ВІСНИК ЛЬВІВ. УН-ТУ *VISNYK LVIV UNIV.*
Серія мех.-мат. 1999. Вип. 55. С. 11–16 *Ser. mech.-math. 1999. No. 55. P. 11–16*

UDC 539.3

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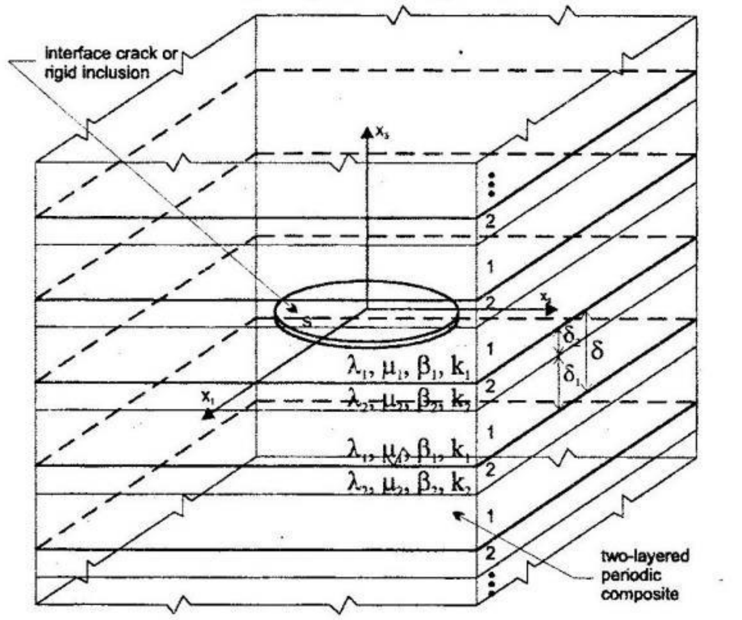
APPLICATIONS OF THE THERMOELASTICITY WITH MICROLOCAL PARAMETERS IN CERTAIN THREE-DIMENSIONAL INTERFACE CRACK AND RIGID INCLUSION PROBLEMS IN COMPOSITES

1. The homogenized model of a microperiodic two-layered space. Let us consider a microperiodic laminated medium consisting of alternating layers of two homogeneous, isotropic and linear-elastic materials, characterized by the Lamй constants λ_l, μ_l, the thermal conductivities k_l, the coefficients of the volume expansion $\beta_l/(\lambda_l+\frac{2}{3}\mu_l)$ and the thicknesses δ_l; here $l=1$ and $l=2$ refer to the sublayers denoted by 1 and 2, forming a thin repeated fundamental layer with the thickness $\delta=\delta_1+\delta_2$ (see Figure).



Two-layered periodic space with an interface plane defect.

Referring to the Cartesian coordinate system (x_1, x_2, x_3) with the x_3 – axis normal to the layering, denote at point $\mathbf{x}=(x_1, x_2, x_3)$ the displacement vector by $\mathbf{u}=(u_1, u_2, u_3)$, the stresses by σ_{11}, σ_{12}, σ_{21}, σ_{22}, σ_{31}, σ_{32}, σ_{33} and the temperature (strictly speaking, the deviation of temperature from a reference state) by θ.

We use the specific method of homogenization called the linear thermoelasticity with microlocal parameters [8, 11] leading to the macro-homogeneous model of the treated body with the following approximations

$$u_i \approx w_i,\ u_{i,\alpha} \approx w_{i,\alpha};\quad \theta \approx \vartheta,\ \theta_{,\alpha} \approx \vartheta_{,\alpha};$$
$$u_{i,3} \approx w_{i,3} + h^{(l)} d_i;\quad \theta_{,3} \approx \vartheta_{,3} + h^{(l)}\Gamma. \tag{1}$$

Here the indices i, j run over 1, 2, 3 and are related to the Cartesian coordinates while the indices α, β run over 1, 2. Subscripts preceded by a comma indicate partial differentiation with respect to the corresponding coordinates. Moreover, w_i, ϑ and d_i, Γ are the unknown functions interpreted as the macrodisplacements, macrotemperature and microlocal parameters, respectively, and

$$h^{(l)} = \begin{cases} 1 & \text{if } l = 1 \ \left(\mathbf{x} \in 1^{\text{st}} \text{ layer}\right), \\ -\eta/(1-\eta) & \text{if } l = 2 \ \left(\mathbf{x} \in 2^{\text{nd}} \text{ layer}\right), \end{cases}$$
$$\eta = \delta_1/\delta \tag{2}$$

is the derivative of the assumed δ – periodic, sectionally linear shape function.

The asymptotic approach to the macro-modelling of this laminated body leads to the governing relations of certain macro-homogeneous medium (homogenized model), given in terms of the macrotemperature ϑ and the macrodisplacements w_i (after eliminating all microlocal parameters and in the absence of heat sources and body forces) [3]:

$$\vartheta_{,\alpha\alpha} + k_0^{-2}\,\vartheta_{,33} = 0, \tag{3a}$$
$$\tfrac{1}{2}(c_{11}+c_{12})\,w_{\beta,\beta\alpha} + \tfrac{1}{2}(c_{11}-c_{12})\,w_{\alpha,\beta\beta} + c_{44}\,w_{\alpha,33} + (c_{13}+c_{44})\,w_{3,3\alpha} = K_1\,\vartheta_{,\alpha},$$
$$(c_{11}+c_{44})\,w_{\alpha,\alpha 3} + c_{44}\,w_{3,\alpha\alpha} + c_{33}\,w_{3,33} = K_3\,\vartheta_{,3}. \tag{3b}$$

The components of stress tensor $\sigma_{ij}^{(l)}$ and heat density vector $\mathbf{q}^{(l)}$ are expressed as follows

$$\sigma_{\alpha 3} = c_{44}\left(w_{\alpha,3} + w_{3,\alpha}\right),\quad \sigma_{33} = c_{13}\,w_{\alpha,\alpha} + c_{33}\,w_{3,3} - K_3\,\vartheta,$$
$$\sigma_{12}^{(l)} = \mu_l\left(w_{1,2} + w_{2,1}\right),$$
$$\sigma_{11}^{(l)} = d_{11}^{(l)}\,w_{1,1} + d_{12}^{(l)}\,w_{2,2} + d_{13}^{(l)}\,w_{3,3} - K_2^{(l)}\,\vartheta,$$
$$\sigma_{22}^{(l)} = d_{12}^{(l)}\,w_{1,1} + d_{11}^{(l)}\,w_{2,2} + d_{13}^{(l)}\,w_{3,3} - K_2^{(l)}\,\vartheta,$$
$$q_\alpha^{(l)} = -k_l\,\vartheta_{,\alpha},\quad q_3 = -K\,\vartheta_{,3}. \tag{4}$$

The positive coefficients appearing in Eqs. (3, 4), describing the material and geometric properties of the composite constituents, are given in the Appendix. Note that the condition of ideal contact between the layers is satisfied. By assuming $\mu_1 = \mu_2 = \mu$, $\lambda_1 = \lambda_2 = \lambda$ and $\beta_1 = \beta_2 = \beta$, $k_1 = k_2 = k$ we get $c_{11} = c_{33} = \lambda + 2\mu$, $c_{12} = c_{13} = \lambda$, $c_{44} = \mu$, $K_1 = K_3 = \beta$, $K = k$, $k_0 = 1$, passing directly to the well-known equations of uncoupled thermoelasticity for a homogeneous isotropic body [9].

According to the results given in [3], the general solution of the governing equations (3) is dependent on the material constants of the sublayers and in the general case $\mu_1 \neq \mu_2$, $t_\alpha \neq k_0$ (the other cases are detailed in [3, 4]; all constants appearing are given in the Appendix) may be expressed in terms of three harmonic potentials $\phi_i(x_1, x_2, z_i)$, $z_i = t_i x_3$ and the temperature harmonic potential $\omega(x_1, x_2, z_0)$, $z_0 = k_0 x_3$ related to the solution of (3a) as follows

$$w_1 = (\phi_1 + \phi_2 + c_1\,\omega)_{,1} - \phi_{3,2}, \quad w_2 = (\phi_1 + \phi_2 + c_1\,\omega)_{,2} + \phi_{3,1},$$

$$w_3 = m_\alpha\, t_\alpha \frac{\partial \phi_\alpha}{\partial z_\alpha} - c_2\, k_0 \frac{\partial \omega}{\partial z_0}. \tag{5}$$

From the constitutive relations (4) we obtain

$$\sigma_{31} = c_{44}\left[(1+m_\alpha)\,t_\alpha \frac{\partial \phi_\alpha}{\partial z_\alpha} + (c_1 - c_2)\,k_0 \frac{\partial \omega}{\partial z_0}\right]_{,1} - t_3 \frac{\partial^2 \phi_3}{\partial z_3 \partial x_2},$$

$$\sigma_{32} = c_{44}\left[(1+m_\alpha)\,t_\alpha \frac{\partial \phi_\alpha}{\partial z_\alpha} + (c_1 - c_2)\,k_0 \frac{\partial \omega}{\partial z_0}\right]_{,2} + t_3 \frac{\partial^2 \phi_3}{\partial z_3 \partial x_1},$$

$$\sigma_{33} = c_{44}\left[(1+m_\alpha)\frac{\partial \phi_\alpha}{\partial z_\alpha} + \alpha_0 \frac{\partial^2 \omega}{\partial z_0^2}\right]. \tag{6}$$

For the purpose of further discussion the remaining stresses $\sigma_{\alpha\beta}^{(l)}$ are not of immediate interest.

2. Interface problem formulation and solution. We are interested in the problem of determining the temperature, heat flux, stress and displacement fields in a bimaterial periodically layered space weakened by a crack (denoted by C) or a rigid sheet-like inclusion (denoted by I) occupying an area S (with a smooth boundary) of the x_1x_2 – plane being one of the interface of materials (see Figure).

Owing to a complicated geometry of the body and complex boundary conditions, the closed solution of the problem under study cannot be obtained. Thus we apply the homogenized model presented in Section 1 to seek an approximate solution and within this model we are faced with the boundary-value problem: find the scalar functions ϑ and w_i suitable smooth on $R^3 - S$ such that Eqs (3) hold and satisfy on S the following global conditions – stress -free faces for crack C or displacement-free faces for inclusion I:

$$\sigma_{31} = \sigma_{32} = \sigma_{33} = 0 \quad \text{for } C, \quad w_1 = w_2 = w_3 = 0 \quad \text{for I}. \tag{7}$$

Moreover, certain conditions resulting from a given external loading (thermal and mechanical) have to be specified.

The procedure for obtaining the solution follows along the same line of reasoning as that used in classical theory of thermal stresses applied to crack (inclusion) problem [1, 6, 7]. The steady-state temperature field is first seeking and a knowledge of potential ω is required to determine the induced thermal stresses by using the displacement representation given by (5). Upon utilizing the appropriate boundary conditions on the region S and using the superposition principle, we focus attention at the non-trivial perturbed problem solution to which tends to zero at infinity and satisfies the necessary boundary condition resulting from the prescribed temperature $T^{(0)}$ (or temperature gradient $q^{(0)}$) and the known stresses $\sigma_{3i}^{(0)}$ and displacements $w_i^{(0)}$, generated at the prospective crack (inclusion) faces in the multilayered space in the absence of the defect with the applied external loads. Proceeding as in the homogeneous case (cf. [7, 10]) it is convenient to resolve the general problem into the symmetric part (denoted by A) and the skew-symmetric problem (denoted by B) related to the half-space $x_3 \geq 0$ with the following thermal and mechanical boundary conditions

$$\mathrm{A}: \begin{cases} \vartheta \,|S = T^{(0)} \\ \vartheta_{,3} \,|\, (Z-S) = 0 \end{cases} \quad \text{or} \quad \begin{cases} \vartheta_{,3} \,|S = q^{(0)} \\ \vartheta_{,3} \,|\, (Z-S) = 0 \end{cases};$$

$$\text{for C:} \begin{cases} \sigma_{31} \,|Z = 0, \quad \sigma_{32} \,|Z = 0, \\ \sigma_{33} \,|S = -\sigma_{33}^{(0)}, \\ w_3 \,|(Z-S) = 0, \end{cases} \qquad \text{for I:} \begin{cases} w_1 \,|Z = 0, \quad w_2 \,|Z = 0, \\ w_3 \,|S = -w_3^{(0)}, \\ \sigma_{33} \,|(Z-S) = 0, \end{cases}$$

$$\mathrm{B}: \begin{cases} \vartheta_{,3} \,|\, S = q^{(0)}, \\ \vartheta \,|\, (Z-S) = 0, \end{cases}; \quad \text{for C:} \begin{cases} \sigma_{33} \,|Z = 0 \\ \sigma_{3\alpha} \,|S = -\sigma_{3\alpha}^{(0)}, \\ w_\alpha \,|(Z-S) = 0, \end{cases} \quad \text{for I:} \begin{cases} w_3 \,|Z = 0, \\ w_\alpha \,|S = -w_\alpha^{(0)}, \\ \sigma_{3\alpha} \,|(Z-S) = 0, \end{cases} \tag{8}$$

where Z denotes the entire x_1x_2 − plane.

The thermoelastic problem is now reducing to its mechanical counterpart by constructing the potential functions well suited to the above boundary conditions. Only the results for the symmetric problem A will be presented (a complete analysis is given in [3, 4]).

An appropriate displacement crack (rigid inclusion) representation in terms of a single harmonic function $f^{C}(f^{I})$, is obtained by taking in (5):

$$\phi_\alpha^{\mathrm{C}} = (-1)^\alpha \left[(1+m_\alpha)\,t_\alpha\right]^{-1} \left[f^{\mathrm{C}} + a_\alpha\, t_\alpha\, \omega\right],$$

$$\phi_\alpha^{\mathrm{I}} = (-1)^\alpha \left[f^{\mathrm{I}} + b_\alpha\, \omega\right], \qquad \phi_3^{\mathrm{C\ or\ I}} = 0. \tag{9}$$

The constants a_α, b_α can be chosen (see the Appendix) so as to reduce the crack (inclusion) problem to the classical mixed problems of potential theory (see [7]) for finding the functions $f^{C}(f^{I})$ as follows

$$f_{,33}^{\mathrm{C}}\Big|_{x_3=0^+} \overset{S}{=} -\frac{t_1 t_2}{c_{44} t_-}\left[\left(a_2 - a_1 + \alpha_0\right)\left(\vartheta\Big|_{x_3=0^+}\right) - \sigma_{33}^{(0)}\right], \qquad f_{,3}^{\mathrm{C}}\Big|_{x_3=0^+} \overset{Z-S}{=} 0;$$

$$f_{,3}^{\mathrm{I}}\Big|_{x_3=0^+} \overset{S}{=} -\frac{\left(m_2 t_2 b_2 - m_1 t_1 b_1 - c_2 k_0\right)\left(\omega_{,3}\Big|_{x_3=0^+}\right) + w_3^{(0)}}{m_2 t_2 - m_1 t_1}, \qquad f_{,33}^{\mathrm{I}}\Big|_{x_3=0^+} \overset{Z-S}{=} 0. \tag{10}$$

Explicit solutions of resulting potential problems are possible to obtain only for elliptical shape of surface of separation S. Similarly to the plane problems [2, 5], stress intensification in the close neighborhood of the crack (rigid inclusion) border is measured by the stress intensity factors. For instance, for a stress-free penny-shaped crack $S = \{(x_1,\ x_2,\ 0) : x_1^2 + x_2^2 = r^2 \le a^2\}$ the stress intensity factor in the symmetric system is given by (see [3])

$$k_1 = \lim_{r \to a^+} [2(r-a)]^{1/2} \sigma_{33}(r,0) = -2(a_2 - a_1 + \alpha_0) c_{44} \pi^{-1} a^{-1/2} \int_0^a r\, T^{(0)}(r) \Big/ \sqrt{a^2 - r^2}\, dr$$

$$\text{or} \quad k_0^{-1} (a_2 - a_1 + \alpha_0) c_{44}\, a^{-1/2} \int_0^a r\, q^{(0)}(r)\, dr. \qquad (11)$$

3. Concluding remarks. In this contribution, the homogenized model with microlocal parameters of two-layered periodic space is applied analyze three-dimensional problems of thermal stresses around interface crack or rigid sheet-like inclusion. Within the framework of this model, the thermoelastic problems are reduced to the corresponding ordinary problems dealing with mechanical loading in homogeneous isothermal elasticity. Formulation in terms of integral equations for an arbitrary shaped rigid inclusion is given in [4].

Appendix

1. Denoting by $B_l = \lambda_l + 2\mu_l \quad (l = 1,2)$, $\eta = \delta_1/\delta$, $\bar{B} = (1-\eta) B_1 + \eta B_2$, $\bar{K} = (1-\eta) k_1 + \eta k_2$, the positive coefficients in Eqs. (3a), (3b), (4) are given by the following formulae:

$$k_0 = \left[\left(\eta k_1 \bar{K} + (1-\eta) k_2 \bar{K}\right) / k_1 k_2\right]^{1/2}, \qquad c_{33} = B_1 B_2 / \bar{B},$$

$$c_{11} = c_{33} + \frac{4\eta(1-\eta)(\mu_1 - \mu_2)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)}{\bar{B}},$$

$$c_{12} = \frac{\lambda_1 \lambda_2 + 2[\eta \mu_2 + (1-\eta)\mu_1][\eta \lambda_1 + (1-\eta)\lambda_2]}{\bar{B}},$$

$$c_{13} = \frac{(1-\eta)\lambda_2 B_1 + \eta \lambda_1 B_2}{\bar{B}}, \qquad c_{44} = \frac{\mu_1 \mu_2}{(1-\eta)\mu_1 + \eta \mu_2},$$

$$K_1 = [\eta \beta_1 \lambda_2 + (1-\eta)\beta_2 \lambda_1]/\bar{B} + 2[(1-\eta)\mu_1 + \eta \mu_2][\eta \beta_1 + (1-\eta)\beta_2]/\bar{B},$$

$$K_3 = [(1-\eta)\beta_2 B_1 + \eta \beta_1 B_2]/\bar{B}, \quad K = k_1 k_2 / \bar{K}, \quad K_2^{(l)} = (2\mu_l \beta_l + \lambda_l K_3)/B_l,$$

$$d_{11}^{(l)} = [4\mu_l(\lambda_l + \mu_l) + \lambda_l c_{13}]/B_l, \quad d_{12}^{(l)} = (2\mu_l \lambda_l + \lambda_l c_{13})/B_l, \quad d_{13}^{(l)} = \lambda_l c_{33}/B_l.$$

2. The constants in Eqs. (6), (9), (10), (11) are defined as

$$t_3 = [(1-\eta)\mu_1 + \eta \mu_2]^{1/2} (c_{44})^{-1/2}; \quad m_\alpha = \left(c_{11} + t_\alpha^{-2} - c_{44}\right) / (c_{13} + c_{44});$$

$$t_1 = (t_+ - t_-)/2, \quad t_2 = (t_+ + t_-)/2$$

provided $t_\pm = \left[(A_+ \pm 2c_{44}) A_\mp / c_{33} c_{44}\right]^{1/2}$, $A_\pm = (c_{11} c_{33})^{1/2} \pm c_{13}$,

$$c_1 = \frac{k_0^2 [(c_{13} + c_{44}) K_3 - c_{33} K_1] + c_{44} K_1}{c_{33} c_{44} \left(k_0^2 - t_1^2\right)\left(k_0^2 - t_2^2\right)},$$

$$c_2 = \frac{k_0 \left[(c_{13} + c_{44}) K_3 - c_{11} K_3 + k_0^2 c_{44} K_3\right]}{c_{33} c_{44} \left(k_0^2 - t_1^2\right)\left(k_0^2 - t_2^2\right)},$$

$$\alpha_0 = \left(-c_1 c_{13} - c_2 c_{33} k_0^2 + K_3\right)/c_{44};$$
$$a_1 = [k_0 (1 + m_1)(c_1 m_2 + c_2)]/t_1 (m_2 - m_1),$$
$$a_2 = [k_0 (1 + m_2)(c_1 m_1 + c_2)]/t_2 (m_2 - m_1);$$
$$b_1 = [c_1 (1 + m_2) - \alpha_0]/(m_2 - m_1),$$
$$b_2 = [c_1 (1 + m_1) - \alpha_0]/(m_2 - m_1).$$

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ЗАСТОСУВАННЯ ТЕРМОПРУЖНОСТІ З МІКРОЛОКАЛЬНИМИ ПАРАМЕТРАМИ ДО ДЕЯКИХ ЗАДАЧ ЩОДО ТРІЩИН І ЖОРСТКИХ ВКЛЮЧЕНЬ У КОМПОЗИТАХ

Подано розв'язки деяких стаціонарних просторових задач для щілин і включень у мікроперіодичних шаруватих композитах. Розв'язки будують з використанням апарату теорії термопружності з мікролокальними параметрами.

Стаття надійшла до редколегії 09.09.99