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SOLVABLE GROUPS WITH MINIMAL AND MAXIMAL CONDITIONS ON NON-"LOCALLY POLYCYCLIC"-BY-FINITE SUBGROUPS

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Artemovych O. D. Solvable groups with minimal and maximal conditions on non-'locally polycyclic'-by-finite subgroups. We characterize the locally solvable groups (respectively the solvable groups) in which the set of non-'locally polycyclic'-by-finite subgroups satisfies the minimal condition (respectively the maximal condition).

0. Let G be a group and $\mathfrak{X}$ a class of groups. We say that G satisfies the minimal condition on non-$\mathfrak{X}$-subgroups (for short Min-$\overline{\mathfrak{X}}$) if for every descending chain $\{G_n \mid n \in \mathbb{N}\}$ of subgroups of G there exists a number $n_0 \in \mathbb{N}$ such that the subgroups G_n are $\mathfrak{X}$-groups for all $n \geqslant n_0$. The maximal condition on non-$\mathfrak{X}$-subgroups of G (for short Max-$\overline{\mathfrak{X}}$) is defined dually, namely, one says that G satisfies Max-$\overline{\mathfrak{X}}$ if there is no infinite ascending chain of non-$\mathfrak{X}$-subgroups of G.

The groups with the minimal condition on non-abelian subgroups have been studied by S.N. Černikov [1] and V.P. Šunkov [2] and the groups with the maximal condition on non-abelian subgroups by D.I. Zaĭtsev and L.A. Kurdachenko [3]. Surveys of known results in this direction can be found in [1,4].

Let us recall that a group G is a minimal non-$\mathfrak{X}$-group if it is not $\mathfrak{X}$-group, while all proper subgroups of G are $\mathfrak{X}$-subgroups. Recall that a group G is called indecomposable if any two proper subgroups of G generate a proper subgroup of G. As proved in [5] an indecomposable hypercentral group is isomorphic to either $\mathbb{C}_{p^n}$ or $\mathbb{C}_{p^\infty}$. If $G \neq G'$ we say that G is a non-perfect group.

In this paper we study the locally solvable groups (respectively the solvable groups) G in which the set of non-"locally polycyclic"-by-finite subgroups satisfies the minimal condition (respectively the maximal condition).

Throughout this paper p will always denote a prime number, τH the periodic part of locally nilpotent group H, $Z(G)$ the centre of group a G; $G', G'', \dots, G^{(n)}$ will indicate the terms of derived series of G, $G^p = \langle x^p \mid x \in G\rangle$, $\mathbb{C}_{p^n}$ and $\mathbb{C}_{p^\infty}$ will stand for the cyclic group of order p^n and for the quasicyclic p-group respectively.

Most of the standard notation can be found in [4].

1. Let $\mathcal{LPF}$ be the class of "locally polycyclic"-by-finite groups.

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Lemma 1.1. *Let G be a $\overline{\mathcal{LPF}}$-group. Then:*
(i) every normal subgroup G is "locally polycyclic G-invariant"-by-abelian;
(ii) if G is non-perfect and indecomposable then every subgroup of G is "locally polycyclic"-by-abelian.

Proof. (i) Let N be a normal subgroup of G. Obviously N contains a locally polycyclic G-invariant subgroup H of finite index. We denote the quotient group G/H by $\overline{G}$. Then $\overline{N} = N/H$ is a finite normal subgroup of $\overline{G}$ and therefore

$$|\overline{G} : C_{\overline{G}}(\overline{N})| = |N_{\overline{G}}(\overline{N}) : C_{\overline{G}}(\overline{N})| < \infty.$$

Since $\overline{G}$ does not contain a subgroup of finite index, we obtain

$$\overline{G} = C_{\overline{G}}(\overline{N})$$

and consequently $\overline{N}$ is abelian.

(ii) Suppose now that G is indecomposable and non-perfect. Then

$$G'K \neq G$$

for every proper subgroup K of G. Hence $G'K$ contains a locally polycyclic G-invariant subgroup F of finite index and the quotient G'/F is abelian. From

$$K/(K \cap F) \cong KF/F \leqslant G'K/F$$

it follows (ii). The lemma is proved.

Lemma 1.2. *Let G be a non-perfect group with "locally polycyclic"-by-finite proper subgroups. If $G/G' \cong \mathbb{C}_{p^\infty}$ then the commutator subgroup G' is locally polycyclic.*

Proof. Suppose that G' is not locally polycyclic. Then by Lemma 1.1 G' contains a G-invariant locally polycyclic subgroup F of finite index. Let $\overline{G} = G/F$. Since $\overline{G'}$ is finite and $\overline{G}/\overline{G'} \cong \mathbb{C}_{p^\infty}$ in view of Lemma 1.1 and Lemma 1.15 of [1] we conclude that $\overline{G}$ is abelian, a contradiction. The lemma is proved.

Corollary 1.3. *An indecomposable locally finite group G with "locally polycyclic"-by-finite proper subgroups is non-simple.*

Proof. Suppose that G is a simple group. By Corollary A1 of [6] G is linear and consequently G must be of Lie type (see [7-9]). Thus G is generated by two proper subgroups, a contradiction. The corollary is proved.

Remark 1.4. The Ol'shanski groups (see [10]) are finitely generated $\overline{\mathcal{LPF}}$-groups.

Proposition 1.5. *Let G be a non-perfect group. If every subgroup of G is "locally polycyclic"-by-finite then G is "locally polycyclic"-by-finite.*

Proof. If either G/G' is not indecomposable or $G/G' \cong \mathbb{C}_{p^n}$ for some prime p and some positive integer n the group G is "locally polycyclic"-by-finite in view of [11].

Now, suppose that $G/G' \cong \mathbb{C}_{p^\infty}$. If, moreover, G is indecomposable then by Lemma 1.1 it is locally polycyclic, a contradiction. Consequently G is not indecomposable and $G = \langle U, V\rangle$ for some proper subgroups U and V of G. This yields that, for example, $G = G'U$. Obviously, G' contains a locally polycyclic G-invariant subgroup A of finite index. Let $\overline{G} = G/A$. Then $\overline{G'}$ is a finite subgroup. By Theorem 1.16 of [1], $\overline{G}$ is abelian, a contradiction. The proposition is proved.

Theorem 1.6. *Let Λ be the class of groups which do not have infinite simple images. Then there are no $\overline{\mathcal{LPF}}$-groups in the class Λ.*

Proof. Let H be a finitely generated proper subgroup of G. Since every element of G is contained in a proper normal subgroup of G, we obtain that H is contained in a normal "locally polycyclic"-by-finite subgroup in view of [11]. By Lemma 1.1, K contains a locally polycyclic G-invariant subgroup A of finite index with the abelian quotient group G/A. Therefore $HA/A \cong H/(A \cap H)$ is abelian and consequently H is polycyclic. Hence G is a locally polycyclic group, as desired.

Corollary 1.7. *Any locally solvable group with proper "locally polycyclic"-by-finite subgroups is "locally polycyclic"-by-finite.*

2. Proposition 2.1.

Proposition 2.1. *A locally solvable group G satisfies the minimal condition on non-"locally polycyclic"-by-finite subgroups if and only if it is "locally polycyclic"-by-finite.*

Proof. In view of Corollary 1.7 suppose that G has a proper subgroup G_1 which is not "locally polycyclic"-by-finite. Then G_1 contains a proper non-"locally polycyclic"-by-finite subgroup. Repeating in this manner we obtain an infinite descending chain of non-"locally polycyclic"-by-finite proper subgroups. This contradiction proves the proposition.

Lemma 2.2. *Let G be a non-perfect non-hypercentral group with the "locally polycyclic"-by-finite commutator subgroup G'. If G satisfies Max-$\overline{\mathcal{LPF}}$ then either G is a "locally polycyclic"-by-finite group or G/G' is finitely generated.*

Proof. It is well known that $\overline{G} = G/G' = \overline{N} \times \overline{D}$, where $\overline{D}$ is the periodic part of $\overline{G}$ and $\overline{N}$ is a reducible abelian subgroup of $\overline{G}$. Let D (respectively N) be the inverse image of $\overline{D}$ (respectively $\overline{N}$) in G. Then D and N are normal in G.

1). Suppose that the periodic part $\overline{D}$ is non-trivial. It is clear that $\overline{D} \cong \mathbb{C}_{p^\infty}$ for some prime p and N is a "locally polycyclic"-by-finite subgroup. Let $H = \langle x_1, ..., x_n \rangle$ be a finitely generated subgroup of D. Since $G'H \neq D$ and the quotient group $H/(G' \cap H)$ is abelian and finite, we conclude that H is a polycyclic subgroup. Thus D is a locally polycyclic subgroup and by the results of [11] G is a "locally polycyclic"-by-finite group.

2). Now suppose that $\overline{D}$ is trivial. Then $\overline{N}^p \neq \overline{N}$ for some prime p. Let A be an inverse image of $\overline{N}^p$ in G. Since G satisfies Max-$\overline{\mathcal{LPF}}$, we obtain that $|G : A| < \infty$ and there is a subgroup F of G such that $G = AF$, $F \geqslant G'$ and the quotient group F/G' is finitely generated. If $F \neq G$ then G/F is a p-divisible abelian group.

Suppose that $B = (G/F)/\tau(G/F)$ is non-trivial. Then by the results of [12] (see also [1, chapter 2, §6]) B contains a p-divisible subgroup H isomorphic to a p-divisible subgroup $\mathbb{Q}^{(p)} = \{\frac{a}{p^n} \mid a \in \mathbb{Z},\ n \in \mathbb{N} \cup \{0\}\}$ of the additive group of rational numbers $\mathbb{Q}$. Let L be a subgroup of H isomorphic to $\mathbb{Z}$. Since $\mathbb{Q}^{(p)}/\mathbb{Z} \cong \mathbb{C}_{p^\infty}$, we see that $B/L \cong \overline{X} \times \overline{Y}$, where $\overline{X} \cong \mathbb{C}_{p^\infty}$ and $\overline{Y}$ is some subgroup of B/L. Hence an inverse image Y of $\overline{Y}$ in G is a "locally polycyclic"-by-finite group. As before we can prove that G is a "locally polycyclic"-by-finite group.

Now suppose that G/F is a periodic π-group. If $|\pi| = \infty$ then there are infinite subsets π_1 and π_2 such that $\pi_1 \cap \pi_2 = \emptyset$ and $\pi = \pi_1 \cup \pi_2$. Consequently $G/F = \overline{X_1} \times \overline{X_2}$, where $\overline{X_i}$ is the Sylow π_i-subgroup of G/F $(i = 1, 2)$. Let X_i be the inverse image of $\overline{X_i}$. Then X_i is a "locally polycyclic"-by-finite normal subgroup of G and by [11] $G = X_1X_2$ is also "locally polycyclic"-by-finite.

Assume that the set π is finite and the quotient G/F is infinite. Then either G is "locally polycyclic"-by-finite or $\overline{G} = G/F = \overline{S} \times \overline{L}$, where $\overline{S}$ is an infinite Sylow s-subgroup of G for some prime $s \in \pi$ and $\overline{L}$ is a finite s'-subgroup. If L is an inverse image of $\overline{L}$ in G then by W we denote the inverse image of $(G/L)^s$. If $|G : W| = \infty$ then G is "locally polycyclic"-by-finite. Therefore we assume that $|G : W| < \infty$. Then there is a subgroup F_1 of F such that $G = WF_1$, $F_1 \lhd G$ and F_1/F is finitely generated. From $F_1 = G$ it follows that G/G' is finitely generated. If $F_1 \neq G$ then G/F_1 is a divisible abelian s-group and consequently G is "locally polycyclic"-by-finite. The lemma is proved.

Corollary 2.3. *Let G be a solvable group. Then G satisfies Max-$\overline{\mathcal{LPF}}$ if and only if G is either a "locally polycyclic"-by-finite group or a finitely generated group with Max-$\overline{\mathcal{LPF}}$.*

Proposition 2.4. *Every finitely generated metabelian group G satisfies Max-$\overline{\mathcal{LPF}}$.*

Proof. Suppose that the group G is not polycyclic. Let K be any non-"locally polycyclic"-by-finite subgroup of G. Then the subgroup $\overline{G'K} = G'K/(G' \cap K) = \overline{G'} \rtimes \overline{K}$ satisfies the maximal condition on normal subgroups Max-n by the Hall Theorem (see e.g. [4, theorem 15.3.1]). If $\overline{S}$ is a subgroup of $\overline{G'K}$ which contains $\overline{K}$, then $\overline{S} = (\overline{G'} \cap \overline{S}) \rtimes \overline{K}$ and consequently $(\overline{G'} \cap \overline{S}) \lhd \overline{G'K}$. This means that every series $\overline{K} \leqslant \overline{K_1} \leqslant \cdots \leqslant \overline{G'K}$ is finite. Hence G satisfies Max-$\overline{\mathcal{LPF}}$, as desired.

Theorem 2.5. *Let G be a finitely generated solvable group. Then G satisfies Max-$\overline{\mathcal{LPF}}$ if and only if at least one of following two cases takes places:*

(1) *G is a polycyclic group;*

(2) *every non-"locally polycyclic"-by-finite subgroup of G is finitely generated.*

Proof. ($\Rightarrow$). If G is a non-polycyclic group with Max-$\overline{\mathcal{LPF}}$ then condition (2) follows from Lemma 2.2.

($\Leftarrow$). Let G be a non-polycyclic finitely generated solvable group of derived lenght $n \geqslant 1$ in which every non-"locally polycyclic"-by-finite subgroup is finitely generated. Since the subgroup $G^{(n-1)}K$ is finitely generated, we obtain that

$$\overline{G^{(n-1)}K} = G^{(n-1)}K/((G^{(n-1)}K') \cap K) = \overline{G^{(n-1)}} \rtimes \overline{K}$$

satisfies Max-n by the Hall Theorem (see e.g. [4, theorem 15.3.1]). This means that every series $K \leqslant K_1 \leqslant \cdots \leqslant G^{(n-1)}K$ is finite. Similarly,

$$\overline{G^{(n-2)}K} = G^{(n-2)}K/((G^{(n-2)}(G^{(n-1)}K)') \cap G^{(n-1)}K) = \overline{G^{(n-2)}} \rtimes \overline{G^{(n-1)}K}$$

satisfies Max-n and thus every series $G^{(n-1)}K \leqslant S_1 \leqslant \cdots \leqslant G^{(n-2)}K$ is finite. In the same manner by finite steps we can prove that every series $K \leqslant K_1 \leqslant \cdots \leqslant G'K$ is finite. Hence G satisfies Max-$\overline{\mathcal{LPF}}$. The theorem is proved.

Corollary 2.6. *Let G be a non-polycyclic finitely generated metabelian group. Then every proper subgroup of G is either "locally polycyclic"-by-finite or finitely generated.*

Corollary 2.7. *Let G be a periodic solvable groups. Then G satisfies Max-$\overline{\mathcal{LPF}}$ if and only if G is a "locally polycyclic"-by-finite group.*

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