УДК 517.576.

ON OPERATOR OF MULTIPLICATION BY THE INDEPENDENT VARIABLE

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Lyantse V. E., Karabin O. O. On operator of multiplication by the independent variable. The framework of this paper is Internal Set Theory of Nelson. We give conditions for nearstandardness of the operator of multiplication by the independent variable and determine its shadow. Since each normal operator is unitarly equivalent to such a multiplication, we obtain conditions for nearstandardness and a way to determine the shadow of an arbitrary normal operator.

1. *Preleminaires.* Recall some definitions ([2],[3],[4]). Let $\mathbb{H}$ be a standard Hilbert space over $\mathbb{C}$. For a vector $x \in \mathbb{H}$ we write $\|x\| \ll \infty$ iff $\exists n \in {}^{st}\mathbb{N} \quad \|x\| < n$ (${}^{st}E$ denote the collection of standard elements of the set E). If $\|x\| \ll \infty$, then there exsists a unique standard vector ${}^{\circ}x$ (the shadow of x) such that $\forall y \in {}^{st}\mathbb{H} \quad (x|y) \approx ({}^{\circ}x|y)$; (for $a, b \in \mathbb{C}$ "$a \approx b$" means that "$\forall n \in {}^{st}\mathbb{N} \quad |a-b| < \frac{1}{n}$"). If $\|x\| \ll \infty$ and $\|x - {}^{\circ}x\| \approx 0$ then x is said to be nearstandard (write $x \in {}^{nst}\mathbb{H}$).

We need also the concept of the shadow ${}^{\circ}E$ for a set E in a standard metric space (X, d). The halo $\mathcal{H}(E)$ of E is the collection of those $x \in X$ for which $\exists y \in {}^{st}E \quad d(x,y) \approx 0$. The shadow ${}^{\circ}E$ is a standard set such that ${}^{st}E = {}^{st}\mathcal{H}(E)$. By the standartization principle of IST, ${}^{\circ}E$ exists and is unique.

Let (X, d_x), (Y, d_y) be standard metric spaces, and f be a map $X \to Y$. We consider its graph $\{(x,y) \in X \times Y : \ x \in \mathrm{dom} f, \ y = f(x)\}$ r as a set in the standard metric space $X \times Y$ (with a distance, e.g., $d_x(x_1, x_2) + d_y(y_1, y_2)$). The map f is said to be *graph-nearstandard* if the shadow ${}^{\circ}[\mathrm{graph} f]$ of its graph is a graph of some map. The last (if it exists) is exactly the shadow ${}^{\circ}f$ of f: ${}^{\circ}[\mathrm{graph}\ f] = \mathrm{graph}({}^{\circ}f)$.

Let $A : X \to Y$ (X, Y are normed spaces) be a linear map. Define $\mathrm{dom}_{nst}A := \{x \in {}^{nst}\mathrm{dom}\ A : \quad Ax \in {}^{nst}Y\}$. We say that for A the $< nst >$ condition holds, iff $(\forall x \in \mathrm{dom}_{nst}A)$ $(x \approx 0 \Rightarrow Ax \approx 0)$. It is known (see, e.g., [5]) that $< nst >$ is necessary and sufficient for graph-nearstandardness of A. If $< nst >$ holds, then

$$x \in {}^{st}\mathrm{dom}({}^{\circ}A) \Longleftrightarrow (\exists x_1 \in \mathrm{dom} A)(x \approx x_1) \tag{1.1}$$

and for such x and x_1 we have

$$({}^{\circ}A)x = {}^{\circ}(Ax_1) \tag{1.2}$$

Since ${}^{\circ}A$ is standard its domain and action are uniquely determined by (1.1), (1.2). Recall that the shadow ${}^{\circ}A$ of a graph-nearstandard linear map A is a *closed* operator.

1991 *Mathematics Subject Classification.* 47S20, 28E05.

2. Let $(T, \mathfrak{P}, p)$ be a standard measure space with a $\sigma-$additive measure. Denote by $\mathbb{H}$ the standard Hilbert space $L_2(T, \mathfrak{P}, p)$ and fix some p-measurable p-bounded function $\lambda \in \mathbb{C}^T$. Define an operator $A \in \mathcal{B}(\mathbb{H})$ by

$$\forall x \in \mathbb{H} \quad Ax = \lambda(\cdot)x \tag{2.1}$$

(i.e, $Ax(t) = \lambda(t)x(t)$ a.e.). Observe that

$$\|A\| = \text{ess sup}_{t\in T}|\lambda(t)|; \tag{2.2}$$

this A is normal (if $\lambda(t) \in \mathbb{R}$ a.e., then A is selfadjoint). The spectrum of A is

$$\sigma(A) = \text{esv } \lambda, \tag{2.3}$$

where esvλ denote the set of essential values of λ, i.e.,

$$\text{esv}\lambda := \{z \in \mathbb{C} : \quad \text{ess inf}_{t\in T}|\lambda(t) - z| = 0\}. \tag{2.3'}$$

We want to discover condition for nearstandardness of A. Denote by $(T_n)_{n\in\mathbb{N}}$ an increasing sequence of sets $T_n \in \mathfrak{P}$ such that

$$T = \bigcup_{n\in\mathbb{N}} T_n \quad \text{and} \quad \forall n \in \mathbb{N} \quad pT_n < \infty \tag{2.4}$$

By the transfer principle (without loss of generality), we can assume that the sequence (T_n) is *standard*. In particular $\forall n \in {}^{st}\mathbb{N} \quad T_n \in {}^{st}\mathfrak{P}$ and $pT_n \in {}^{st}\mathbb{R}$.

2.1 Definition. A p-measurable p-bounded function $\lambda \in \mathbb{C}^T$ is said to be *p-nearstandard* iff there exists a *standard* p-measurable function $\mu \in \mathbb{C}^T$ such that

$$\forall n \in {}^{st}\mathbb{N} \quad \text{ess sup}_{t\in T_n}|\lambda(t) - \mu(t)| \approx 0. \tag{2.5}$$

If this holds, μ is called the *shadow* of λ and denoted by $^\circ\lambda$.

2.2 Remark. The above definition determines the shadow $\mu = {}^\circ\lambda$ uniquely a.e. Indeed, let (2.5) holds for another standard $(\widetilde{T}_n)$ and $(\tilde{\mu})$. Then $\forall m, n \in {}^{st}\mathbb{N}$ ess $\sup_{t\in T_m\cap\widetilde{T}_n}|\mu(t)-\tilde{\mu}(t)| \approx 0$. Since this ess sup is a standard number, it equals zero. Since $\bigcup_{m,n\in\mathbb{N}} T_m \cap \widetilde{T}_n = T$ by the transfer principle, we get $\mu(t) = \tilde{\mu}(t)$ a.e on T.

2.3 Warning. The shadow $^\circ\lambda$ (of a p-bounded λ) may be *unbounded*. But for (2.5) and by the transfer principle,

$$\forall n \in {}^{st}\mathbb{N} \quad \text{ess sup}_{t\in T_n}|({}^\circ\lambda)(t)| \in {}^{st}\mathbb{R}. \tag{2.6}$$

2.4 Lemma. . *Let $\lambda \in \mathbb{C}^T$ be a p-nearstandard function. Then*

$$(\forall x \in {}^{nst}\mathbb{H})\,(\|Ax\| \ll \infty \Longrightarrow {}^\circ(Ax) = ({}^\circ\lambda)(\cdot){}^\circ x). \tag{2.7}$$

In particular, if $x \in \text{dom}_{nst}A$, then

$$\int_T |({}^\circ\lambda)(t)({}^\circ x)(t)|^2\, p(dt) < \infty. \tag{2.8}$$

Proof. Let $x \in {}^{nst}\mathbb{H}$, $\|Ax\| \ll \infty$, and $y \in {}^{st}\mathbb{H}$. Then $({}^\circ(Ax)|y) \approx (Ax|y) = (\lambda(\cdot)x|y)$. Therefore,

$$\forall n \in {}^{st}\mathbb{N} \quad \int\limits_{T_n} {}^\circ(Ax)(t)\overline{y(t)}\,p(dt) \approx \int\limits_{T_n} \lambda(t)x(t)\overline{y(t)}\,p(dt) \approx$$

$$\approx \int\limits_{T_n} ({}^\circ\lambda(t))x(t)\overline{y(t)}\,p(dt) \approx \int\limits_{T_n} ({}^\circ\lambda)(t)({}^\circ x)(t)\overline{y(t)}\,p(dt).$$

Indeed, $\|x - {}^\circ x\| \approx 0$ and by (2.6),

$$\forall n \in {}^{st}\mathbb{N} \quad \int\limits_{T_n} |({}^\circ\lambda)(t)y(t)|^2\,p(dt) \ll \infty.$$

Since the first and the last members of the chain above are standard numbers, they are equal. By the transfer we find

$$\forall y \in \mathbb{H}\,\forall n \in \mathbb{N} \int\limits_{T_n} {}^\circ(Ax)(t)\overline{y(t)}\,p(dt) = \int\limits_{T_n} ({}^\circ\lambda)(t)({}^\circ x)(t)\overline{y(t)}\,p(dt),$$

whence (2.7) follows.

2.5 Theorem. *Let a function λ be p-nearstandard. Then the operator A defined by (2.1) is graph-nearstandard. Its shadow ${}^\circ A$ is a densely defined (possibly unbounded) operator given by*

$$\operatorname{dom}{}^\circ A = \{x \in \mathbb{H} : \quad ({}^\circ\lambda)(\cdot)x \in \mathbb{H}\}, \quad \forall x \in \operatorname{dom}{}^\circ A \quad ({}^\circ A)x = ({}^\circ\lambda)(\cdot)x. \tag{2.9}$$

Proof. Let $x \in \operatorname{dom}_{nst}A$ and $x \approx 0$. Since ${}^\circ x = 0$ by (2.7) ${}^\circ(Ax) = 0$. Therefore $Ax \approx 0$, what means that for A the $< nst >$ condition holds (see section 1), i.e. A is graph-nearstandard.

Assume that $x \in {}^{st}(\operatorname{dom}{}^\circ A)$. Then by (1.2) and *lemma 2.4* $({}^\circ A)x = {}^\circ(Ax) = ({}^\circ\lambda)(\cdot)x$. Therefore $({}^\circ\lambda)(\cdot)x \in \mathbb{H}$. By the transfer, $\forall x \in \operatorname{dom}{}^\circ A \quad ({}^\circ\lambda)(\cdot)x \in \mathbb{H}$ and $({}^\circ A)x = ({}^\circ\lambda)(\cdot)x$.

Conversely let $x \in {}^{st}\mathbb{H}$ and

$$\int\limits_T |({}^\circ\lambda)(t)x(t)|^2\,p(dt) < \infty.$$

In view of (2.5), if $n \in {}^{st}\mathbb{N}$, then

$$\int\limits_{T_n} |({}^\circ\lambda)(t) - \lambda(t)|^2|x(t)|^2\,p(dt) \approx 0.$$

By the Robinson lemma this holds for some $n = n_0 \in \mathbb{N} \setminus {}^{st}\mathbb{N}$. Define $x_1 \in \mathbb{H}$ by $x_1(t) = x(t)$ for $t \in T_{n_0}$ and $x_1(t) = 0$ for $t \in T \setminus T_{n_0}$. Then

$$\|x - x_1\|^2 = \int\limits_{T\setminus T_{n_0}} |x(t)|^2\,p(dt) \approx 0$$

because the standard sequence

$$\left(\int_{T_n} |x(t)|^2 \, p(dt)\right)_{n\in\mathbb{N}}$$

converges. Therefore $x_1 \in {}^{nst}\mathbb{H}$, $\;{}^{\circ}x_1 = x$. Besides,

$$\|Ax_1 - ({}^{\circ}\lambda)(\cdot)x\|^2 = \int_{T_{n_0}} |\lambda(t) - ({}^{\circ}\lambda)(t)|^2 |x(t)|^2 \, p(dt) + \int_{T\setminus T_{n_0}} |({}^{\circ}\lambda)(t)x(t)|^2 \, p(dt) \approx 0.$$

Thus $Ax_1 \in {}^{nst}\mathbb{H}$, what means that $x_1 \in \mathrm{dom}_{nst}A$. By (1.1) $x \in \mathrm{dom}({}^{\circ}A)$. Using the transfer principle, we get: $\forall x \in \mathbb{H}$ if $({}^{\circ}\lambda)(\cdot)x \in \mathbb{H}$, then $x \in \mathrm{dom}({}^{\circ}A)$.

3. Now we give conditions for

$$\sigma({}^{\circ}A) = {}^{\circ}[\sigma(A)], \tag{3.1}$$

where A is defined by (2.1) and $\sigma(\cdot)$ denotes the spectrum of $(\cdot)$.

Assume that a standard metric d is defined on T and for any $n \in {}^{st}\mathbb{N}$ the set T_n is d-compact (see(2.4)). Suppose that the following holds

(i) λ is p-nearstandard and its shadow (see definition 2.1) ${}^{\circ}\lambda$ is a d-continuous function;
(ii) if $|z| \ll \infty$ and $z \in \mathrm{esv}\lambda$, then $\exists n \in {}^{st}\mathbb{N}\quad \mathrm{ess\,inf}_{t\in T_n}|\lambda(t) - z| \approx 0$;
(iii) $\forall z \in {}^{st}\mathbb{C}\quad \mathrm{ess\,inf}_{t\in T}|\lambda(t) - z| \approx 0$ implies $\exists z_1 \approx z \quad z_1 \in \mathrm{esv}\lambda$.

3.1 Theorem. *In the above conditions (3.1) holds.*

Proof. Let $z \in {}^{st}\mathbb{C} \cap {}^{\circ}[\sigma(A)]$. Thus $z \approx z_1$ for some $z_1 \in \mathrm{esv}\lambda$. By (ii) $\mathrm{ess\,inf}_{t\in T_{n_0}}|\lambda(t) - z_1| = 0$ for some $n_0 \in {}^{st}\mathbb{N}$. Condition (2.5) implies $\mathrm{ess\,inf}_{t\in T_{n_0}}|\mu(t) - z| \approx 0$, where $\mu := {}^{\circ}\lambda$. Since T_{n_0} is d-compact and μ is standard and d-continuous there exists $t_0 \in {}^{st}T_{n_0}$ such that $\mu(t_0) \approx z$, hence $\mu(t_0) = z$ and $z \in \sigma({}^{\circ}A)$. By the transfer principle, ${}^{\circ}[\sigma(A)] \supseteq \sigma({}^{\circ}A)$.

Conversely, let $z \in {}^{st}\mathbb{C} \cap \sigma({}^{\circ}A)$. Since ${}^{\circ}A$ is the operator of multiplication by a standard continuous function μ $\forall \varepsilon > 0\, \exists t \in T \quad |\mu(t) - z| < \varepsilon$. For standard $\varepsilon > 0$ the above t may be choosen *standard*. Therefore there exists $n \in {}^{st}\mathbb{N}$ such that $|\mu(t_0) - z| < \varepsilon$ for some $t_0 \in {}^{st}T_n$. By (2.5) $\mathrm{ess\,inf}_{t\in T_n}|\lambda(t) - z| < 2\varepsilon$. Let $E := \{\varepsilon > 0 : \quad \mathrm{ess\,inf}_{t\in T}|\lambda(t) - z| < 2\varepsilon\}$. We have proved that E contains all $\varepsilon \gg 0$. By the permanence principle there exists an infinitesimal $\varepsilon \in E$. Therefore $\mathrm{ess\,inf}_{t\in T}|\lambda(t) - z| \approx 0$. By (iii) $\exists z_1 \approx z \quad z_1 \in \mathrm{esv}\lambda$. For (2.3) $z_1 \in \sigma(A)$, thus $z \in {}^{\circ}[\sigma(A)]$. By the transfer, $\sigma({}^{\circ}A) \subseteq {}^{\circ}[\sigma(A)]$.

4. *Example.* Denote by $\mathbb{H}$ the standard Hilbert spase $L_2(\mathbb{R})$. For a fixed infinitesimal $h > 0$ define

$$\forall x \in \mathbb{H} \quad D_h x(t) = \frac{1}{h^2}\left[x(t+2h) - 2x(t+h) + x(t)\right], \quad t \in \mathbb{R}. \tag{4.1}$$

Obviously, $D_h \in \mathcal{B}(\mathbb{H})$ but $\|D_h\| \approx \infty$ (namely $\|D_h\| = 4/h^2$; see below). We claim that the operator D_h is *graph-nearstandard* and its shadow $D := {}^{\circ}D_h$ is given by

$$\mathrm{dom}D = H^2(\mathbb{R}), \quad \forall x \in H^2(\mathbb{R}) \quad Dx(t) = \frac{d^2x}{dt^2}(t), \quad t \in \mathbb{R}; \tag{4.2}$$

where $H^2(\mathbb{R}) := \{x \in L_2(\mathbb{R}) : \quad \frac{dx}{dt}, \frac{d^2x}{dt^2} \in L_2(\mathbb{R})\}$.

For proof use the unitary transformation $\mathcal{F}$

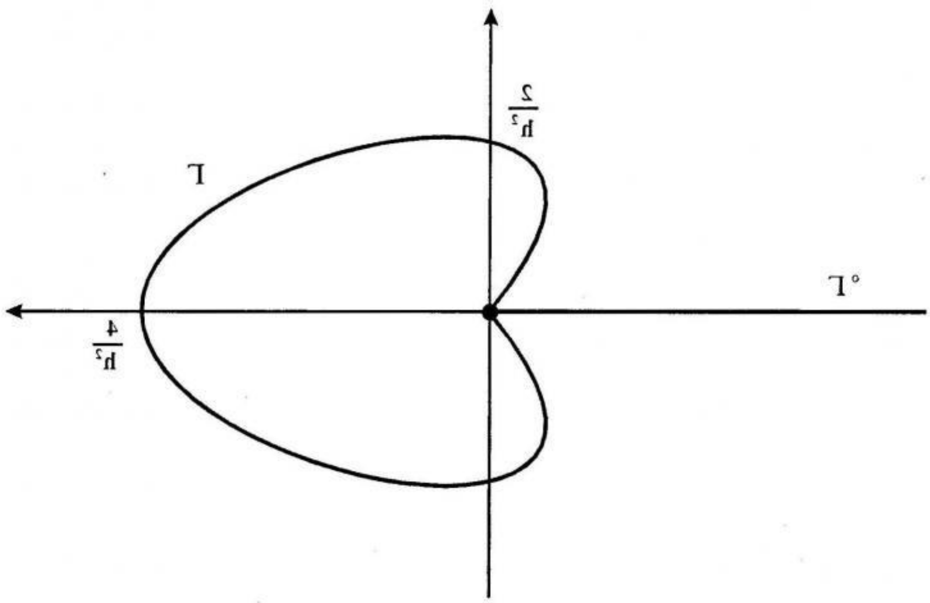
$$\forall x \in L_2(\mathbb{R}) \quad \hat{x}(\tau) = \mathcal{F}x(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} x(t)e^{-it\tau}\, dt, \quad \tau \in \mathbb{R}.$$

The Fourier-image $\widehat{D}_h$ of D_h (i.e. the operator $\mathcal{F}D_h\mathcal{F}^{-1}$) is the operator of multiplication by a function

$$\widehat{D}_h \hat{x} = \lambda(\cdot)\hat{x}, \tag{4.3}$$

where $\lambda(\tau) = \frac{1}{h^2}(e^{i\tau h} - 1)^2$. Since $|\tau| \ll \infty$ implies $\frac{1}{h}(e^{i\tau h} - 1) \approx i\tau$, this function λ is nearstandard with respect to the Lebesque measure on $\mathbb{R}$ and its shadow ${}^\circ\lambda$ is (see definition 2.1)

$${}^\circ\lambda(\tau) = -\tau^2. \tag{4.4}$$



By *theorem 2.5* $\widehat{D}_h$ is graph-nearstandard and by (2.9),

$$\operatorname{dom}^\circ \widehat{D}_h = \left\{ \hat{x} \in L_2(\mathbb{R}) : \int\limits_{\mathbb{R}} |\tau^2 \hat{x}(\tau)|^2 \, d\tau < \infty \right\},$$

${}^\circ\widehat{D}_h\hat{x}(\tau) = -\tau^2\hat{x}(\tau), \quad \tau \in \mathbb{R}$. Using $\mathcal{F}^{-1}$ we get (4.2).

Remark. Since for the above function λ the conditions (i), (ii), (iii) of section **3** are satisfied, we have $\sigma(D) = {}^\circ[\sigma(D_h)]$. But we can see this immediately. Indeed, let $z = \frac{1}{h}(e^{i\tau h} - 1), \quad \tau \in \mathbb{R}$. Then $|hz + 1| = 1$ i.e. values of z form a circle with centre $\left(-\frac{1}{h}, 0\right)$ and radius $\frac{1}{h}$, with a polar equation $\rho = -\frac{2}{h}\cos\varphi, \quad \frac{\pi}{2} \leqslant \varphi \leqslant \frac{3\pi}{2}$. Values $\lambda = \frac{1}{h}(e^{i\tau h} - 1)^2$ fill in a curve Γ which arises from this circle by the transformation $z \longmapsto z^2$ or in polar coordinates, $\rho \longmapsto \rho^2, \quad \varphi \longmapsto 2\varphi$. Thus the polar equation of Γ is $\sqrt{\rho} = -\frac{2}{h}\cos\frac{\varphi}{2}$ i.e., $\rho = \frac{2}{h^2}(1 + \cos\varphi), \quad \pi \leqslant \varphi \leqslant 3\pi$. We see that the spectrum $\sigma(D_h) = \sigma(\widehat{D}_h)$ is an infinitely large *cardioid*. Its shadow is the semiaxe $^\circ\Gamma =]-\infty, 0]$.

Remark. Obviously, the operator D (see (4.2)) is the shadow not only of the (normal) difference operator D_h. For instance (instead of (4.1)) consider the (selfadjoint) difference operator $\widetilde{D}_h$ given by $\widetilde{D}_h x(t) = \frac{1}{h^2}[x(t+h) - 2x(t) + x(t-h)]$. It is easy to check that $\widetilde{D}_h$ is graph-nearstandard with shadow $^\circ\widetilde{D}_h = D$. Now the spectrum of $\widetilde{D}_h$ is simply the segment $\left[-\frac{4}{h^2}, 0\right]$, but as before $\sigma(^\circ\widetilde{D}_h) = {}^\circ[\sigma(\tilde{D}_h)]$ (because $^\circ\left[-\frac{4}{h^2}, 0\right] = [-\infty, 0]$).

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Стаття надійшла до редколегії 29.05.1998