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VITALIJ BONDARENKO

INFINITELY REPRESENTED BUNDLES OF TWO SEMICHAINS HEREDITARY RINGS

In this paper we study representations of bundles of semichains. These representations were classified (in the invariant form) in [1]. A history of solving this classification problem and its applications was presented in [2].

We associate a bigraph with a bundle of two semichains and give (in its terms) a necessary and sufficient condition that a bundle is infinitely represented, i.e. that it has (up to isomorphism) infinitely many indecomposable representations.

Throughout the paper, k denotes an arbitrary field; all partially ordered sets (posets) are finite and all vector spaces are finite-dimensional and right. Considering linear maps, morphisms and so on, we use the right-side notation.

1. The category of representations of a bundle of two semichains. For a poset A with involution $*$ and a field k, $\mathrm{mod}_{(A,*)}k$ denotes (by analogy with the category of finite-dimensional vector k-spaces $\mathrm{mod}\, k$) the category of $(A,*)$-spaces over k [3], i.e. the category with objects the vector k-spaces $U = \oplus_{a\in A} U_a$, where $U_{a^*} = U_a$ (for all $a \in A$), and with morphisms $\delta : U \to U'$ the linear maps $\delta \in \mathrm{Hom}_k(U, U')$ for which $\delta_{a^*a^*} = \delta_{aa}$ and $\delta_{bc} = 0$ if $b \not\leq c$. Here δ_{xy} denotes (as usual in analogous situations) the linear map of U_x into U'_y induced by δ. We identify $(A,*)$ with A if the involution $*$ is trivial.

Recall that a semichain is a poset of the form $Y = \overset{s}{\underset{i=1}{\cup}} Y_i$, where each Y_i consists of either one point or two incomparable points, and $Y_1 < Y_2 < \cdots < Y_s$ (i.e. $y_1 < y_2 < \cdots < y_s$ for all $y_i \in Y_i$); the subsets Y_i are called the links of the semichain Y (if each link consists of one point, the set Y is called a chain).

Let A and B be disjoint semichains ($A \cup B \neq \varnothing$). A bundle of the semichains A and B is a triple $\overline{S} = (A, B, *)$, where $*$ is an involution of $A \cup B$ such that $x^* = x$ for each x belonging to a two-point link. The representations of the bundle $\overline{S} = (A, B, *)$ over k are the triples (U, V, φ), where $U \in \mathrm{mod}_A k$, $V \in \mathrm{mod}_B k$, $U \oplus V \in \mathrm{mod}_{(A\cup B,*)} k$ and φ is a linear map of U into V ($A \cup B$ is the poset with the smallest order relation containing the order relations of A and B). A morphism from (U, V, φ) to (U', V', φ') is determined by a pair (α, β) of linear maps $\alpha : U \to U'$ and $\beta : V \to V'$ such that $\alpha \in \mathrm{mod}_A k$, $\beta \in \mathrm{mod}_B k$, $\alpha \oplus \beta \in \mathrm{mod}_{(A\cup B,*)} k$ and $\varphi\beta = \alpha\varphi'$. The representations of the bundle $\overline{S} = (A, B, *)$ over k form a (Krull–Schmidt) category which we will denote by $\mathcal{B}_k(\overline{S})$ or $\mathcal{B}_k(A, B, *)$.

A bundle of two semichains $\overline{S} = (A, B, *)$ is called finitely represented (over k) if the category $\mathcal{B}_k(\overline{S})$ has only finitely many isomorphism classes of indecomposable objects; otherwise, $\overline{S}$ is called infinitely represented.

2. Formulation of the main result. Let $\overline{S}$ be a bundle of semichains A and B. Define two symmetric binary relations, $\sim$ and $-$, on $A \cup B$ by putting $x \sim y$ if and only if $x^* = y$, $x \neq y$, and $x - y$ if and only if either $x \in A$, $y \in B$ or $x \in B$, $y \in A$.

With a bundle $\overline{S} = (A, B, *)$ we associate the following (nonoriented) bigraph $G = G(\overline{S})$:

a) the vertices of G are the symbols e_x, $x \in A \cup B$;

b) G has edges of two type – "$\sim$" and "$-$"; the edge $e_x \sim e_y$ (respectively, $e_x - e_y$) exists if and only if $x \sim y$ (respectively, $x - y$) in $A \cup B$.

A subgraph of G is determined as usual: it is a bigraph with a set of vertices $E \subset \{e_x | x \in A \cup B\}$ and some edges from G (between vertices $e_x, e_y \in E$). If the bigraph $G(\overline{S})$ is geometrically given, we identify e_x with x.

In the sequel, we denote links of semichains by lower case letters, and identify the one-poits links with the points themselves; the points of a two-point link x is denoted by x^+ and x^-.

The main theorem. *A bundle $\overline{S}$ of two semichains is infinitely represented if and only if the bigraph $G(\overline{S})$ contains one of the following subgraphs:*

$$1)\ x \simeq y; \quad 2)\ \begin{array}{ccc} x^+ & - & y^+ \\ | & & | \\ y^- & - & x^- \end{array}; \quad 3)\ \begin{array}{ccc} x_1 & \sim & x_2 \\ | & & | \\ y_1 & \sim & y_2 \end{array}; \quad 4)\ \begin{array}{ccccc} & & y^+ & & \\ & / & & \backslash & \\ x_1 & & \sim & & x_2 \\ & \backslash & & / & \\ & & y^- & & \end{array}.$$

3. Invariants of indecomposable representations of the bundle $\overline{S}$. Let $\overline{S} = (A, B, *)$ be a bundle of semichains A and B; let $L(A)$ or $L(B)$ be the set of links of the semichain A or B, respectively; denote by $L(S)$, or simply L, the union of the sets $L(A)$ and $L(B)$. We denote the number of points of a link x by $r(x)$.

Define two symmetric binary relations, α and β, on the set L by putting $x\alpha y$ if and only if either $x \neq y$, $r(x) = r(y) = 1$ and $x^* = y$, or $x = y$ and $r(x) = 2$; and $x\beta y$ if and only if either $x \in L(A)$ and $y \in L(B)$, or $x \in L(B)$ and $y \in L(A)$.

We call an L-chain (respectively, L-cycle) an expression g of the form $x_1\lambda_1x_2\lambda_2 \dots x_{m-1}\lambda_{m-1}x_m$, $m \geqslant 1$ (respectively, $x_1\lambda_1x_2\lambda_2 \dots x_{m-1}\lambda_{m-1}x_m\lambda_m$, $m \geqslant 2$), where $x_i \in L$, $\lambda_j \in \{\alpha, \beta\}$, $x_j\lambda_jx_{j+1}$ in L and $\lambda_j \neq \lambda_{j+1}$ for all $i = 1, \dots, m$ and $j = 1, \dots, m-1$ (respectively, $i, j = 1, \dots, m$); notice that for cycles the subscripts $p > m$ and $p < 1$ are considered modulo m (in particular $x_{m+1} = x_1$ and $\lambda_{m+1} = \lambda_1$). The number m is called the length of an L-chain (respectively, L-cycle) and is denoted by $|g|$. Denote by g^*, where g is an L-chain (respectively, L-cycle), the L-chain $x_m\lambda_{m-1}x_{m-1} \dots \lambda_2x_2\lambda_1x_1$ (respectively, the L-cycle $x_m\lambda_{m-1}x_{m-1} \dots x_2\lambda_1x_1\lambda_m$); for an L-cycle g, $g(i)$ denotes the L-cycle $x_i\lambda_i \dots x_{i+m-1}\lambda_{i+m-1} = x_i\lambda_i \dots x_m\lambda_mx_1\lambda_1 \dots x_{i-1}\lambda_{i-1}$.

L-chains (respectively, L-cycles) g and h are called isomorphic if either $g = h$ or $g = h^*$ (respectively, either $g = h(i)$ or $g = h^*(i)$ for some i). An L-chain (respectively, L-cycle) is called symmetric if $|g| > 1$ and $g = g^*$ (respectively, $g(i) = g^*$ for some i); an L-cycle g is called periodic if $g = g(i)$ for some i, $1 < i \leqslant m$.

L-subchains (or, simply, subchains) of an L-chain (respectively, L-cycle) g is called the L-chains h of the form $x_i\lambda_{i+1} \dots \lambda_{i+s-1}x_{i+s}$, where $0 \leqslant s \leqslant m - i$ (respectively, $0 \leqslant s \leqslant m + i - 1$).

An L-chain g will be called admissible if $x_i\alpha y$ in L, $x_i \neq y$, imply either $\lambda_{i-1} = \alpha$ or $\lambda_i = \alpha$. The left end x_1 (respectively, the right end x_m) of an L-chain g of length $m > 1$ will be called double if $\lambda_1 = \beta$ and $x_1\alpha x_1$ in L (respectively, $\lambda_{m-1} = \beta$ and

$x_m \alpha x_m$ in L); for $m = 1$, the end x_1 will be called double if $x_1 \alpha x_1$ (in L). The number of double ends of g will be denoted by $d(g)$ (if $m = 1$ and $x_1 \alpha x_1$, then $d(g) = 1$).

In the case when h is an L-chain and $d(h) = 2$, we denote by $h^{[s]}$ the L-chain of the form $h^{(1)} \alpha h^{(2)} \alpha \dots \alpha h^{(s)}$, where $h^{(i)} = h$ for odd i and $h^{(i)} = h^*$ for even i; if only the right end of h is double, then $h^{[s]}$ can be constructed only for $s = 1, 2$ (in the remaining cases only for $s = 1$). An L-chain g will be called composite if it can be represented in the form $g = h^{[s]}$ for $s > 1$, and simple otherwise. An L-cycle is called simple if it is nonperiodic.

Denote by $G_1(L)$ the set of simple admissible L-chains, and by $G_2(L)$ the set of simple L-cycles; put $G(L) = G_1(L) \cup G_2(L)$. For an L-cycle $g \in G_2(L)$ denote by $\delta(g)$ the number of $i \in \{1, \dots, m\}$ such that $x_i \neq x_{i+1}$ and either $x_i, x_{i+1} \in L(A)$, or $x_i, x_{i+1} \in L(B)$ ($m = |g|$); put $\delta_0(g) = \delta(g)/2$ for an symmetric L-cycle g, and $\delta_0(g) = \delta(g)$ otherwise.

Let k be an arbitrary field. In [2] (see § 1) we associate to the L-chains $g \in G_1(L)$ and L-cycles $g \in G_2(L)$ certain special representations (over k) of the bundle $\overline{S} = (A, B, *)$. Namely, to an L-chain $g \in G_1(L)$, we associate the representation $U_1(g)$ if $d(g) = 0$, the representations $U_s(g)$, $s = 1, 2$, if $d(g) = 1$, and the representations $U_s(g, p)$, $s = 1, 2, 3, 4$, if $d(g) = 2$, where p is any natural number. To an L-cycle $g \in G_2(L)$, we associate the representation $U(g, f)$, where $f = f(t)$ is a power of a monic polynomial f_0, irreducible over k, such that $f_0 \neq t$ if g is nonsymmetric, and $f_0 \neq t, t+1$ (respectively, $f_0 \neq t, t-1$) if g is symmetric for an even (respectively, odd) $\delta_0(g)$.

These representations (whose explicit form were indicated in [1, 2]) are all the indecomposable representations of the bundle $\overline{S} = (A, B, *)$. More precisely, the following statement holds.

Theorem. *Choose one representative in each isomorphism class of L-chains and L-cycles belonging to $G(L)$. Then the set of representations of the form $U_s(g)$, $U_s(g, p)$ and $U(g, f)$ associated to the chosen L-chains and L-cycles is a complete set of pairwise nonequivalent indecomposable representations of the bundle $\overline{S} = (A, B, *)$.*

The Theorem was proved in [1].

4. Proof of the main theorem. Prove first the following lemma.

Lemma. *The following conditions are equivalent:*

a) $G_2(L) = \varnothing$;

b) $G_1(L)$ contains only finitely many L-chains and does not contain L-chains with two double ends.

Proof. Obviously, conditions a) and b) are equivalent, respectively, to the following conditions:

a') the set $\overline{G}_2(L)$ of all (not necessarily simple) L-cycles is empty;

b') the set $\overline{G}_1(L)$ of all (not necessarily simple) admissible L-chains contains only finitely many elements.

$a') \Rightarrow b'$). Let $\overline{G}_2(L) = \varnothing$. Then $\overline{G}_1(L)$ does not contain an L-chain $g = (x_1 \lambda_1 x_2 \dots \lambda_{m-1} x_m)$ such that $x_i = x_{i+s}$ and $\lambda_i = \lambda_{i+s}$ for some $s > 0$ (otherwise, $(x_i \lambda_i x_{i+1} \dots x_{i+s-1} \lambda_{i+s-1}) \in \overline{G}_2(L)$). Hence $|\overline{G}_1(L)| < \infty$.

$b') \Rightarrow a'$). Let $|\overline{G}_1(L)| < \infty$. Show that $\overline{G}_2(L) = \varnothing$. Assume the contrary and consider some L-cycle $g = (x_1 \lambda_1 x_2 \dots x_m \lambda_m)$ in $\overline{G}_2(L)$; denote by g' the L-chain $x_1 \lambda_1 x_2 \dots x_m$. Then $g^n g' \in \overline{G}_1(L)$ for any natural n, contradicting the assumption that $|\overline{G}_1(L)| < \infty$. The lemma is proved.

Proposition. *A bundle $\overline{S} = (A, B, *)$ is infinitely represented if and only if the set $G_2(L)$ contains an L-cycle of length 2 or 4.*

Proof. It follow from the Theorem and Lemma that $\overline{S}$ is infinitely represented if and only if $G_2(L) \neq \varnothing$. Show that if the set $G_2(L)$ is not empty, then it contains some L-cycle of length 2 or 4.

Let $(x_1\lambda_1x_2\ldots x_m\lambda_m) \in G_2(L)$, where $m > 4$; obviously, we can assume that $\lambda_1 = \alpha$. If for some odd i either $x_i \in L(A)$ and $x_{i+1} \in L(B)$, or $x_i \in L(B)$ and $x_{i+1} \in L(A)$, then $x_i\alpha x_{i+1}\beta$ is a simple L-cycle (of length 2). Otherwise, $x_1\alpha x_2\beta x_3\alpha x_4\beta$ is a simple L-cycle (of length 4).

The proposition is proved.

The Main theorem follows from the proposition: if $(x\alpha y\beta) \in G_2(L)$, then the bigraph $G(\overline{S})$ contains a subgraph of the form 1); if $(x\alpha x\beta y\alpha y\beta) \in G_2(L)$, then $G(\overline{S})$ contains a subgraph of the form 2); if $(x_1\alpha x_2\beta y_1\alpha y_2\beta) \in G_2(L)$, where $x_1 \neq x_2$ and $y_1 \neq y_2$, then $G(\overline{S})$ contains a subgraph of the form 3); if, finally, $(x_1\alpha x_2\beta y\alpha y\beta) \in G_2(L)$, where $x_1 \neq x_2$, then $G(\overline{S})$ contains a subgraph of the form 4).

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1. *Bondarenko V.M.* Bundles of semichained sets and their representations // Kiev: Inst. Math. Acad. Sci. Ukrainian SSR, 1988 (Preprint 88.60). – 32 p. (in Russian).
2. *Bondarenko V.M.* Representations of bundles of semichained sets, and their applications // Algebra and Analysis. – 1991. – Vol.3. – N 5. – P. 973–996 (in Russian).
3. *Bondarenko V.M.* On classifications of linear operators up to S-similarity // Dokl. Acad. Sci. of Ukraine. – 1997. – N 10. – P. 16–20 (in Russian).

Бондаренко В.

В'ЯЗКИ ДВОХ НАПІВЛАНЦЮГІВ НЕСКІНЧЕННОГО ТИПУ

У статті кожній в'язці двох напівланцюгів поставлено у відповідність деякий біграф і в його термінах сформульовано необхідні і достатні умови того, щоб в'язка мала нескінченне число нерозкладних зображень.

Institute of Mathematics, National Academy of Science of Ukraine, Kiev.

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