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NST-RIESZ BASIS IN A HILBERT SPACE

1. Notion of *nst*-equivalent bases. Denote by $\mathbb{H}$ a standard separable complex Hilbert space. Bases $(\varphi_i)_{i\in\mathbb{N}}$, $(\tilde{\varphi}_i)_{i\in\mathbb{N}}$ of $\mathbb{H}$ are said to be *equivalent* iff $\forall i \in \mathbb{N}\ \tilde{\varphi}_i = U\varphi_i$ for some $U \in \mathcal{B}(\mathbb{H})$ such that $\ker U = \{0\}$ and $U^{-1} \in \mathcal{B}(\mathbb{H})$. If this holds U is called the equivalency of (φ_i) and $(\tilde{\varphi}_i)$. Equivalent bases (φ_i), $(\tilde{\varphi}_i)$ are said to be *nst-equivalent* iff its equivalency U and U^{-1} are uniformly nearstandard operators, i.e. there exist standard operators ${}^\circ U$, $({}^\circ U)^{-1} \in \mathcal{B}(\mathbb{H})$ (${}^\circ U$ is the shadow of U) such that $\|U - {}^\circ U\| \approx 0$, $\|U^{-1} - ({}^\circ U)^{-1}\| \approx 0$ (read "is infinitesimal" for "≈ 0"). Note that the above defined relation is a genuine equivalency.

Observe that bases (ψ_i), $(\tilde{\psi}_i)$, associated biorthogonal to nst-equivalent bases (φ_i), $(\tilde{\varphi}_i)$ are also nst-equivalent.

For $x \in \mathbb{H}$ such that $\|x\| \ll \infty$ there exists a unique standard vector ${}^\circ x \in \mathbb{H}$ (*shadow* of x) such that for any standard $y \in \mathbb{H}$ $\quad ({}^\circ x|y) \approx (x|y)$. For a given sequence $(\varphi_i) \subset \mathbb{H}$ such that for standard $i \in \mathbb{N}$ $\quad \|\varphi_i\| \ll \infty$, there exists, by standardization principle, a unique standard sequence $(\mathring{\varphi}_i)_{i\in\mathbb{N}}$, such that $\mathring{\varphi}_i = {}^\circ\varphi_i$ for standard $i \in \mathbb{N}$. This sequence $(\mathring{\varphi}_i)$ is called the *shadow* of the sequence (φ_i).

Let (φ_i) be a basis of $\mathbb{H}$ for which $\|\varphi_i\| \ll \infty$ for standard $i \in \mathbb{N}$. Suppose that its shadow $(\mathring{\varphi}_i)$ is also a basis and (φ_i), $(\mathring{\varphi}_i)$ are equivalent with equivalency U, such that $\|U - I\| \approx 0$. Then (φ_i) is said to be a *nearstandard* basis.

It is easy to see that a basis (ψ_i) assosiated to a nearstandard basis (φ_i) is nearstandard too. Indeed, $\|U - I\| \approx 0$ implies $\|(U^*)^{-1} - I\| \approx 0$.

1.1. Proposition. *Let (φ_i) and $(\tilde{\varphi}_i)$ are nst-equivalent bases of $\mathbb{H}$. Then (φ_i) is nearstandard iff so is $(\tilde{\varphi}_i)$. The shadow ${}^\circ U$ of the equivalency U of φ_i and $\tilde{\varphi}_i$ is the equivalency of $(\tilde{\varphi}_i)$ and $\left(\mathring{\tilde{\varphi}}_i\right)$.*

Proof. Assume that $\mathring{\varphi}_i = V\varphi_i$, where $\|V - I\| \approx 0$ and $\tilde{\varphi}_i = U\varphi_i$, where U is uniformly nearstandard. Define $\forall i \in \mathbb{N}$ $\quad \hat{\varphi}_i = ({}^\circ U)\mathring{\varphi}_i$. Then $(\hat{\varphi}_i)$ is a standard basis of $\mathbb{H}$ which is equivalent to $(\mathring{\varphi}_i)$ with the standard equivalency ${}^\circ U$. Set $V_1 := ({}^\circ U)VU^{-1}$. Then ${}^\circ V_1 = ({}^\circ U)I^\circ(U^{-1}) = I$ and $\|V_1 - I\| \approx 0$. It is easy to check that $\hat{\varphi}_i = V_1\tilde{\varphi}_i$. Therefore, $(\tilde{\varphi}_i)$ is nearstandard and $(\hat{\varphi}_i) = (\mathring{\tilde{\varphi}}_i)$.

It is also so easy to prove the following

1.2. Proposition. *The shadow of a nearstandard orthonormal basis is an orthonormal basis.*

Let (φ_i) be a nearstandard basis of $\mathbb{H}$. Consider an arbitrary vector $x \in \mathbb{H}$ and denote by (c_i) the sequence of coordinates of x: $x = \sum_{i \in \mathbb{N}} c_i \varphi_i$. Suppose that $\|x\| \ll \infty$, then $|c_i| \ll \infty$ for standard $i \in \mathbb{N}$ and there exists a unique standard ${}^\circ c_i \in \mathbb{C}$ such that $c_i \approx {}^\circ c_i$ By standardization principle of IST there exists a unique standard sequence $(\mathring{c}_i)$ in $\mathbb{C}$ such that $\mathring{c}_i = {}^\circ c_i$ for standard $i \in \mathbb{N}$.

1.3. Proposition. *In the above assumption we have*

$$ {}^\circ x = \sum_{i \in \mathbb{N}} \mathring{c}_i \mathring{\varphi}_i, \tag{1.1} $$

where $(\mathring{\varphi}_i)$ is the shadow of the basis (φ_i).

Proof. As it was noted the assosiated basis (ψ_i) is nearstandard too. Hence $\forall i \in \mathbb{N}$ $\|\mathring{\psi}_i - \psi_i\| \approx 0$. Whence $\|\psi_i\| \ll \infty$ for standard $i \in \mathbb{N}$. Because $\|x\| \ll \infty$ we have $|(x|\psi_i)| \ll \infty$ for standard $i \in \mathbb{N}$. Therefore indeed $|c_i| \ll \infty$ and the sequence $(\mathring{c}_i)$ is well defined above. Observe that ${}^\circ x = \sum_{i \in \mathbb{N}} ({}^\circ x|\mathring{\psi}_i)\mathring{\varphi}_i$, but for standard $i \in \mathbb{N}$ we have $({}^\circ x|\mathring{\psi}_i) = {}^\circ(x|\mathring{\psi}_i) = {}^\circ(x|\psi_i) = \mathring{c}_i$.

2. nst-Riesz basis. Recall that a basis (φ_i) is said to be a Riesz basis iff it is equivalent to an orthonormal basis [2]. A basis which is nst-equvalent to a Riesz basis is called an *nst-Riesz basis*. Obviously, each nst-Riesz basis is a Riesz basis.

2.1. Proposition. *(i) A basis is an nst-Riesz basis iff it is nearstandard and its shadow is a Riesz basis.*

(ii) A basis is an nst-Riesz basis iff it is nst-equivalent to some standard orthonormal basis.

Proof. (i) Let (φ_i) be an nst-Riesz basis. Denote by U the (uniformly nearstandard) equivalency of (φ_i) and a standard Riesz basis $(\tilde{\varphi}_i)$. Set $\hat{\varphi}_i := ({}^\circ U)^{-1}\tilde{\varphi}_i$. Since $U\varphi_i = \tilde{\varphi}_i$, we have $({}^\circ U)^{-1}U\varphi_i = \hat{\varphi}_i$. Therefore $({}^\circ U)^{-1}U$ is an equivalency of (φ_i) and $(\hat{\varphi}_i)$. But $\|({}^\circ U)^{-1}U - I\| \approx 0$ and $\hat{\varphi}_i$ is standard. Hence $(\hat{\varphi}_i)$ is the shadow $(\mathring{\varphi}_i)$. Because $(\tilde{\varphi}_i)$ and $(\hat{\varphi}_i)$ are equivalent, $(\mathring{\varphi}_i)$ is a Riesz basis. The converse is evident.

(ii) Let (φ_i) be an nst-Riesz basis. Then $(\mathring{\varphi}_i)$ is a standard Riesz basis. By definition and transfer principle, $(\mathring{\varphi}_i)$ is equivalent to the standard orthonormal basis (e_i) with a standard equivalency. By transitivity of the nst-equivalence, (φ_i) and (e_i) are nst-equivalent. Since each orthonormal basis is a Riesz basis, the converse is evident.

2.2. Corollary. *Let (φ_i) be an nst-Riesz basis. Then the constants γ_1, γ_2 in the Parseval inequality*

$$ \forall x \in \mathbb{H} \quad \gamma_1\|x\|^2 \leqslant \sum_{i \in \mathbb{N}} |(x|\psi_i)|^2 \leqslant \gamma_2\|x\|^2, \tag{2.1} $$

are appreciables numbers i.e. $0 \ll \gamma_1 < \gamma_2 \ll \infty$.

Proof. Let U be the uniformly nearstandard equivalency of (φ_i) and standard orthonormal basis (e_i). Therefore $\sum_{i \in \mathbb{N}} |(x|\psi_i)|^2 = \|Ux\|^2$. Because $\|Ux\| \leqslant \|U\|\|x\|$ and

$\|x\| \leqslant \|U^{-1}\|\|Ux\|$, (2.1) holds for $\gamma_1 = \|U^{-1}\|^{-1}$ and $\gamma_2 = \|U\|$. Since U and U^{-1} are uniformly nearstandard, the constants γ_1 and γ_2 are appreciables.

Recall that a vector $x \in \mathbb{H}$ such that $\|x\| \ll \infty$ is said to be (strongly) nearstandard iff $\|x - {}^\circ x\| \approx 0$.

2.3. Theorem. *Let (φ_i) be an nst-Riesz basis. A vector $x \in \mathbb{H}$, such that $\|x\| \ll \infty$ is nearstandard iff for any infinite $n \in \mathbb{N}$*

$$\sum_{i>n} |c_i|^2 \approx 0, \tag{2.2}$$

where (c_i) is the sequence of coordinates of x: $x = \sum_{i\in\mathbb{N}} c_i\varphi_i$.

Proof. As it is well known (see e.g. [4] or [5]), a vector $x \in \mathbb{H}$, such that $\|x\| \ll \infty$ is nearstandard iff for a standard orthonormal basis (e_i) $\sum_{i>n} |(x|e_i)|^2 \approx 0$ holds for any infinite $n \in \mathbb{N}$. Let (ψ_i) be the associated basis for (φ_i) and a standard orthonormal basis (e_i). It easy to check that $c_i = (x|\psi_i) = (Ux|e_i)$. Therefore (2.2) is a necessary and sufficient condition for Ux to be nearstandard. Because the equivalency U (and also U^{-1}) is uniformly nearstandard, Ux is nearstandard iff so is x. (Note that ${}^\circ(Ux) = ({}^\circ U)({}^\circ x)$).

2.4 Remark. For each orthonormal sequence (e_i) in $\mathbb{H}$, by standardization principle we can construct a unique standard sequence $(\hat{e}_i)$, such that $\hat{e}_i = {}^\circ e_i$ for standard $i \in \mathbb{N}$. By transfer principle, $(\hat{e}_i)$ is orthonormal. It is not difficult to prove (using Robinson's lemma) that $(\hat{e}_i)$ is a basis iff so is (e_i) and $\sum_{i>n} |(x|e_i)|^2 \approx 0$ holds for any standard x and any infinite $n \in \mathbb{N}$.

2.5 Remark. Let (φ_i) be a basis of $\mathbb{H}$. As it is known (see e.g. [2]), (φ_i) is a Riesz basis iff for any bijection $\pi : \mathbb{N} \to \mathbb{N}$ $(\varphi_{\pi(i)})_{i\in\mathbb{N}}$ is a basis as well. Suppose that the basis (φ_i) is nearstandard. Then $(\varphi_{\pi(i)})_{i\in\mathbb{N}}$ is a nearstandard basis iff the bijection π is standard. For proof use the following

Remark. Let f, g be bijections with $\mathrm{dom}f = \mathrm{im}g$. Suppose that f is standard. Then $f \circ g$ is standard iff so is g.

3. Infinitesimal perturbation of basis. Introduce the following notion. A sequence (γ_i) in $\mathbb{H}$ is said to be *uniformly infinitesimal* iff for arbitrary $(c_i) \in \mathbb{C}$, $n \in \mathbb{N}$

$$\sum_{i\leqslant n} |c_i|^2 \leqslant 1 \implies \|\sum_{i\leqslant n} c_i\gamma_i\| \approx 0. \tag{3.1}$$

3.1. Lemma. *Let (γ_i) be an uniformly infinitesimal sequence in $\mathbb{H}$. Then for any $c = (c_i) \in \ell_2$ the series $\sum_{i\in\mathbb{N}} c_i\gamma_i$ is convergent. The operator Γ, defined by*

$$\forall c \in \ell_2 \; \Gamma c = \sum_{i\in\mathbb{N}} c_i\gamma_i \tag{3.2}$$

belongs to $\mathcal{B}(\ell_2;\mathbb{H})$ and $\|\Gamma\| \approx 0$.

Proof. Let ε be the lest upper bound of $\|\sum_{i\leqslant n} c_i\gamma_i\|$ for $n \in \mathbb{N}$ and $\sum_{i\in\mathbb{N}} |c_i|^2 \leqslant 1$. By (3.1) $\varepsilon \approx 0$. At first define Γ by (3.2) for c satisfying $(\exists n \in \mathbb{N})(\forall i > n)(c_i = 0)$. Then $\|\Gamma c\| \leqslant \varepsilon\|c\|$. Because such c are dense in ℓ_2 we can extend Γ onto all $c \in \ell_2$ and get $\Gamma \in \mathcal{B}(\ell_2, \mathbb{H})$, $\|\Gamma\| \approx 0$.

3.2 Definition. Sequences (φ_i), $(\tilde{\varphi}_i)$ in $\mathbb{H}$ are said to be (uniformly) infinitely close iff the sequence $(\varphi_i - \tilde{\varphi}_i)$ is uniformly infinitesimal.

3.3. Proposition. *Let (φ_i) be a Riesz basis in $\mathbb{H}$. Suppose that for an equivalency U of (φ_i) and some orthonormal basis (e_i)*

$$\|U\| \ll \infty \tag{3.3}$$

holds. Then each infinitely close to (φ_i) sequence $(\tilde{\varphi}_i)$ is a Riesz basis.

Proof. Denote by (ψ_i) the basis associated whith (φ_i) and set

$$\forall x \in \mathbb{H} \quad \Delta x = ((x|\psi_i))_{i\in\mathbb{N}}. \tag{3.4}$$

Since $\psi_i = U^* e_i$ $\|\Delta x\|^2 = \sum_{i\in\mathbb{N}} |(Ux|e_i)|^2 = \|Ux\|^2$. In particular $\|\Delta\| = \|U\| \ll \infty$. Let Γ be operator (3.2) for $\gamma_i := \varphi_i - \tilde{\varphi}_i$. It is easy to check that

$$\tilde{\varphi}_i = (I + \Gamma\Delta)\varphi_i. \tag{3.5}$$

But $\|\Gamma\Delta\| \approx 0$. Therefore $\ker(I + \Gamma\Delta) = \{0\}$ and $(I + \Gamma\Delta)^{-1} \in \mathcal{B}(\mathbb{H})$. Since $\tilde{\varphi}_i = (I + \Gamma\Delta)U^{-1}e_i$, $(I + \Gamma\Delta)U^{-1}$ is an equivalency of (e_i) and $(\tilde{\varphi}_i)$.

3.4. Theorem. *A sequence $(\tilde{\varphi}_i)$ which is infinitely close to nst-Riesz basis (φ_i) is an nst-Riesz basis.*

Proof. Let U be the uniformly nearstandard equivalency of (φ_i) and a standard orthonormal basis (e_i) (see 2.1 (ii)). Since $U\varphi_i = e_i$, by (3.5) we have $U(I+\Gamma\Delta)^{-1}\tilde{\varphi}_i = e_i$ (notation as above). Thus $U(I + \Gamma\Delta)^{-1}$ is an equivalency of $(\tilde{\varphi}_i)$ and (e_i). But $\|U(I + \Gamma\Delta) - {}^\circ U\| \approx 0$ (because $\|\Gamma\Delta\| \approx 0$). Therefore $U(I + \Gamma\Delta)^{-1}$ is uniformly nearstandard.

Remark. In condition of theorem 3.4, (φ_i) and $(\tilde{\varphi}_i)$ have common shadow. This follows from proposition 1.1, because ${}^\circ(U(I + \Gamma\Delta)) = {}^\circ U$.

Example.(Contribution to some Paley–Wiener theorem; see [6], [7]). Consider the functions $\varphi_n(t) = (2\pi)^{-1/2}\exp(i\lambda_n t)$ as elements of the standard Hilbert spase $\mathbb{H} = L_2(-\pi, \pi)$. Suppose that $\max_{n\in\mathbb{Z}} |\lambda_n - n| \ll \pi^{-1}\ln 2$. Denote by $(\mathring{\lambda}_n)_n \in \mathbb{Z}$ the (unique) standard sequence of complex numbers such that $\mathring{\lambda}_n \approx \lambda_n$ for standard $n \in \mathbb{N}$. Assume that $\sum_{i\in\mathbb{N}} |\lambda_n - \mathring{\lambda}_n| \approx 0$. Then (φ_n) is an nst-Riesz basis of $L_2(-\pi, \pi)$ with the shadow which consists of $(2\pi)^{-1/2}\exp(i\mathring{\lambda}_n t), n \in \mathbb{Z}$. The proof is almost the same as in [7].

4.Diagonal operators. Note that each operator $A \in \mathcal{B}(\mathbb{H})$ such that $\|A\| \ll \infty$, has a shadow ${}^\circ A$ which is defined as a standard element of $\mathcal{B}(\mathbb{H})$, such that $({}^\circ Ax|y) \approx (Ax|y)$ for arbitrary standard $x, y \in \mathbb{H}$. If for any standard $x \in \mathbb{H}$ $\|(A - {}^\circ A)x\| \approx 0$ then A is said to be *nearstandard*, and if $\|A - {}^\circ A\| \approx 0$ then A is said to be *uniformly nearstandard*. Let (φ_i), $(\tilde{\varphi}_i)$ be biorthogonal bases in $\mathbb{H}$. Then to $A \in \mathcal{B}(\mathbb{H})$ there corresponds the matrix with elements $a_{i,j} = (A\psi_j|\varphi_i)_{j\in\mathbb{N}}$. Suppose that the basis (φ_i) is nearstandard (therefore so is (ψ_i)), and $\|A\| \ll \infty$. It is easy to check that $(\mathring{a}_{i,j})$, defined as a standard matrix such that for standard $i, j \in \mathbb{N}$ $\mathring{a}_{i,j} = {}^\circ(a_{i,j})$, is the matrix of the shadow ${}^\circ A$ with respect to the shadows of bases (φ_i), (ψ_i), i.e. $\mathring{a}_{i,j} = ({}^\circ A\mathring{\psi}_{i,j}|\mathring{\varphi}_i)$.

Somewhat more concrete information concerns operators of the form

$$\forall x \in \mathbb{H} \quad Ax = \sum_{i\in\mathbb{N}} \lambda_i(x|\psi_i)\varphi_i. \tag{4.1}$$

4.1. Theorem. *Let (the eigenvectors of A)* φ_i *forms an nst-Riesz basis of* $\mathbb{H}$*. Then A is (strongly) nearstandard, iff* $\forall i \in \mathbb{N} \quad |\lambda_i| \ll \infty$*. Denote by* $(\overset{\circ}{\lambda}_i)$ *a standard sequence in* $\mathbb{C}$ *such that* $\overset{\circ}{\lambda}_i \approx \lambda_i$ *for standard* $i \in \mathbb{N}$*. Then A is uniformly nearstandard iff* $\forall i \in \mathbb{N}$ $\overset{\circ}{\lambda}_i \approx \lambda_i$*.*

Proof is not difficult, it is based only on theorem 2.3 and Robinson lemma.

4.2 Warning. Let p be some bijection $\mathbb{N} \to \mathbb{N}$. For operator (4.1) define $(pA)x := \sum_{i\in\mathbb{N}} \lambda_{p(i)}(x|\psi_i)\varphi_i$. Then A and pA are *similar*. Suppose $\forall i \in \mathbb{N} \quad |\lambda_i| \ll \infty$ and (φ_i) is an nst-Riesz basis. Assume that the bijection p is standard. Then A and pA a both (strongly) nearstandard and so are their shadows $^\circ A$ and $^\circ(pA)$. But in general this is not true.

Remark. Recall that an operator $A \in \mathcal{B}(\mathbb{H})$ is said to be S-compact iff $\|x\| \ll \infty$ implies that Ax is nearstandard. Suppose that (φ_i) is an nst-Riesz basis. Then operator (4.1) is S-compact, iff $\forall i \in \mathbb{N} \quad |\lambda_i| \ll \infty$ and $\lambda_i \approx 0$ for nonstandard $i \in \mathbb{N}$. If this holds, then $^\circ A$ is a standard *compact* operator.

5. Unbounded operators. In order to define the shadow $^\circ A$ whenever $A \in \mathcal{B}(\mathbb{H})$ but $\|A\| \approx \infty$ use the concept of *graph-nearstandardness* [9]. Denote by $\mathrm{dom}_{nst}A$ the set of nearstandard $x \in \mathbb{H}$ for which Ax is nearstandard. Suppose that for each infinitesimal $u \in \mathrm{dom}_{nst}A$ the vector Au is infitesimal. Then the shadow of the graph of A is the graph of some standard map, which by definition is the shadow $^\circ A$ of A. This $^\circ A$ is optionally an element of $\mathcal{B}(\mathbb{H})$, but it is a *standard closed* operator. Thus $A \in \mathcal{B}(\mathbb{H})$ is graph-nearstandard iff $^\circ(\mathrm{graph}A) = \mathrm{graph}(^\circ A)$.

5.1. Theorem. *Let* (φ_i) *be a nst-Riesz basis and* $|\lambda| \ll \infty$ *for any standard* $i \in \mathbb{N}$*. Then the operator (4.1) is graph-nearstandard. Its shadow is the standard closed densely defined operator* $^\circ A$ *such that* $\mathrm{dom}(^\circ A)$ *is the set of* $x \in \mathbb{H}$ *for which the series* $\sum_{i\in\mathbb{N}} |\overset{\circ}{\lambda}_i(x|\overset{\circ}{\psi}_i)|^2$ *converges, and for* $x \in \mathrm{dom}{}^\circ A \quad (^\circ A)x = \sum_{i\in\mathbb{N}} \overset{\circ}{\lambda}_i(x|\overset{\circ}{\psi}_i)\overset{\circ}{\varphi}_i$*.*

Proof. Let $u \approx 0$ and $u \in \mathrm{dom}_{nst}A$. By Robinson lemma there exists an infinite $k \in \mathbb{N}$ such that $\sum_{i\leqslant k} |\lambda_i(u|\psi_i)|^2 \approx 0$. By theorem 2.3 $\sum_{j>k} |\lambda_j(u|\psi_j)|^2 \approx 0$. By the Parseval inequality for nst-Riesz basis $Au \approx 0$. Thus A is graph-nearstandard. Let $x \in \mathbb{H}$ be standard and the series $\sum_{i\in\mathbb{N}} |\overset{\circ}{\lambda}_i(x|\overset{\circ}{\psi}_i)|^2$ converges. Set $y := \sum_{i\in\mathbb{N}} \overset{\circ}{\lambda}_i(x|\overset{\circ}{\psi}_i)\overset{\circ}{\varphi}_i$, this y is standard (as a sum of a standard convergent series). Find infinite $k \in \mathbb{N}$ for which $\sum_{i\leqslant k} \|(x|\overset{\circ}{\psi}_i)\overset{\circ}{\varphi}_i - (x|\psi_i)\varphi_i\| \approx 0$ and $\sum_{i\leqslant k} \|\overset{\circ}{\lambda}_i(x|\overset{\circ}{\psi}_i)\overset{\circ}{\varphi}_i - \lambda_i(x|\psi_i)\varphi_i\| \approx 0$. For this k define $x_1 := \sum_{i\leqslant k}(x|\psi_i)\varphi_i$. Then $x_1 \approx x$, in particular x_1 is nearstandard. It is easy to check that $Ax_1 \approx y$. This means that $(x,y) \in {}^\circ(\mathrm{graph}A)$, i.e. $(^\circ A)x = y$. For the transfer principle the part ($\Longrightarrow$) is proved. Conversely, suppose that $x \approx x_1$, where $x_1 \in \mathrm{dom}_{nst}A$. Then $(^\circ A)x = {}^\circ(Ax_1) = {}^\circ(\sum_{i\in\mathbb{N}} \lambda_i(x|\psi_i)\varphi_i)$. Since $^\circ(\lambda_i(x|\psi_i)) = \overset{\circ}{\lambda}_i(x|\overset{\circ}{\psi}_i)$, for standard $i \in \mathbb{N}$ by *Proposition 1.3* $^\circ(Ax_1) = \sum_{i\in\mathbb{N}} \overset{\circ}{\lambda}_i(x|\overset{\circ}{\psi}_i)\overset{\circ}{\varphi}_i$. In particular we see that $\sum_{i\in\mathbb{N}} |\overset{\circ}{\lambda}_i(x|\overset{\circ}{\psi}_i)|^2$ converges.

5.2 Example. Denote by T the interval $[0, 2\pi[$ considered as an additive group with the addition $\mathrm{mod}(2\pi)$. Let $\mathbb{H} = L_2(T)$ with the standard Lebesgue measure. For an infinitesimal $h > 0$ define

$$\forall x \in \mathbb{H} \quad Ax(t) = \frac{1}{h^2}[x(t+2h) - 2x(t+h) + x(t)],\ t \in T. \tag{5.1}$$

Then $A \in \mathcal{B}(\mathbb{H})$ but $||A|| = \frac{1}{h^2} \approx +\infty$. Rewrite (5.1) as $A = \frac{1}{h^2}(S-I)^2$, where S is the shift $Sx(t) = x(t+h)$. The eigenvalues and eigenfunctions of S are e^{int} and $t \longmapsto e^{int}$, $n \in \mathbb{Z}$. Therefore the eigenvalues of A are $\lambda_n = \frac{1}{h^2}(e^{inh} - I)^2$ with the same eigenfunctions. We see that A is unitary equivalent to the operator $\widehat{A}$ in $\widehat{H} = \ell_2(\mathbb{Z})$ of multiplication by λ_n. Since $\lambda_n \approx -n^2$ for standard $n \in \mathbb{Z}$, $|\lambda_n| \ll \infty$ for such n. Therefore $\widehat{A}$ (and A) is graph-nearstandard. Its shadow $^\circ\widehat{A}$ is the multiplication by $\overset{\circ}{\lambda}_n = -n^2$ in $\widehat{H}$ and $\mathrm{dom}(^\circ\widehat{A}) = \{(c_n) \in \ell_2(\mathbb{Z}) : \ \sum |n^2 c_n|^2 \text{ converges }\}$.

Hence $\mathrm{dom}(^\circ A) = \{x \in \mathbb{H} : \ \ x', x'' \in \mathbb{H}\}$, $^\circ Ax = x''$. Observe that λ_n are placed on *infinitely large cardioid* with the equation $\rho = \frac{2}{h^2}(1 + cos\varphi)$ in polar coordinates. The shadow of this curve is the union of the right and left *shores* of negative real semiaxis $\mathbb{R}_-$ on which the eigenvalues $\overset{\circ}{\lambda}_n = -n^2$ of $^\circ A$ are placed.

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1. *Krein M.* On the Bary bases of the Hilbert spases// Uspekhi mat. nauk. – 1957. – Vol. 12. – N 3. – C. 333-341 (in Russian).
2. *Gohberg I.C., Krein M.G.* Introduction to the theory of linear non-self-adjoint operators. – M., 1965 (in Russian).
3. *Nelson E.* Internal set theory: a new approach to nonstandard analysis// Bull. Amer. Math. Soc. – 1977. – Vol. 83. – P.1165-1198.
4. *Davis M.* Applied nonstandard analysis. – M., 1980 (in Russian).
5. *Diener F., Reeb G.* Analyse Nonstandard. – Hermann, Paris, 1989.
6. *Paley R.E.A., Wiener N.* Fourier transforms in complex domain. – New York, 1934.
7. *Duffin R.J., Eachus J.J.* Some notes on an expansion theorem of Paley and Wiener// Bul. Amer. Math. Soc. – 1942. – Vol. 48. – P.850-855.
8. *Bary N.K.* Biorthogonal systems and bases in Hilbert spase// Uch. Zap. Moscow State University. – 1951. – Vol. 4. – N 148. – C.69-107.
9. *Lyantse V.* Nearstandardness on finite set. – Dissert. Math. CCCLXIX, Warszawa, 1997.
10. *Lyantse V., Kudryk T.* Introduction to nonstandard analysis.– Mathematical Studies, Monograph Series, 1998.

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NST-БАЗИ РІСА У ГІЛЬБЕРТОВОМУ ПРОСТОРІ

Розглянуто деякі нестандартні аспекти теорії баз, введено поняття nst-бази Ріса. На підставі праці Нельсона про внутрішню теорію множин наведено умови колостандартності вектора в базі Ріса, розглянуто бази, нескінченно близькі до nst-баз Ріса, означена тінь необмеженого оператора.

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