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RIGID DIFFERENTIALLY TRIVIAL HEREDITARY RINGS

0. A ring R having no nontrivial derivations will be called differentially trivial (see [1]). A ring R is called rigid if it has only the trivial ring endomorphisms: that is, identity id_R and zero 0_R. The problem of investigating rigid rings has been posed by C. Maxson [2]. C. Maxson [2] and K. McLean [3] described the rigid right Artinian rings. M. Friger [4] constructed an example of a noncommutative ring R with the additive group R^+ of finite Prüfer rank. The rigid ring R with the additive group R^+ of finite Prüfer rank are characterized in [5].

In this note we characterize the rigid differentially trivial right hereditary rings.

Throughout the paper all rings are associative and, as a rule, contain an identity element. $Q(A)$ will always denote the field of quotients of a commutative domain A, $\mathrm{char}(R)$ the characteristic of a ring R, $\mathcal{J}(R)$ the Jacobson radical of R and $\phi|_R$ the restriction of a homomorphism ϕ to R. Any unexplained terminilogy is standard, as in [6].

1. C. Maxson [2, theorem 2.5] proved that a commutative domain R of prime characteristic p is rigid if and only if it is isomorphic to the finite field $\mathbb{Z}_p$ with p elements. Some rigid fields of characteristic 0 were investigated in [2] and [5]. In particular, C. Maxson [2, theorem 4.2] characterized the rigid finitely generated algebraic extension of the rational numbers field $\mathbb{Q}$. To date, the problem of a description of the rigid domains (in particular, fields) A of characteristic 0 with the additive group A^+ of infinite Prüfer rank remains open. The field of real numbers $\mathbb{R}$ is an example of a rigid field with the additive group $\mathbb{R}^+$ of infinite Prüfer rank.

Let K, L, N be the commutative domains, $K \subseteq L$ and $\sigma : K \longrightarrow N$ is a homomorphism. If an element α is algebraic over $Q(K)$ then by $f_{(Q(K),\alpha)}(x)$ we denote a minimal polynomial of α over $Q(K)$. A homomorphism σ extends to a homomorphism $\overline{\sigma} : Q(K) \longrightarrow Q(N)$. By $g^\sigma(x)$ we denote the polynomial

$$\overline{\sigma}(a_0)x^n + \ldots + \overline{\sigma}(a_n) \in Q(N)[x],$$

where $g(x) = a_0x^n + \ldots + a_n \in Q(K)[x]$.

The following lemmas are generalizations of the results from [7, chapter VI, §41] and [8].

Lemma 1.1. *Let K, L, N be the commutative domains, $K \subseteq L$, the field $Q(L)$ be algebraic over $Q(K)$ and $\alpha \in L \setminus K$. Then a homomorphism $\sigma : K \longrightarrow N$ can be extended to a homomorphism $\tau : K[\alpha] \longrightarrow N$ if and only if the polynomial $f^\sigma_{(Q(K),\alpha)}(x)$ has at least one root in N.*

Proof. ($\Rightarrow$) Suppose that σ extended to a homomorphism $\tau : K[\alpha] \longrightarrow N$. Then $\tau(\alpha) \in N$ and $f_{(Q(K),\alpha)}(\alpha) = 0$ implies that $f^\sigma_{(Q(K),\alpha)}(\tau(\alpha)) = 0$.

($\Leftarrow$) Let β be a root of

$$f^{\sigma}_{(Q(K),\alpha)}(x) \in Q(N)[x],$$

which is contained in N. Then the map $\tau : K[\alpha] \longrightarrow N$ given by the rule

$$\tau(\sum_{i=0}^{n} a_i\alpha^{n-i}) = \sum_{i=0}^{n} \sigma(a_i)\beta^{n-i}$$

determines a homomorphism $\tau : K[\alpha] \longrightarrow N$. The lemma is proved.

Recall that a domain L is finitely generated over K if $L = K[\alpha_1, \dots, \alpha_n]$ for some elements $\alpha_1, \dots, \alpha_n \in L \quad (n \in \mathbb{N})$.

Lemma 1.2. *Let K, L, N be the commutative domains, $K \subseteq L$ and $Q(L)$ be algebraic over $Q(K)$. Then a homomorphism $\sigma : K \longrightarrow N$ can be extended to a homomorphism $\tau : L \longrightarrow N$ if and only if σ extends to a homomorphism $\sigma_M : M \longrightarrow N$ for all finitely generated extension M of K in L.*

Proof. ($\Rightarrow$) is obvious.

($\Leftarrow$) Let $\{K_\alpha\}_{\alpha\in I}$ be the set of all finitely generated extensions of K, every of which is contained in L and, moreover, $I = \{0, \alpha, \beta, \dots\}$, $K_0 = K$ and $\alpha \leqslant \beta$ if and only if $K_\alpha \subseteq K_\beta$. Then I is a partially ordered directed set, $L = \bigcup_{\alpha\in I} K_\alpha$.

Let

$$A_i = \{\phi_\nu^{(i)} \mid \nu \in I_i = \{\nu, \mu, \dots\}\} \ (i \in I)$$

be the set of all homomorphisms

$$\phi_\nu^{(i)} : K_i \longrightarrow N,$$

where $\phi_\nu^{(i)}$ is an extension of σ. By our hypothesis the set A_i is nonempty and finite. Define the map $\pi_{ji} : A_i \longrightarrow A_j \ (i, j \in I, \ j \leqslant i)$ by the rule

$$\pi_{ji}\phi_\nu^{(i)} = \phi_\nu^{(i)}|_{K_j}.$$

Since every set A_i is finite and nonempty, by Lemma of [10, chapter II, §2] there exists a nonempty inverse limit $A = \varprojlim A_i$ of the inverse system $\{A_i, \pi_{ji}\}$. Let $\phi \in A$. Then $\phi : L \longrightarrow N$ is a homomorphism, which extended σ. The lemma is proved.

Corollary 1.3. *Let K, L, N be the commutative domains of prime characteristic p, $K \subseteq L$, the field $Q(L)$ be a purely inseparable extension of $Q(K)$. Then a homomorphism $\sigma : K \longrightarrow N$ can be extended to a homomorphism $\tau : L \longrightarrow N$ if and only if the polynomial $f^{\sigma}_{(Q(K),\alpha)}(x)$ has at least one root in N for every element $\alpha \in L \setminus K$.*

Proof. ($\Rightarrow$) is obvious.

($\Leftarrow$) Let $Q(L)$ be a purely inseparable extension of $Q(K)$ and $M = K[\alpha_1, \dots, \alpha_n]$ for some elements $\alpha_1, \dots, \alpha_n$ in L. We shall prove by induction on n.

Suppose that σ can be extended to a homomorphism $\sigma_1 : M_1 \longrightarrow N$, where $M_1 = K[\alpha_1, \dots, \alpha_n]$. Then $M = M_1[\alpha_n]$. Let

$$f(x) = f^{\sigma}_{(Q(M_1),\alpha_n)}(x) = x^{p^s} + a = (x - \alpha_n)^{p^s} \in Q(M_1)[x],$$

$$g(x) = f^{\sigma}_{(Q(K),\alpha_n)}(x) = x^{p^m} + b = (x - \alpha_n)^{p^m} \in Q(K)[x].$$

Hence

$$g^{\sigma}(x) = x^{p^m} + \overline{\sigma}(a) = (x - \beta)^{p^m}$$

for some element β of N. The polynomial $f^{\sigma_1}(x)$ divides the polynomial $g^{\sigma}(x)$ over the field $Q(\sigma_1(M))$. Since $g^{\sigma}(x)$ has a root in N, we conclude that $\beta \in N$ and consequently β is a root of $f^{\sigma_1}(x)$. So by Lemma 1.1 a homomorphism σ_1 can be extended to a homomorphism $\sigma_M : M \longrightarrow N$. Finally, we apply Lemma 1.2 to complete the proof.

Corollary 1.4 ([8, corollary 1]). *Let K, L, N be the fields of prime characteristic p, $K \subseteq L$, L be a separable (relatively purely inseparable) algebraic extension of K. Then a homomorphism $\sigma : K \longrightarrow N$ can be extended to a homomorphism $\tau : L \longrightarrow N$ if and only if the polynomial $f^{\sigma}_{(Q(K),\alpha)}(x)$ has at least one root in N for every element $\alpha \in L \setminus K$.*

Lemmas 1.1 and 1.2 yield

Theorem 1.5. *Let F be a field of zero characteristic algebraic over its prime subfield P. Then F is a rigid field if and only if the minimal polynomial $f_\xi(x) \in P[x]$ of ξ has exactly one root in F for every element ξ of F.*

2. In this section we characterize the rigid differentially trivial right hereditary rings.

Proposition 2.1 (see [1] or [5]). *A commutative domain R is differentially trivial if and only if at least one of the following two cases takes place:*
(*i*) $\mathrm{char}(R) = 0$ *and the field of quotients $Q(R)$ is algebraic over its prime subfield;*
(*ii*) $\mathrm{char}(R) = p$ *is a prime and $R = \{ x^p \mid x \in R \}$.*

Lemma 2.2. *Let R be a right hereditary ring. Then R is differentially trivial if and only if it is a ring direct sum of differentially trivial domains.*

Proof. ($\Rightarrow$) Since R is commutative, we conclude that R not contains nontrivial nilpotent elements (see, for example, [6, chapter 8, exercises]). In view of [10] (see also 8.23.9(a) of [6]) R is a ring direct sum of differentially trivial domains.

($\Leftarrow$) Let R be a ring direct sum of differentially trivial domains, i.e.

$$R = \sum\nolimits^{\oplus} R_i.$$

Suppose that R has a nontrivial derivation D. Let $r = (r_i)_{i \in I}$ be an element of R such that

$$D(r) = (a_i)_{i \in I} \neq 0.$$

and j be a fixed element of I such that

$$a_j \neq 0.$$

Then the map $d : R_j \longrightarrow R_j$ given by the rule

$$d(r_j) = a_j \ (r_j \in R_j)$$

determines a nontrivial derivation of R_j, a contradiction. The lemma is proved.

Lemma 2.3. *Let R be a ring with the identity element 1 and nontrivial nilpotent element a of the nilpotency index n. If R has a nontrivial derivation D then the rule*

$$\tau(r) = \sum\nolimits_{i=0}^{n} \frac{D^{(i)}(r)}{i!} a^i, \ D^{(0)}(r) = r, \ a^0 = 1 \ (r \in R)$$

determines a nontrivial automorphism of R.

Proof. Straightforward.

Recall that a ring R is called reduced if $x^2 = 0$ implies $x = 0$ for any $x \in R$.

Lemma 2.4. *Any rigid right hereditary ring R is reduced.*

Proof. If R contains a nontrivial nilpotent element then by Lemma 2.3 it is a differentially trivial ring. Consequently R is commutative, a contradiction. The lemma is proved.

Corollary 2.5. *Let R be a commutative hereditary ring. Then R is a rigid ring if and only if $R \cong \mathbb{Z}_p$ for some prime p or R is a rigid domain of zero characteristic.*

Proof. ($\Leftarrow$) is obvious.

($\Rightarrow$) In view of Lemma 2.4 and the results of [10] R is a Dedekind domain. By Corollary of [11, application, §4, $n^\circ 3$] R is a local domain. If $\mathrm{char}(R) = p$ is a prime then by Theorem 2.5 of [2] $R \cong \mathbb{Z}_p$, and the proof is complete.

Lemma 2.6. *Let R be a differentially trivial domain of zero characteristic. If R contains a subfield then the Jacobson radical $\mathcal{J}(R)$ is trivial.*

Proof. Since R contains a subfield, its prime subfield is isomorphic to the field of rational numbers $\mathbb{Q}$. Then by Proposition 2.1 for every element j of $\mathcal{J}(R)$ there exists a nontrivial polynomial

$$g(x) = \sum_{i=0}^{m} a_i x^{m-i} \in P[x]$$

such that

$$g(j) = 0.$$

Hence

$$a_m = -\sum_{i=0}^{m-1} a_i j^{m-i} \in (P \cap \mathcal{J}(R)),$$

and this yields that

$$j \cdot (\sum_{i=0}^{m-1} a_i j^{m-i-1}) = 0,$$

a contradiction. The lemma is proved.

Theorem 2.7. *Let R be a right hereditary ring. Then the following statements are equivalent.*

(*a*) *R is a rigid differentially trivial ring.*

(*b*) *R is of one of the following types:*

(*i*) $R \cong \mathbb{Z}_p$;

(*ii*) *R is a rigid field of zero characteristic algebraic over its prime subfield;*

(*iii*) *R is a rigid local domain of zero characteristic with the residue field $R/\mathcal{J}(R)$ of prime characteristic p and the field $Q(R)$ is algebraic over its prime subfield.*

Proof. (b) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (b) From Corollary 2.5 it follows that $R \cong \mathbb{Z}_p$ or R is a rigid differentially trivial local domain of zero characteristic. Let $\mathrm{char}(R) = 0$. By Proposition 2.1 $Q(R)$ is algebraic over its prime subfield. From $\mathrm{char}(R/\mathcal{J}(R)) = 0$ in view of Lemma 2.6 it follows that R is of type (ii). If $\mathrm{char}(R/\mathcal{J}(R)) = p$ is a prime then R is a ring of type (iii). The theorem is proved.

Recall [12], that a v-ring is a commutative unramified complete regular rank one local domain of zero characteristic with a residue field of prime characteristic.

Remark 2.8. If a ring R of type *(iii)* (see Theorem 2.7) is complete in $\mathcal{J}(R)$-adic topology then R is a v-ring with the quotient field $R/\mathcal{J}(R) \cong \mathbb{Z}_p$.

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1. *Артемович О.Д.* Ідеально-диференціальні і досконалі жорсткі кільця// Доп. АН УРСР. – 1985. – N 4. – C.3-5.
2. *Maxson C.J.* Rigid rings// Proc. Edinburgh Math. Soc.– 1979. – Vol.21. – N 1. – P.95-101.
3. *McLean K.R.* Rigid Artinian Rings// Proc. Edinburgh Math. Soc. – 1982. –Vol. 25. – N 1. – P.97-99.
4. *Фригер М.Д.* О жестких кольцах без кручения// Сиб. мат. журн. – 1986. –Т. 27. – N 1. – С.217-219.
5. *Artemovych O.D.* Differentially trivial and rigid rings of finite rank// Periodica Math. Hungar. – 1998. – Vol.36. – N 1. – P. 1-16.
6. *Фейс К.* Алгебра: кольца, модули и категории. Т.1. – М.,1977.
7. *Ван дер Варден Б.Л.* Алгебра. – М., 1977.
8. *Сергеев Э.А.* О продолжении мономорфизмов полей. – Краснодар, 1980. – Депонировано ВИНИТИ, N 50-81.
9. *Серр Ж.-П.* Курс арифметики. – М., 1972.
10. *Levy L.* Unique direct sums of prime rings// Trans. Amer. Math. Soc. – 1963. – Vol. 106. – N 1. – P.64-76.
11. *Грауэрт Г., Реммерт Р.* Аналитические локальные алгебры. – М., 1988.
12. *Cohen I.S.* On the structure and ideal theory of complete local rings// Trans. Amer. Math. Soc. – 1946. – Vol. 59. – N 1. – P.54-106.

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ЖОРСТКІ ДИФЕРЕНЦІАЛЬНО-ТРИВІАЛЬНІ СПАДКОВІ КІЛЬЦЯ

Охарактеризовано праві спадкові кільця, які володіють тільки нульовими диференціюваннями і тривіальними кільцевими ендоморфізмами.

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