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ON PROJECTIVE FUNCTORS IN THE CATEGORY OF COMPACTA

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Teleiko A.B. On projective functors in the category of compacta. We characterize projective functors in the class of normal functors of finite degree: only such ones admit functorial extensions of semigroup operations.

This note is devoted to the general problem of lifting of endofunctors in the category of compacta onto categories of compacta with some algebraic structures (see, e.g., [3, 7]). As remarked in [7] the problem of extension of a binary operation $X \times X \to X$ onto FX (F is a functor) is related to existence of a "nice" natural transformation $FX \times FX \to F(X \times X)$. It turns out that in the class of normal functors of finite degree only projective functors admit such transformations. As a consequence, we shall obtain that in this class only projective functors lift onto the category of compact semigroups.

Remark also that some characterizations of projective functors in the category of compacta are known. The interested reader may look through the articles [4, 5].

1. An endofunctor F on the category $\mathcal{C}$ is said to be *projective* if there exist natural transformations $\eta: 1_{\mathcal{C}} \to F$ and $\pi: F \to 1_{\mathcal{C}}$ with $\pi \circ \eta = \mathrm{id}$ (see [2] for related notions).

Denote by $\mathcal{C}omp$ the category of compacta and their continuous maps.

A functor $F: \mathcal{C}omp \to \mathcal{C}omp$ is called *normal* [1] if it is continuous, monomorphic, epimorphic, preserves weight, intersections, preimages, singletons, and empty set.

Remark 1. If F is a projective normal endofunctor in $\mathcal{C}omp$, then $F \times 1_{\mathcal{C}omp}$ and every epimorphic subfunctor of F are also projective (and normal).

For a normal functor F we shall denote by η a unique natural transformation $1_{\mathcal{C}omp} \to F$ [1].

Let F be a normal functor, $X \in \mathcal{C}omp$, $a \in FX$. The *support* $\mathrm{supp}(a)$ of the point a is defined by the formula [1]: $\mathrm{supp}(a) = \bigcap\{A \mid A \text{ is a closed set in } X, a \in FA\}$ (here we identify FA and $Fj(A)$, where $j: A \to X$ is the natural embedding). Denote by $\deg(a)$ the *degree* of a: $\deg(a) = |\mathrm{supp}(a)|$. The *degree* $\deg F$ of the functor F is said to be the cardinal number $\sup\{|\mathrm{supp}(a)| \mid a \in FX, X \in \mathcal{C}omp\}$. Recall that normal functors F preserve supports, i.e., $\mathrm{supp}\, Ff(a) = f(\mathrm{supp}(a))$, $a \in FX$, $f: X \to Y$, $X, Y \in \mathcal{C}omp$.

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Let $(-)^2: \mathcal{C}omp \to \mathcal{C}omp$ be the power functor: $(-)^2X = X \times X$, $(-)^2 f = f \times f$, $f: X \to Y$, $X, Y \in \mathcal{C}omp$.

We identify a number $n \in \mathbb{N}$ with the set $\{0, 1, \dots, n-1\}$.

Theorem 1. *Let F be a normal functor of finite degree. Then F is projective if and only if there exists a natural transformation $\xi: (-)^2 F \to F(-)^2$ with $\xi \circ (-)^2\eta = \eta(-)^2$.*

Proof. ($\Longrightarrow$) Let $\pi: F \to 1_{\mathcal{C}omp}$ be a projection. Set $\xi = \eta(-)^2 \circ (\pi \times \pi)$.
($\Longleftarrow$) Let X_1 and X_2 be compacta. Let $j_i: X_i \to X_1 \sqcup X_2$, $i = 1, 2$, be the natural embeddings. We identify $a_i \in FX_i$ with $Fj_i(a_i) \in F(X_1 \sqcup X_2)$, $i = 1, 2$. Writing $\xi(X \sqcup Y)(a, b)$, we always suppose that $a \in FX \subset F(X \sqcup Y)$ and $b \in FY \subset F(X \sqcup Y)$.

Denote by pr_i the natural projection $(-)^2 \to 1_{\mathcal{C}omp}$ onto the i-th factor, $i = 1, 2$.

Claim 1. *The following equalities hold: $F\,\mathrm{pr}_2 \circ \xi(X_1 \sqcup X_2)\big(a, \eta X_2(x)\big) = \eta X_2(x)$, $a \in FX_1, x \in X_2$, and $F\,\mathrm{pr}_1 \circ \xi(X_1 \sqcup X_2)\big(\eta X_1(x), b\big) = \eta X_1(x)$, $b \in FX_2, x \in X_1$.* Prove the first equality. Let $x_1 \in X_1$ and $r: X_1 \sqcup X_2 \to \{x_1\} \sqcup X_2$ be the retraction $r(X_1) = \{x_1\}$. Then $Fr(a) = \eta X_1(x_1)$, $Fr\big(\eta X_2(x)\big) = \eta X_2(x)$. Therefore, $Fr \circ \xi(X_1 \sqcup X_2)\big(a, \eta X_2(x)\big) = \eta(X_1 \sqcup X_2)^2(x_1, x)$. Since F preserves supports, we see that $\mathrm{supp}\,\xi(X_1 \sqcup X_2)\big((a, \eta X_2(x)\big) \subset X_1 \times \{x\}$. This inclusion implies the first equality. The second one is obtained in the similar manner.

Claim 2. *For every $a, a', a'' \in FX_1$, $b, b', b'' \in FX_2$ one has*

$$F\,\mathrm{pr}_1 \circ \xi(X_1 \sqcup X_2)(a, b') = F\,\mathrm{pr}_1 \circ \xi(X_1 \sqcup X_2)(a, b'') \text{ and}$$
$$F\,\mathrm{pr}_2 \circ \xi(X_1 \sqcup X_2)(a', b) = F\,\mathrm{pr}_1 \circ \xi(X_1 \sqcup X_2)(a'', b).$$

Show only the first equality. Since $\xi \circ (-)^2\eta = \eta(-)^2$, we have $\mathrm{supp}\,\xi(a, b') \subset \mathrm{supp}(a) \times \mathrm{supp}(b')$. Let $x_0 \in X_2$. Consider the retraction $r: X_1 \sqcup X_2 \to X_1 \sqcup \{x_0\}$ such that $r(X_2) = \{x_0\}$. We obtain

$$\begin{aligned} F\,\mathrm{pr}_1 \circ \xi(X_1 \sqcup X_2)(a, b') &= Fr \circ F\,\mathrm{pr}_1 \circ \xi(X_1 \sqcup X_2)(a, b') = \\ &= F\,\mathrm{pr}_1 \circ F(r \times r) \circ \xi(X_1 \sqcup X_2)(a, b') = F\,\mathrm{pr}_1 \circ \xi(X_1 \sqcup X_2)\big(a, \eta X_2(x_0)\big). \end{aligned}$$

Hence, for every $a \in FX_1$ the point $F\,\mathrm{pr}_1 \circ (\xi) X_1 \sqcup X_2(a, b')$ does not depend on $b' \in FX_2$.

Let $\deg F = n$. For $m \leqslant n$ denote by F_m the subfunctor $F_m X = \{a \in FX \mid \deg(a) \leqslant m\}$, $X \in \mathcal{C}omp$, of F. Set

$$m_1 = \max\big\{\deg F\,\mathrm{pr}_1 \circ \xi(n \sqcup n)\big(a, \eta n(0)\big) \mid a \in Fn\big\},$$
$$m_2 = \max\big\{\deg F\,\mathrm{pr}_2 \circ \xi(n \sqcup n)\big(\eta n(0), b\big) \mid b \in Fn\big\}.$$

Consider the following case: $\max\{m_1, m_2\} < n$.
Without restricting generality, we may suppose that $m_1 \geqslant m_2$. Let $a \in Fn$ be such that $m_1 = \deg F\,\mathrm{pr}_1 \circ \xi(n \sqcup n)\big(a, \eta n(0)\big)$. Set $n_1 = m_2$.

Now we desire to construct a natural transformation $p: F \to F_{n_1}$ with $p \circ \eta = \eta$. For $k \leqslant n$ let $pk: Fk \to F_{n_1}k$ act by the formula $pk(b) = F\,\mathrm{pr}_2 \circ \xi(n \sqcup k)(a, b)$, $b \in Fk$. For a map $f: k_1 \to k_2$, $k_1, k_2 \leqslant n$, setting $h: n \sqcup k_1 \to n \sqcup k_2$, $h|n = \mathrm{id}$, $h|k_1 = f$, we obtain ($b \in Fk_1$)

$$\begin{aligned} Ff \circ pk_1(b) &= Fh \circ F\,\mathrm{pr}_2 \circ \xi(n \sqcup k_1)(a, b) = F\,\mathrm{pr}_2 \circ \xi(n \sqcup k_2)\big(Fh(a), Fh(b)\big) = \\ &= F\,\mathrm{pr}_2 \circ \xi(n \sqcup k_2)\big(a, Ff(b)\big) = pk_2 \circ Ff(b). \end{aligned}$$

Moreover, by Claim 1, $pk \circ \eta k(x) = \eta k(x)$, $x \in k$, $k \leqslant n$. Therefore, there exists a natural transformation $p: F \to F_{n_1}$ with $p \circ \eta = \eta$ (see, e.g., Pr.3.10 of ch.1 from [6]).

Now set $\xi_1 = p(-)^2 \circ \xi$. Then ξ_1 is a natural transformation $(-)^2 F \to F_{n_1}(-)^2$ with $\xi_1 \circ (-)^2\eta = \eta(-)^2$. Hence, if $\max\{m_1, m_2\} < n$, we obtain the number $n_1 < n$ and the natural transformation ξ_1.

Therefore, without restricting generality, we can suppose that there exist a number N, a point $a \in FN$, and a natural transformation $\xi': (-)^2 F \to F_N(-)^2$ such that $\xi' \circ (-)^2\eta = \eta(-)^2$ and

$$N = \deg F_N \operatorname{pr}_1 \circ \xi'(N \sqcup n)\big(a, \eta n(0)\big).$$

Remark that for a point $A \in FN$, $\deg A = N$, the functor

$$(F_N/A)X = \{b \in F_N(N \times X) \mid F_N \operatorname{pr}_1(b) = A\}$$

is projective [5]. To obtain this fact it is sufficient to consider the natural transformation $\pi_0: F_N/A \to 1_{Comp}$, $\pi_0 X(b) = y$, where $(0, y) \in \operatorname{supp}(b)$, $b \in (F_N/A)(X)$, $X \in Comp$.

Now set $\pi X: FX \to F_N(N \times X)$, $\pi X(b) = \xi'(N \sqcup X)(a, b)$, $b \in FX$, $X \in Comp$. Let $A = F_N \operatorname{pr}_1 \circ \xi'(N \sqcup n)\big(a, \eta n(0)\big)$. Applying Claim 2, we obtain that $\pi X(FX) \subset (F_N/A)(X)$. Hence, π is a natural transformation $F \to F_N/A$. By Claim 1 for the natural transformation $\pi_0 \circ \pi: F \to 1_{Comp}$ we have $\pi_0 \circ \pi \circ \eta = \mathrm{id}$.

2. Denote by $\mathcal{CS}$ the category of compact semigroups and their continuous homomorphisms. Let $U: \mathcal{CS} \to Comp$ be the forgetful functor. A functor $\bar{F}: \mathcal{CS} \to \mathcal{CS}$ is called a *lifting* of F onto the category $\mathcal{CS}$ if $U\bar{F} = FU$. A lifting $\bar{F}$ is *natural* if for every $(S, m) \in \mathcal{CS}$ the mapping ηS is a homomorphism $(S, m) \to \bar{F}(S, m)$.

Theorem 2. *Let F be a normal functor of finite degree. Then F has a natural lifting onto $\mathcal{CS}$ if and only if F is projective.*

Proof. ($\Longleftarrow$) Let $\pi: F \to 1_{Comp}$ be a projection. For every $(S, m) \in \mathcal{CS}$ it is sufficient to consider the following multiplication on FS: $\overline{m}: FS \times FS \to FS$, $\overline{m}(a, b) = \eta S \circ m\big(\pi(a), \pi(b)\big)$, $a, b \in FS$.

($\Longrightarrow$) Let $i: 1_{Comp} \to (-)^2$ be the natural transformation $iX(x) = (x, x)$, $x \in X$, $X \in Comp$. For every $X \in Comp$ consider the following multiplication mX on $X \times X$:

$$mX\big((x, y), (z, t)\big) = (x, t).$$

It is easy to see that mX is associative. Let $\bar{F}(X \times X, mX) = \big(F(X \times X), \overline{m}X\big)$. Set $\xi X = \overline{m}X \circ (FiX \times FiX): FX \times FX \to F(X \times X)$. Show that ξ is a natural transformation $(-)^2 F \to F(-)^2$ with $\xi \circ (-)^2\eta = \eta(-)^2$. Indeed, $\xi \circ (\eta \times \eta) = \overline{m} \circ (Fi \times Fi) \circ (\eta \times \eta) = \overline{m} \circ \big(\eta(-)^2 \times \eta(-)^2\big) \circ (i \times i) = \eta(-)^2 \circ m \circ (i \times i) = \eta(-)^2$, because $m \circ (i \times i) = \mathrm{id}$. Moreover, let $f: X \to Y$, $X, Y \in Comp$, be arbitrary. Then $\xi Y \circ (Ff \times Ff) = \overline{m}Y \circ (-)^2(FiY \circ Ff) = \overline{m}Y \circ (-)^2 F(f \times f) \circ (-)^2 FiX$. Since $f \times f: (X \times X, mX) \to (Y \times Y, mY)$ is a homomorphism, we have $\xi Y \circ (-)^2 Ff = F(f \times f) \circ \overline{m}X \circ (-)^2 FiX = F(f \times f) \circ \xi X$. By Theorem 1, we immediately obtain that F is projective.

Remark 2. It is interested whether Theorems 1 and 2 hold for weakly normal functors. (Recall that a functor is weakly normal if it satisfies all conditions of the definition of normal functor excepting the condition of preserving preimages.) The author knows only some answers to this question.

Let F be a weakly normal functor of finite degree n. A point $a \in FX$ is called invariant if $Fh(a) = a$ for every automorphism h of X such that $h(\mathrm{supp}(a)) = \mathrm{supp}(a)$.

It is easy to prove the following statement: let $a \in Fn$ be an invariant point with $\deg(a) > \sqrt{n}$; then there exists no natural transformation $\xi: (-)^2 F \to F(-)^2$ with $\xi \circ (-)^2\eta = \eta(-)^2$. In particular, this implies that the functors $\lambda_n, G_n, (N_m)_n$, $m \geqslant 2$, (see for definitions [6]) have no natural lifting onto $\mathcal{CS}$.

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