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ON EXTENSION OF THE CONTRAVARIANT FUNCTOR C_p ONTO CATEGORIES OF MULTIVALUED MAPS

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Levyts'ka V. S. On extension of the contravariant functor C_p onto categories of multivalued maps. It is proved that the contravariant functor C_p (the pointwise convergence function functor) has an extension onto the category of Tychonov spaces and finite-valued maps and has no extension onto the category of Tychonov spaces and compact-valued maps.

1°. INTRODUCTION

The general problem of extension of (covariant) functors onto the Kleisli category of a monad (see the definitions below) has been investigated by many authors (see [1],[2] for categorical results and [3] for the case of categories of compacta). In [4] the author considered the problem of extension of contravariant functors to the Kleisli categories and found a criterion for existence of such an extension.

In this note we consider the contravariant functor C_p acting in the category $Tych$ of Tychonov spaces and continuous maps and the problem of extension of this functor onto the categories of finite-valued and compact-valued maps. These categories can be naturally identified with the Kleisli categories of the finite hyperspace monad and the hyperspace monad respectively.

2°. DEFINITION AND AUXILARY RESULTS

A monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$, where $T : \mathcal{C} \to \mathcal{C}$ is a covariant functor and $\eta : 1_{\mathcal{C}} \to T$, $\mu : T^2 \to T$ are natural transformations satisfying the conditions: $\mu \circ \eta T = \mu \circ T\eta = 1_T$ and $\mu \circ \mu T = \mu \circ T\mu$. The *Kleisli category* of $\mathbb{T}$ is the category $\mathcal{C}_{\mathbb{T}}$ defined as follows: $|\mathcal{C}_{\mathbb{T}}| = |\mathcal{C}|$, $\mathcal{C}_{\mathbb{T}}(X,Y) = \mathcal{C}(X, TY)$, and the composition $g * f$ of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X,Y)$, $g \in \mathcal{C}_{\mathbb{T}}(Y,Z)$ is given by $g * f = \mu Z \circ Tg \circ f$.

Define the functor $I: \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ by $IX = X$, $X \in |\mathcal{C}|$ and $If = \eta Y \circ f$ for $f \in \mathcal{C}(X,Y)$.

A functor $\overline{F} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ called an extension of the functor $F : \mathcal{C} \to \mathcal{C}$ on the Kleisli category $\mathcal{C}_{\mathbb{T}}$ iff $IF = \overline{F}I$.

The following proposition gives a criterion for existence of extension of contravariant functors onto the Kleisli categories [2].

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Proposition 1. *There exists a bijective correspondence between the extensions of a contravariant functor F onto the category $\mathcal{C}_{\mathbb{T}}$ and the natural transformations $\xi : F \to TFT$ satisfying the conditions:*
(i) $TF\eta \circ \xi = \eta F$;
(ii) $TF\mu \circ \xi = \mu FT^2 \circ T\xi T \circ \xi$.

The proof is given in [4]; here we only note that the extension which corresponds to ξ is defined as follows: $\overline{F}f = TFf \circ \xi Y$, for $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$

In this situation, the natural transformation ξ is called *associated* to the extension $\overline{F}$.

Let $Tych$ denote the category of Tychonov spaces and their continuous maps. For a Tychonov space X we denote by C_pX the space of real-valued functions on X endowed with the topology of pointwise convergence. This construction determines a contravariant functor in $Tych$: for a map $f : X \to Y$ we have $C_pf(\varphi) = \varphi \circ f$, $\varphi \in C_pY$.

3°. Finite hyperspace monad

Let X be a Tychonov space. We denote by $\exp X$ the space of all non-empty compact subsets of X equipped with the Vietoris topology. Recall that the sets

$$\langle U_1, \dots, U_n \rangle = \{A \in \exp X \mid A \subset U_1 \cup \dots \cup U_n, A \cap U_i \neq \emptyset,$$

for all $i = 1, \dots, n\}$, where U_i run over the topology of X, form a base of the Vietoris topology.

For a continuous mapping $f : X \to Y$ the mapping $f : \exp X \to \exp Y$ is defined by the formula: $\exp f(A) = f(A) \in \exp Y$, $A \in \exp X$. Define the natural transformations $s : 1_{Tych} \to \exp$ and $u : \exp^2 \to \exp$ as follows: $sX(x) = \{x\}$ for each $x \in X$; $uX(\mathcal{A}) = \cup\mathcal{A}$, $\mathcal{A} \in \exp^2 X$. This construction determines the hyperspace monad $\mathbb{H} = (\exp, s, u)$ on the category $Tych$ (as well as on the category $Comp$)(see [5]).

We consider also its submonad $\mathbb{H}_f = (\exp_f, s, u)$ of hyperspace of finite sets on the category $Tych$. Here $\exp_f X = \{A \in \exp X \mid A \text{ is a finite set}\}$.

Note that the morphisms of the Kleisli category of the monad $\mathbb{H}$ (respectively $\mathbb{H}_f$) are compact-valued (respectively finite-valued) maps.

We consider the following problem: is there an extension of the contravariant functor C_p on the Kleisli category of the monad $\mathbb{H}_f = (\exp_f, s, u)$?

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $Tych$ and $(\mathbb{R}, \alpha)$ a $\mathbb{T}$-algebra. Define the map $\xi X : C_pX \to TC_pTX$ as follows:

$$\xi X(\varphi) = \eta C_pTX(\alpha \circ T\varphi) \quad \varphi \in C_pX.$$

Lemma 1. *Suppose ξX is the continuous mapping for every Tychonov space X. Then $\xi = (\xi X)_{X \in |Tych|}$ is a natural transformation which satisfies conditions (i) and (ii) from Proposition 1.*

Proof. Check that ξ is a natural transformation. Let $f \in Tych(X, Y)$ and $\varphi \in C_pY$. Then

$$\begin{aligned}\xi X(C_pf(\varphi)) &= \eta C_pTX(\alpha \circ TC_pf(\varphi)) = \eta C_pTX \circ C_pTf(\alpha \circ T\varphi) \\ &= TC_pTf \circ \eta C_pTY(\alpha \circ T\varphi) = TC_pTf(\xi Y(\varphi)).\end{aligned}$$

Show that ξ satisfies conditions (i) and (ii) from Proposition 1. Let $\varphi \in C_pX$, then

$$\begin{aligned}TC_p\eta X \circ \xi X(\varphi) &= TC_p\eta X \circ \eta C_pTX(\alpha \circ T\varphi) = \eta C_pX \circ C_p\eta X(\alpha \circ T\varphi) \\ &= \eta C_pX(\alpha \circ T\varphi \circ \eta X) = \eta C_pX(\alpha \circ \eta\mathbb{R} \circ \varphi) = \eta C_pX(\varphi),\end{aligned}$$

thus, (i) is satisfied.

We have to check (ii):

$$TC_p\mu X \circ \xi X(\varphi) = TC_p\mu X \circ \eta C_pTX(\alpha \circ T\varphi) = \eta C_pT^2X \circ C_p\mu X(\alpha \circ T\varphi)$$
$$= \eta C_pT^2X(\alpha \circ T\varphi \circ \mu X) = \eta C_pT^2X(\alpha \circ \mu\mathbb{R} \circ T^2\varphi) = \eta C_pT^2X(\alpha \circ T\alpha \circ T^2\varphi),$$

and

$$\mu C_pT^2X \circ T\xi TX \circ \xi X(\varphi) = \mu C_pT^2X \circ T\eta C_pT^2X(\eta C_pT^2X(\alpha \circ T(\alpha \circ T\varphi)))$$
$$= \eta C_pT^2X(\alpha \circ T\alpha \circ T^2\varphi).$$

Consider the mapping $\alpha : \exp_f \mathbb{R} \to \mathbb{R}$, $\alpha(A) = \max A$, where $A \in \exp_f \mathbb{R}$. It is easy to see that the pair $(\mathbb{R}, \alpha)$ is an $\mathbb{H}_f$-algebra (see [5]).

By Lemma 1, the natural transformation ξ associated to the extension of C_p onto $Tych_{\mathbb{H}_f}$, can be considered as the composition of mappings,

$$\xi = \eta C_p \exp_f \circ \xi',$$

where $\xi'X : C_pX \to C_p \exp_f X$ is defined as follows: $\xi'(\varphi) = \alpha \circ \exp_f(\varphi), \varphi \in C_pX$.

Show that the mapping $\xi'X$ is continuous.

Let $\varphi_0 \in C_pX$ i $\xi'(\varphi_0) = \Phi_0$. Base neighbourhoods of Φ_0 in $C_p \exp_f X$ are sets of the form

$$O(\Phi_0; A_1, \dots, A_k; \varepsilon) = \{\Phi \in C_p \exp_f X | |\Phi(A_i) - \Phi_0(A_i)| < \varepsilon \text{ for all } i = 1, \dots, k\}.$$

Let $A_1 \cup \dots \cup A_k = \{x_1, \dots, x_l\}$. Consider the neighbourhood

$$O(\varphi_0; x_1, \dots, x_l; \varepsilon) = \{\varphi \in C_pX | |\varphi(x_i) - \varphi_0(x_i)| < \varepsilon \text{ for all } i = 1, \dots, l\}$$

of φ_0 in C_pX. It is easy to see that $\xi'(O(\varphi_0; x_1, \dots, x_l; \varepsilon)) \subset O(\Phi_0; A_1, \dots, A_k; \varepsilon)$.

Summing up, we have

Theorem 1. *There exists an extension of contravariant functor C_p onto the Kleisli category of the monad $\mathbb{H}_f$.*

Note that the structure of $\mathbb{H}_f$-algebra on $\mathbb{R}$ is not uniquely determined (we can consider, e. g., min instead of max in the above expressions). Thus, the extension of C_p onto $Tych_{\mathbb{H}_f}$ is not unique.

4°. Hyperspace monad

The following question naturally arizes: is there an extension C_p onto the category $Tych_{\mathbb{H}}$? Recall that the category $Tych_{\mathbb{H}}$ is a category of Tychonov spaces and compact-valued maps.

Lemma 2. *Suppose there is a natural transformation $\xi : C_p \to \exp C_p \exp$ associated to an extension of C_p onto the category $Tych_{\mathbb{H}}$. For $c \in \mathbb{R}$ let $\varphi \in C_pX$ be such that $\varphi(x) = c$ for all $x \in X$. Then $\xi X(\varphi) = \{\Psi\}$, where $\Psi \in C_p \exp X$ is such that $\Psi(A) = c$ for every $A \in \exp X$. (Thus, ξX preserves the constants.)*

Proof. Fix any one-point space $\{*\}$ and consider the only mapping $f : X \to \{*\}$. Since ξ is a natural transformation, we have $\xi X \circ C_pf = \exp C_p \exp f \circ \xi\{*\}$. Denote by $\chi_c \in C_p\{*\}$, $\chi'_c \in C_p \exp\{*\}$ the constant functions with the value $c \in \mathbb{R}$. Let $\chi_c \in C_p\{*\} \equiv \mathbb{R}$. Considering condition (i) of Proposition 1, we obtain:

$$\exp C_p\eta\{*\}(\xi\{*\}(\chi_c)) = \exp C_p\eta\{*\}(\{\chi'_{c_\alpha} \in C_p \exp\{*\}|\alpha \in \Gamma\}) = \{C_p\eta\{*\}(\chi'_{c_\alpha})|\alpha \in \Gamma\}$$
$$= \{\chi'_{c_\alpha} \circ \eta\{*\}|\alpha \in \Gamma\} = \eta C_p\{*\}(\chi_c) = \{\chi'_c\}.$$

Hence $\xi\{*\}(\chi_c) = \{\chi'_c\}$.

Obviously, $C_p f(\chi_c) \in C_p X$ is the constant function with the value c on X. We denote this mapping by φ. Then

$$\xi X \circ C_p f(\chi_c) = \xi X(\varphi) = \exp C_p \exp f(\xi\{*\}(\chi_c)) = \exp C_p \exp f(\{\chi'_c\}) = \{C_p \exp f(\chi'_c)\}$$
$$= \{\chi'_c \circ \exp f\} = \{\Psi\},$$

where $\Psi(A) = c$ for each $A \in \exp X$.

Theorem 2. *There is no extension of C_p onto $Tych_{\mathbb{H}}$.*

Proof. Suppose the opposite. Let K denote the middle-third Cantor set and let $\varphi \in C_pK$ be a function for which $\varphi(K) = \{0,1\}$. Obviously there exist two sequences of homeomorphisms $h_i: K \to K, g_i: K \to K$ such that $\varphi \circ h_i \longrightarrow 0, \varphi \circ g_i \longrightarrow 1$, if $i \longrightarrow \infty$.

Let $\xi K(\varphi) = \{\Phi_\alpha \in C_p \exp K|\alpha \in \Gamma\}$. Since ξ is a natural transformation, we have:

$$\exp C_p \exp h_i \circ \xi K(\varphi) = \exp C_p \exp h_i\{\Phi_\alpha|\alpha \in \Gamma\} = \{\Phi_\alpha \circ \exp h_i|\alpha \in \Gamma\} = \xi K(\varphi \circ h_i).$$

Consider the element $K \in \exp K$. We have

$$\xi K(\varphi \circ h_i)(K) = \{\Phi_\alpha \circ \exp h_i(K)|\alpha \in \Gamma\} = \{\Phi_\alpha(K)|\alpha \in \Gamma\}.$$

Similary for the sequence (g_i) we have

$$\exp C_p \exp g_i \circ \xi K(\varphi) = \{\Phi_\alpha \circ \exp g_i|\alpha \in \Gamma\} = \xi K(\varphi \circ g_i).$$

Since $\exp C_p \exp h_i \circ \xi K(\varphi)(K) = \exp C_p \exp g_i \circ \xi K(\varphi)(K)$, we see that $\xi K(\varphi \circ g_i)(K) = \xi K(\varphi \circ h_i)(K)$ for any $i \in \mathbb{N}$.

By Lemma 2 we have: $\{\Phi(K)|\Phi \in \xi K(\varphi \circ g_i)\} \underset{i\to\infty}{\longrightarrow} \{\xi K(\chi'_1)(K)\} = \{1\}$, and, on the other hand, $\{\Phi(K)|\Phi \in \xi K(\varphi \circ h_i)(K)\} \underset{i\to\infty}{\longrightarrow} \{\xi K(\chi'_0)(K) = \{0\}$. We have obtained a contradiction.

5°. Remarks and open question

Note that the method used in the proof of Theorem 1 works also in some other situations. Given a monad $\mathbb{T} = (T, \eta, \mu)$ on $Tych$ such that $\mathbb{R}$ is a $\mathbb{T}$-algebra and T is a functor with finite supports (see, e. g., [6] for the results concerning functors in $Tych$) we can argue similarly as in the proof of Theorem 1 in order to prove the existence of extensions of C_p onto $Tych_{\mathbb{T}}$.

It is well-known that the probability measure functor determines (a unique) monad onto the category $Comp$ (see, e. g., [7]). In [8] it is shown that the related functor P_τ of τ-smooth measures determines a unique monad on $Tych$ that extends the probability measure monad.

Question. *Is there an extension of the functor C_p onto the Kleisli category of the monad of τ-smooth probability measures?*

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