$ then

$$\mathfrak{X}(\langle b \rangle) = \langle b \rangle \cap \mathfrak{X}(G) = \langle b \rangle.$$

Therefore, for any element a of $\mathcal{B}(G) \setminus \mathfrak{X}(G)$ we have

$$\mathfrak{X}(\langle a \rangle) = \langle a \rangle \cap \mathfrak{X}(G) \neq \langle a \rangle,$$

a contradiction with Corollary 1.3.

Proposition 1.6. *Any hyperabelian torsion-free locally nilpotent (respectively torsion-free Baer) group G has only the trivial hereditary proper radical (in the sense of Kurosh).*

Proof. Let G be a hyperabelian torsion-free locally nilpotent group with normal series

$$E = A_0 \leqslant A_1 \leqslant \dots A_\alpha \leqslant A_{\alpha+1} \leqslant \dots \leqslant A_\gamma = G \tag{1}$$

such that $A_\beta = \bigcap_{\delta<\beta} A_\delta$ if β is a limit ordinals, the quotient $A_{\alpha+1}/A_\alpha$ is an Abelian torsion-free group for all $\alpha < \gamma$ and any term of the series (1) is a normal subgroups of G. Let $\mathfrak{X}(G)$ be an arbitrary nontrivial hereditary radical (in the sense of Kurosh) of G and A an arbitrary nontrivial subnormal cyclic subgroup of $\mathfrak{X}(G)$. Then for every nontrivial element b of A_1 in view of isomorphism $A \cong \langle b \rangle$ and $\langle b \rangle \lhd\lhd G$ we have $\langle b \rangle \leqslant \mathfrak{X}(G)$ and consequently $A_1 \leqslant \mathfrak{X}(G)$. By induction it can be readily verified that $\mathfrak{X}(G) = G$.

Corollary 1.7. *Let G be a torsion-free locally nilpotent group, $\mathfrak{X}(G)$ a hereditary radical (in the sense of Kurosh) of G. If*

$$I_{\mathfrak{X}(G)}(\mathfrak{X}(G)') \neq \mathfrak{X}(G)$$

then the subgroup $\mathfrak{X}(G)$ is isolated in G.

Corollary 1.8. *Let G be a torsion-free group, $\mathfrak{X}(G)$ a hereditary radical (in the sense of Kurosh) of G. If the Baer radical $\mathcal{B}(\mathfrak{X}(G))$ of G is nontrivial then $\mathfrak{X}(G)$ is isolated in G.*

2. In accord with [8] a mapping $\theta : \mathfrak{N} \longrightarrow \mathfrak{N}$ is called a functorial if it assings to each group G of $\mathfrak{N}$ some subgroup $\theta(G)$ of G and

$$\theta(\phi(G)) = \phi(\theta(G))$$

for every isomorfism $\phi : G \longrightarrow \phi(G)$. A functorial θ is hereditary radical in the class $\mathfrak{N}$ (in the sense of Plotkin) if it satisfies the following condition:

$(P1)$ $\theta(\theta(G)) = \theta(G)$ for all $G \in \mathfrak{N}$;
$(P2)$ $\theta(H) = H \cap \theta(G)$ for every normal subgroup H of G, $G \in \mathfrak{N}$;
$(P3)$ $\pi(\theta(G)) \leqslant \theta(D)$ for all epimorphisms $\pi : G \longrightarrow D$ with $G, D \in \mathfrak{N}$.

Lemma 2.1. *Let $\theta(G)$ be a hereditary radical (in the sense of Plotkin) of torsion-free group G. Then either $\mathcal{B}(G) \leqslant \theta(G)$ or $\mathcal{B}(G) \cap \theta(G) = E$.*

Corollary 2.2. *Any Baer (in particular a Fitting) torsion-free group G has only the trivial hereditary proper radical (in the sense of Plotkin).*

Corollary 2.3. *Any torsion-free hypercentral group of lenght at most ω has only the trivial proper hereditary radicals (in the sense of Plotkin).*

Proposition 2.4. *The Baer radical $\mathcal{B}(G)$ (respectively the Fitting radical $\mathcal{F}(G)$) of G is isolated in a torsion-free locally nilpotent group G.*

Proof. If $a^n \in \mathcal{B}(G)$ for some nontrivial element a of $G \setminus \mathcal{B}(G)$ and some integer n then G has a normal series

$$\langle a^n \rangle \leqslant I_1 \leqslant \dots I_n = G$$

and consequently G has a normal series

$$I_G(\langle a^n \rangle) \leqslant I_G(I_1) \leqslant \dots \leqslant G.$$

Since the isolator $I_G(\langle a^n \rangle)$ is Abelian [6, p.413], we conclude that the subgroup $\langle a \rangle$ is subnormal in G, and consequently $a \in \mathcal{B}(G)$.

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