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BARRIERS ON CONES FOR DEGENERATE QUASILINEAR ELLIPTIC OPERATORS

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Borsuk M. V., Portnyagin D. V. Barriers on Cones for Degenerate Quasilinear Elliptic Operators. Barrier functions of boundary value problems are constructed for quasilinear elliptic second order operator of divergent form on the cone.

Lately many mathematicians have been considering nonlinear problems for elliptic degenerate equations (see e.g. [1] and the extensive bibliography in it). In the present paper we take the first step to the investigation of the behaviour of solutions of boundary value problems for a quasilinear elliptic second order equation with triple degeneracy:

$$Lu \equiv \frac{d}{dx_i}\left(|x|^\tau |u|^q |\nabla u|^{m-2} u_{x_i}\right) = \mu |x|^\tau |u|^{q-1} \mathrm{sgn}\, u |\nabla u|^m, \quad x \in G_0, \qquad (1)$$
$$-1 < \mu \leqslant 0, \quad q \geqslant 0, \quad m > 1, \quad \tau > m - n,$$

where G_0 is an $n-$dimensional *convex* circular cone with the vertex at the origin of coordinates O, Γ_0 its lateral area, and $\Omega = G_0 \cap S^{n-1}$ is a domain on the unit sphere S^{n-1} with a smooth boundary $\partial\Omega$. Exactly, we shall construct functions playing the fundamental role in the study of behaviour of solutions to elliptic boundary value problems in the neighbourhood of the irregular boundary point (see [2-6]). The special structure of the solution near a conical point is of particular interest for physical applications (cf. [7-9]). It can be used also to improve numerical algorithms (cf. [10-12]).

The proof of the estimates for the solution itself are based on the observation that the function $|x|^\lambda \Phi(\omega)$ is usable as barrier for the problem. By weak comparison principle (see Theorems from chapt. 10 [2]; it is possible to verify that assumptions of this principle are fulfilled if we observe that the equation (1) is equivalent to

$$\frac{d}{dx_i}\left(|\nabla u|^{m-2} u_{x_i}\right) + \tau |x|^{-2} |\nabla u|^{m-2}(x\nabla u) + (q-\mu)|u|^{-1} \mathrm{sgn}\, u |\nabla u|^m = 0,$$
$$x \in G_0, \quad -1 < \mu \leqslant 0, \quad q \geqslant 0, \quad m > 1, \quad \tau > m - n$$

on the set where $u \neq 0$), one might obtain then the bound of solution near conical boundary point. In this connection the finding of exact value of the exponent λ is very important and most difficult. In the case of planar bounded domain with corner boundary points the exact value of the exponent λ will be calculated explicitly.

Let us transfer to the spherical coordinates with the pole at the point O:

$$x_1 = r\cos\omega_1, \; x_2 = r\cos\omega_2 \sin\omega_1, \; \ldots,$$

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$$x_{n-1} = r\cos\omega_{n-1}\sin\omega_{n-2}\dots\sin\omega_1,\ x_n = r\sin\omega_{n-1}\dots\sin\omega_1;$$

$0 \leqslant r = |x| < \infty;\quad 0 \leqslant \omega_k \leqslant \pi,\quad k \leqslant n-2;\quad 0 \leqslant \omega_{n-1} \leqslant 2\pi$. The differential operator takes the form:

$$Lu = \frac{1}{J}\sum_{i=1}^{n}\frac{d}{d\xi_i}\left(r^{\tau}|u|^{q}|\nabla u|^{m-2}\frac{J}{H_i^2}\frac{\partial u}{\partial\xi_i}\right),$$

where

$$J = r^{n-1}\sin^{n-2}\omega_1\dots\sin\omega_{n-2},\ H_1 = 1,\quad \xi_1 = r,\quad \xi_{i+1} = \omega_i,\qquad H_{i+1} = r\sqrt{q_i},$$
$$i = \{\overline{1,n-1}\},\quad q_1 = 1,\quad q_i = (\sin\omega_1\dots\sin\omega_{i-1})^2,\quad i = \{\overline{2,n-1}\}.$$

We shall seek the solution of the problem (1) as $u = r^{\lambda}\Phi(\omega)$ with $\Phi(\omega) \geqslant 0$. Then $\Phi(\omega)$ satisfies the equation:

$$\frac{1}{j(\omega)}\sum_{k=1}^{n-1}\frac{d}{d\omega_k}\left(\frac{j(\omega)}{q_k}\left(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2\right)^{\frac{m-2}{2}}|\Phi|^q\frac{\partial\Phi}{\partial\omega_k}\right) +$$
$$+\lambda[\lambda(q+m-1)+\tau+n-m]\left(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2\right)^{\frac{m-2}{2}}\Phi|\Phi|^q =$$
$$= \mu\Phi|\Phi|^{q-2}\left(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2\right)^{\frac{m}{2}}, \quad (2)$$

where $|\nabla_\omega\Phi|^2 = \sum\limits_{j=1}^{n-1}\frac{1}{q_j}\left(\frac{\partial\Phi}{\partial\omega_j}\right)^2$; $j(\omega) = \sin^{n-2}\omega_1\dots\sin\omega_{n-2}$.

The Dirichlet problem

Let $G_0 = \{x|\ 0 \leqslant \omega_1 \leqslant \frac{\omega_0}{2},\ \omega_0 \in (0,\pi)\}$, $\cos\omega_1 = x_1|x|^{-1}$. First we consider the Dirichlet problem for the equation (1): $u|_{\Gamma_0} = 0$. Hence, it obviously follows that: $\Phi(\omega) = 0,\ \omega \in \partial\Omega$. Multiplying (2) by $\Phi(\omega)$ and integrating by parts over Ω we get:

$$\int\limits_{\Omega}\left(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^q|\nabla_\omega\Phi|^2 d\Omega =$$
$$= \lambda[\lambda(q+m-1)+\tau+n-m]\int\limits_{\Omega}\left(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^{q+2}d\Omega -$$
$$-\mu\int\limits_{\Omega}|\Phi|^q\left(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2\right)^{m/2}d\Omega \equiv$$
$$\equiv \int\limits_{\Omega}\left(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^q\left\{\lambda[\lambda(q+m-1)+\tau+n-m]\Phi^2 -\right.$$
$$\left.-\mu(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2)\right\}d\Omega.$$

Hence it follows that

$$(1+\mu)\int\limits_{\Omega}\left(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^q|\nabla_\omega\Phi|^2 d\Omega =$$
$$= \lambda[\lambda(q+m-1-\mu)+\tau+n-m]\int\limits_{\Omega}\left(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^{q+2}d\Omega.$$

Since $\Phi(\omega) \not\equiv 0$, $\mu > -1$, we have

$$\lambda[\lambda(q+m-1-\mu)+\tau+n-m] > 0. \tag{3}$$

We shall consider the case of $\Phi(\omega)$ not depending on $\omega_2, \dots, \omega_{n-1}$ and so Φ be a function of single angular coordinate $\omega_1 = \omega \in (0, \frac{\omega_0}{2}), 0 < \omega_0 < \pi$. Such $\Phi(\omega)$ satisfies the boundary value problem for ordinary differential equation:

$$\begin{cases} \left[(m-1)\Phi'^2 + \lambda^2\Phi^2\right]\Phi\Phi'' + \left(\lambda^2\Phi^2 + \Phi'^2\right)\left\{(q-\mu)\Phi'^2 + \right. \\ \left. +\lambda[\lambda(q+m-1-\mu)+\tau+n-m]\Phi^2 + (n-2)\Phi\Phi'\text{ctg } \omega\right\} + \\ +(m-2)\lambda^2\Phi^2\Phi'^2 = 0, \quad \omega \in \left(0, \frac{\omega_0}{2}\right); \\ \Phi'(0) = \Phi(\omega_0/2) = 0. \end{cases} \tag{ODE}$$

By making the substitution $y = \Phi'/\Phi$, $y' + y^2 = \Phi''/\Phi$, we arrive at:

$$\begin{gathered} \left[(m-1)y^2 + \lambda^2\right]y' + (m-1+q-\mu)(y^2+\lambda^2)^2 + \\ +[\lambda(\tau+n-m) + (n-2)y\text{ctg } \omega](y^2+\lambda^2) = 0; \quad \omega \in \left(0, \frac{\omega_0}{2}\right). \end{gathered} \tag{4}$$

Since ctg $\omega > 0$ on $(0, \omega_0/2)$, from equation (4) and condition (3) it follows that:

$$[(m-1)y^2+\lambda^2]y' + (n-2)y(y^2+\lambda^2)\text{ctg } \omega < 0, \quad \omega \in \left(0, \frac{\omega_0}{2}\right). \tag{5}$$

Let us solve the Cauchy problem:

$$\begin{cases} [(m-1)\overline{y}^2 + \lambda^2]\overline{y}' + (n-2)\overline{y}(\overline{y}^2+\lambda^2)\text{ctg } \omega = 0, \quad \omega \in \left(0, \frac{\omega_0}{2}\right); \\ \overline{y}(0) = 0. \end{cases}$$

We get:

$$\int \frac{(m-1)\overline{y}^2+\lambda^2}{\overline{y}(\overline{y}^2+\lambda^2)} d\overline{y} = -(n-2)\int \text{ctg } \omega d\omega + \text{const} \qquad \Longrightarrow$$

$$\begin{cases} \overline{y}(\overline{y}^2+\lambda^2)^{\frac{m-2}{2}} = C\sin^{(2-n)}\omega \\ \overline{y}(0) = 0 \end{cases} \quad \Rightarrow \quad C = 0 \quad \Rightarrow \quad \overline{y} \equiv 0.$$

Comparing the solution of inequality (5) with that of the Cauchy problem, we deduce that $y(\omega) \leqslant 0$.

Since ctg $\omega > 0$ and $y \leqslant 0$ on our interval, by (4) we have:

$$\begin{aligned} &[(m-1)y^2+\lambda^2]y' + [(m-1+q-\mu)(\lambda^2+y^2) + \lambda(\tau+n-m)](\lambda^2+y^2) = \\ &\qquad = -(n-2)y(y^2+\lambda^2)\text{ctg } \omega \geqslant 0, \quad \omega \in \left(0, \frac{\omega_0}{2}\right). \end{aligned}$$

Thus:

$$\begin{cases} [(m-1)y^2+\lambda^2]y' \geqslant -[(m-1+q-\mu)(\lambda^2+y^2) + \lambda(\tau+n-m)](\lambda^2+y^2), \\ y(0) = 0, \quad \omega \in \left(0, \frac{\omega_0}{2}\right). \end{cases}$$

By the comparison theorem, similarly we obtain $y(\omega) \geqslant z(\omega)$, where $z(\omega)$ is the solution of the following Cauchy problem:

$$\begin{cases} [(m-1)z^2+\lambda^2]z' = -[(m-1+q-\mu)(\lambda^2+z^2) + \lambda(\tau+n-m)](\lambda^2+z^2), \\ z(0) = 0, \quad \omega \in \left(0, \frac{\omega_0}{2}\right). \end{cases}$$

Solving the latter, we obtain the expression for z in the implicit form:

$$\frac{\left(\frac{m-1}{m-1+q-\mu}+\lambda\frac{m-2}{\tau+n-m}\right)}{\sqrt{\lambda^2+\lambda\frac{\tau+n-m}{m-1+q-\mu}}}\text{arctg}\,\frac{z}{\sqrt{\lambda^2+\lambda\frac{\tau+n-m}{m-1+q-\mu}}}+\omega+\frac{m-2}{m-n-\tau}\text{arctg}\,\left(\frac{z}{\lambda}\right)=0. \quad (6)$$

By joining the results obtained, we arrive at the conclusion that:

$$0\geqslant y(\omega)\geqslant z(\omega). \quad (7)$$

Let us now return to the equation for $y(\omega)$. On making the substitution $\Psi=\ln\Phi$, $w(\Psi)=y^2(\Psi)$, $w'(\Psi)=2yy'(\Psi)=2y\frac{d\omega}{d\Psi}y'(\omega)=2y'(\omega)$, we get:

$$\frac{1}{2}[(m-1)w+\lambda^2]w'+[(m-1+q-\mu)(\lambda^2+w)+\lambda(\tau+n-m)](\lambda^2+w)-$$
$$-(n-2)\sqrt{w}(w+\lambda^2)\text{ctg}\,\omega=0$$

(here we use $y=\pm\sqrt{w}$ and $y<0$). Acting similarly, as it have been shown above, we get the differential inequality for w :

$$\frac{1}{2}[(m-1)w+\lambda^2]w'+[(m-1+q-\mu)(\lambda^2+w)+\lambda(\tau+n-m)](\lambda^2+w)>0$$

Integrating the respective differential equation:

$$\frac{1}{2}[(m-1)\overline{w}+\lambda^2]\overline{w}'+[(m-1+q-\mu)(\lambda^2+\overline{w})+\lambda(\tau+n-m)](\lambda^2+\overline{w})=0$$

we get:

$$\lambda\frac{m-2}{m-n-\tau}\ln\,(\lambda^2+\overline{w})+\Big(\frac{m-1}{m-1+q-\mu}+$$
$$+\lambda\frac{m-2}{\tau+n-m}\Big)\ln\,((m-1+q-\mu)(\lambda^2+\overline{w})+\lambda(\tau+n-m))+2\ln\,\Phi=\ln\,C.$$

Solving the latter expression with the respect to Φ we obtain:

$$\Phi^2(\omega)=C^2\left(\frac{(m-1+q-\mu)(\lambda^2+\overline{w})+\lambda(\tau+n-m)}{\lambda^2+\overline{w}}\right)^{\lambda\frac{m-2}{m-n-\tau}}\times$$
$$\times\left[(m-1+q-\mu)(\lambda^2+\overline{w})+\lambda(\tau+n-m)\right]^{-\frac{m-1}{m-1+q-\mu}}.$$

Now it's evident that $\overline{w}=z^2(\Psi)$, $w=y^2(\Psi)$. From (7) it follows that $w\leqslant\overline{w}$. Then we can rewrite: $\Phi^2(\omega)=$

$$=C^2(z^2+\lambda^2)^{\frac{1-m}{m-1+q-\mu}}\left(m-1+q-\mu+\frac{\lambda(\tau+n-m)}{(z^2+\lambda^2)}\right)^{\lambda\frac{m-2}{m-n-\tau}-\frac{m-1}{m-1+q-\mu}}.$$

Whence it follows that:

$$\Phi(\omega)\sim\frac{1}{|z|^{\frac{(m-1)}{m-1+q-\mu}}}\quad\text{for}\quad|z|\to+\infty$$

Since $y^2\leqslant z^2$, then $1/z^2\leqslant 1/y^2$, and it's now clear that $\lim\limits_{\omega\to\frac{\omega_0}{2}-0}z(\omega)=-\infty$ (since $\Phi(\frac{\omega_0}{2})=0$). Further, since $y=\frac{\Phi'}{\Phi}<0$ and $\Phi>0$ on $(0,\frac{\omega_0}{2})$, $\Phi'<0$ on $(0,\frac{\omega_0}{2})$, i.e. $\Phi(\omega)$ decreases on $(0,\frac{\omega_0}{2})$

from some positive value $\Phi(0)$ up to $\Phi(\frac{\omega_0}{2}) = 0$. Φ doesn't vanish anywhere else in $(0, \frac{\omega_0}{2})$, otherwise it should increase somewhere. From the equation we have:

$$y' = -[(m-1+q-\mu)(y^2+\lambda^2) + \lambda(\tau+n-m)]\frac{(y^2+\lambda^2)}{(m-1)y^2+\lambda^2} -$$
$$-(n-2)y\frac{y^2+\lambda^2}{(m-1)y^2+\lambda^2}\operatorname{ctg}\omega \to -\infty \quad \text{for} \quad y \to -\infty.$$

That's to say $y(\omega)$ decreases in the vicinity of the point $\overline{\omega}$, where $y \to -\infty$. It is possible only at $\overline{\omega} = \omega_0/2$(passing $\omega \to \frac{\omega_0}{2} - 0$). Passing to the limit $\omega \to \frac{\omega_0}{2} - 0$ in (6) and taking account of the fact that $z \to -\infty$, we get:

$$\frac{\omega_0}{\pi} + \frac{m-2}{\tau+n-m} = \frac{\frac{m-1}{m-1+q-\mu} + \lambda\frac{m-2}{\tau+n-m}}{\sqrt{\frac{\lambda[\lambda(m-1+q-\mu)+\tau+n-m]}{m-1+q-\mu}}}.$$

Hence we obtain the explicit expression for λ
if $m \geqslant 2$ or $1 < m < 2$, $\frac{\tau+n-m}{2-m} \neq \frac{2\pi}{\omega_0}$

$$\lambda = \frac{\pi}{2\omega_0(m-1+q-\mu)}\left\{\frac{m(m-2)-2(m-2)t-t^2}{t+2(m-2)} + \right.$$
$$\left. + \frac{\sqrt{[t^2+2(m-2)t+m^2][t^2+2(m-2)t+(m-2)^2]}}{t+2(m-2)}\right\}, \tag{8}$$

where $t = \frac{\omega_0}{\pi}(\tau+n-m)$;
if $1 < m < 2$, $\frac{\tau+n-m}{2-m} = \frac{2\pi}{\omega_0}$

$$\lambda = \frac{2\pi(m-1)^2}{\omega_0 m(m-1+q-\mu)}. \tag{9}$$

In the case of $n = 2$, $\tau = 0$ *we obtain the result*

$$\lambda = \frac{(m-1)}{(m-1-\mu+q)} + \frac{(\pi-\omega_0)[m(\pi-\omega_0) + \sqrt{(m-2)^2(\pi-\omega_0)^2 + 4(m-1)\pi^2}]}{2\omega_0(2\pi-\omega_0)(m-1-\mu+q)}. \tag{10}$$

In the case of $\tau = \mu = q = 0$, $n = 2$ *we get the known result. If* $m = n = 2$ *we have from (8)*

$$\lambda = \frac{\sqrt{\left(\frac{2\pi}{\omega_0}\right)^2 + \tau^2} - \tau}{2(1+q-\mu)}$$

Now we consider the case $n = 3$. We shall assume the $\tau = 0$. We shall seek solution of a form $u = r^\lambda \Phi(\omega)\sin^\lambda\varphi$, where $\varphi \in (0,\pi)$; $\omega \in (-\omega_0/2, \omega_0/2)$. Then we obtain for $\Phi(\omega)$ the problem which coincides with (ODE) for $n = 2$ and so we have for λ the expression (10).

The mixed boundary value problem

Now we consider te mixed boundary value problem in planar domain $G_0 = \{(r,\omega)|\ r > 0,\ 0 < \omega < \omega_0 < \pi\}$ with corner boundary point:

$$\begin{cases} \frac{d}{dx_i}\left(|u|^q|\nabla u|^{m-2}u_{x_i}\right) = \mu|u|^{q-1}\operatorname{sgn} u|\nabla u|^m, & x \in G_0, \\ u\big|_{\omega=\omega_0} = 0, \quad \left.\frac{\partial u}{\partial x_2}\right|_{\omega=0} = 0, \end{cases}$$

where ω_0 is an angle with the vertex at the point O. By analogy with abovestated we come to the following expression for λ :

$$\lambda = \frac{(m-1)}{(m-1-\mu+q)} + \frac{(\pi-2\omega_0)[m(\pi-2\omega_0)+\sqrt{(m-2)^2(\pi-2\omega_0)^2+4(m-1)\pi^2}]}{8\omega_0(\pi-\omega_0)(m-1-\mu+q)}.$$

It is clear that this expression coincides with (9) for the Dirichlet problem, if in the latter we set $2\omega_0$ instead of ω_0.

Thus, there are constructed barrier functions $w = r^\lambda\Phi(\omega)$ *of the first boundary value problem for the equation* (1) *and also of the mixed boundary value problem for (1) by* $\tau = 0$.

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