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A DISPERSIVE MODEL OF HONEYCOMB BASED COMPOSITES

The paper deals with investigations of overall dynamic behavior of the linear-elastic honeycomb based microstructured composite solids made of an isotropic homogeneous matrix reinforced by a hexagonal lattice of fibres or by a hexagonal skeleton of thin slender walls as shown in Fig. 1. Problems similar to this one were investigated in a series of papers. In most cases the material structures under consideration were described by means of certain equivalent homogeneous solid equations [1, 4, 5], which can be also obtained by the known asymptotic homogenization method, cf. [6, 9]. However, the modelling procedures used in the aforementioned papers lead to a nondispersive continuum equations and hence are not able to describe the effect of microstructure size on the overall dynamic solid behavior. Dispersive models of honeycomb-type structures were discussed in [2, 3, 8, 9] but the attention was restricted to the hexagonal beam–like systems like gridworks and latticed or perforated plates. So far, accordingly to the authors' knowledge, the dispersive models for overall response of honeycomb based composites have not been investigated. The tensorial notation is used; all small Greek characters run over 1,2 and are related to the plane orthogonal Cartesian coordinate system $0x_1x_2$; summation convention holds. Symbols t^A stand for Fig. 1 the unit vectors parallel to the A-th family of reinforcement elements where here and in the sequel $A = 1,\ 2,\ 3$; we shall also assume that $t^1 + t^2 + t^3 = 0$. The vector basis d^1, d^2 in Fig. 1 determines the periodic structure of the solid under consideration.

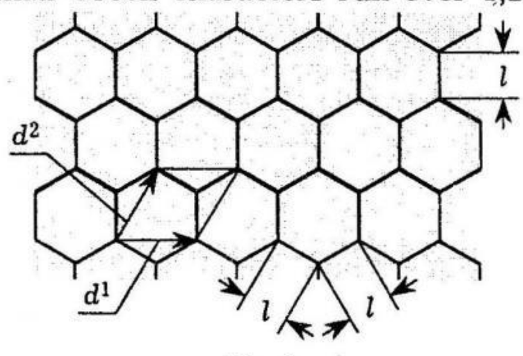

Fig. 1.

In order to formulate an averaged continuum model of honeycomb based composites under consideration we have to restrict ourselves to the cases in which only special kinds of deformations are investigated. First, the smallest wavelength L of a deformation pattern has to be sufficiently large compared to the microstructure length parameter l. Second, it is assumed that on these long-wave deformations there are superimposed locally-periodic deformations which within every repeated cell of a solid periodic structure can be approximated by the periodic ones, see [9], Chapter 6. In the first approximation which is applied in this contribution we shall assume that the locally-periodic superimposed deformations of all finite triangular elements of the triangulation lattice, shown in Fig. 2, can be treated as uniform. Let $u_\alpha(x, t)$ be a displacement vector field. Moreover, let

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$\delta(x)$ stand for the scalar periodic continuous function, which at the nodes of periodic triangulation lattice takes the values 0, $+0.5l$, $-0.5l$ indicated on the right hand side of Fig. 2 and is linear in every triangular element. From the kinematic assumptions formulated above it follows that

$$u_\alpha(\mathbf{x},t) = v_\alpha(\mathbf{x},t) + \delta(\mathbf{x})\, q_\alpha(\mathbf{x},t)\,, \tag{1}$$

where $v_\alpha(\mathbf{x},t), q_\alpha(\mathbf{x},t)$ are slowly varying vector functions, i.e., together with all spatial derivatives can be approximately treated as constant in every parallelogram spanned on vectors d^1, d^2 and hence constant in every fibre segment of the length l. That is why the distribution of displacement gradients in every repetive element will be approximated by [10]:

$$u_{\beta,\alpha}(\mathbf{x},t) \cong v_{\beta,\alpha}(\mathbf{x},t) + \delta_{,\alpha}(\mathbf{x})\, q_\beta(\mathbf{x},t)\,, \tag{2}$$

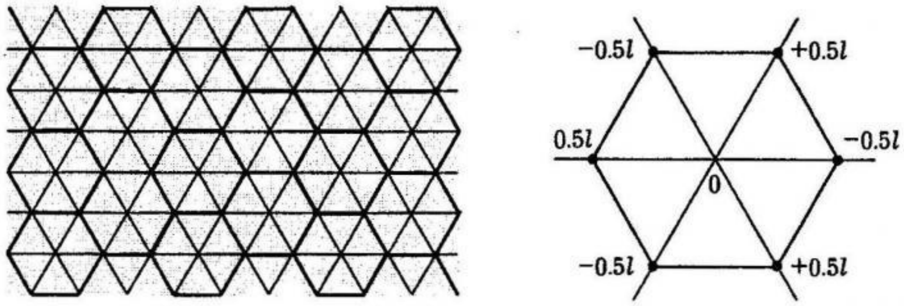

Fig. 2.

It has to be emphasized that the formulae (1), (2) represent only the first approximation of superimposed locally-periodic deformations being related to the triangulation lattice shown in Fig. 2; the possible higher approximations will be discussed separately.

Independently of the aforementioned kinematic assumptions, leading to the formulas (1), (2), we shall postulate that:

1° The mass distribution in composite solid under consideration can be approximated by a system of concentrated masses assigned to the nodes of triangulation lattice,

2° The thickness of reinforcement in Ox_1x_2-plane can be treated as negligibly small in the description of geometry of a solid. Roughly speaking, the fibres and walls of the honeycomb reinforcement lattice are infinitely thin in Ox_1x_2-plane.

We begin with the calculations of averaged values of strain and kinetic energy densities, both for the matrix and reinforcement. Let $a_{\alpha\beta\gamma\delta} = \lambda_m \delta_{\alpha\beta}\delta_{\gamma\delta} + \mu_m(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$ stand for the matrix elasticity tensor with λ_m, μ_m as Lame module either for the plane-strain or plane-stress problem. The matrix averaged strain energy density of σ_m by means of Eq. (2) will be given by

$$\sigma_m = \frac{1}{2}[\lambda_m \delta_{\alpha\beta}\delta_{\gamma\delta} + \mu_m(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})]\, v_{\alpha,\beta}\, v_{\gamma,\delta} + \frac{1}{2}(\lambda_m + 3\mu_m)\,\delta_{\alpha\beta} q_\alpha q_\beta\,. \tag{3}$$

Let ρ_m stand for the matrix mass density. The known expression for the matrix kinetic energy density, under the mass distribution assumption formulated above and by using formula (1), after simple calculations yields

$$\kappa_m = \frac{1}{2}\rho_m\left(\dot{v}_\alpha \dot{v}_\alpha + \frac{l^2}{6}\dot{q}_\alpha \dot{q}_\alpha\right) \tag{4}$$

as the averaged kinetic energy of the matrix.

Let v be a fractional concentration of reinforcement and λ_f, μ_f stand for Lame module of the reinforcement material. Introducing tensors

$$T^4_{\alpha\beta\gamma\delta} = \sum_{A=1}^{3} t^A_\alpha t^A_\beta t^A_\gamma t^A_\delta, \quad T^3_{\alpha\beta\gamma} = \sum_{A=1}^{3} t^A_\alpha t^A_\beta t^A_\gamma, \quad T^2_{\alpha\beta} = \sum_{A=1}^{3} t^A_\alpha t^A_\beta,$$

we obtain

$$\sigma_f = \frac{1}{2}v(\lambda_f + 2\mu_f)(T^4_{\alpha\beta\gamma\delta} v_{\alpha,\beta}\, v_{\gamma,\delta} + 2T^3_{\alpha\beta\gamma}\, v_{\alpha,\beta}\, q_\gamma + T^2_{\alpha\beta}\, q_\alpha\, q_\beta) \tag{5}$$

for the averaged strain energy density of the honeycomb reinforcement lattice.

Let ρ_f be the mass density of reinforcement material. Hence the averaged mass density of the reinforcement is equal to $v\rho_f$. Bearing in mind that the mass of reinforcement lattice has to be assigned exclusively to the nods of the lattice and using formula (1) we arrive to the expression

$$\kappa_f = \frac{1}{2}\nu\,\rho_f\left(\dot{v}_\alpha \dot{v}_\alpha + \frac{l^2}{4}\dot{q}_\alpha \dot{q}_\alpha\right) \tag{6}$$

for the averaged kinetic energy density of the honeycomb reinforcement lattice.

The crucial point of the theoretical considerations in this contribution is to show that it is possible to obtain an isotropic continuum model of honeycomb based composites. It can be proved that tensors $T^2_{\alpha\beta}$, $T^4_{\alpha\beta\gamma\delta}$ are isotropic and have the form

$$T^4_{\alpha\beta\gamma\delta} = \frac{3}{8}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}), \; T^2_{\alpha\beta} = \frac{3}{2}\delta_{\alpha\beta}.$$

At the same time the third order tensor $T^3_{\alpha\beta\gamma}$ in the two-dimensional space cannot be isotropic. Let us introduce the coordinate system Ox_1x_2 such that t_1 is the versor of Ox_1-axis. In this case $t^1 = (1, 0)$, $t^2 = (-1/2, \sqrt{3}/2)$, $t^3 = (-1/2, -\sqrt{3}/2)$. In the above coordinate system we obtain $T^3_{111} = 3/4$, $T^3_{112} = -3/4$, $T^3_{122} = 0$, $T^3_{222} = 0$; it has to be remembered that $T^3_{\alpha\beta\gamma}$ is symmetric with respect to all subscripts and hence the above equalities determine all components of this tensor. Let us define the second order tensor $W_{\alpha\beta}$ which is traceless, $W_{\alpha\alpha} = 0$, symmetric $W_{\alpha\beta} = W_{\beta\alpha}$, and its components in the coordinate system Ox_1x_2 introduced above are given by $W_{11} = -W_{22} = q_1$, $W_{12} = W_{21} = -q_2$. In this coordinate system

$$T^3_{\alpha\beta\gamma} v_{\alpha,\beta}\, q_\gamma = \frac{3}{4}\left(v_{1,1} W_{11} + v_{2,2} W_{22} + v_{1,2} W_{12} + v_{2,1} W_{21}\right) = \frac{3}{4} v_{\alpha,\beta} W_{\alpha\beta}.$$

Similarly, it can be shown that

$$\delta_{\alpha\delta}\, q_\alpha\, q_\beta = \frac{1}{2}\delta_{\alpha\gamma}\,\delta_{\beta\delta}\, W_{\alpha\beta}\, W_{\gamma\delta} = \frac{1}{2} W_{\alpha\beta}\, W_{\alpha\beta}\,.$$

It follows that the expressions (3)–(6) can be respectively replaced by the following ones

$$\begin{aligned}
\sigma_m &= \frac{1}{2}\Big[\lambda_m v_{\alpha,\alpha}\, v_{\beta,\beta} + \mu_m (v_{\alpha,\beta}\, v_{\alpha,\beta} + v_{\alpha,\beta}\, v_{\beta,\alpha}) + \frac{1}{4}(\lambda_m + 3\mu_m) W_{\alpha\beta} W_{\alpha\beta}\Big],\\
\kappa_m &= \frac{1}{2}\rho_m \left(\dot v_\alpha\, \dot v_\alpha + \frac{l^2}{6}\dot q_\alpha\, \dot q_\alpha\right),\\
\sigma_f &= \frac{3}{16}\kappa\left(\lambda_f + 2\mu_f)(v_{\alpha,\alpha}\, v_{\beta,\beta} + v_{\alpha,\beta}\, v_{\alpha,\beta} + v_{\alpha,\beta}\, v_{\beta,\alpha} + 4 v_{\alpha,\beta}\, W_{\alpha\beta} + 2 W_{\alpha\beta} W_{\alpha\beta}\right),\\
\kappa_f &= \frac{1}{2}\nu\,\rho_f\, \dot v_\alpha\, \dot v_\alpha + \frac{l^2}{16}\rho_f\, \dot W_{\alpha\beta}\, \dot W_{\alpha\beta}.
\end{aligned} \tag{7}$$

Under the extra denotations

$$\xi = \frac{3}{8}\nu(\lambda_f + 2\mu_f) \tag{8}$$

let us define the following material and inertial modulae

$$\lambda = \lambda_m + \xi,\ \mu = \mu_m + \xi,\ \alpha = \frac{1}{4}(\lambda_m + 3\mu_m) + \xi,\ \rho = \rho_m + \frac{2\sqrt{3}}{3}\rho_f,\ \eta = \frac{\sqrt{3}}{12}\rho_f. \tag{9}$$

Also define the linearized strain tensor by means of the well known expression

$$E_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}). \tag{10}$$

The principle of stationary action, based on formulae (7), after neglecting body forces, leads to the equations of motion

$$S_{\alpha\beta,\beta} - \rho\ddot U_\alpha = 0\,, \tag{11}$$

and what will be called dynamic evolution equation

$$l^2\eta\,\ddot W_{\alpha\beta} + H_{\alpha\beta} = 0\,, \tag{12}$$

together with the constitutive equations

$$\begin{aligned}
S_{\alpha\beta} &= \lambda\,\delta_{\alpha\beta} E_{\gamma\gamma} + 2\mu\, E_{\alpha\beta} + 2\xi\, W_{\alpha\beta},\\
H_{\alpha\beta} &= \alpha W_{\alpha\beta} + 2\xi D_{\alpha\beta},
\end{aligned} \tag{13}$$

where

$$D_{\alpha\beta} = E_{\alpha\beta} - \frac{1}{2}\delta_{\alpha\beta} E_{\gamma\gamma},\quad W_{\alpha\beta} = W_{\beta\alpha},\quad W_{\alpha\alpha} = 0. \tag{14}$$

Equations (10)–(14) with denotations (8), (9) represent the averaged model of honeycomb based composites under consideration. The main features of the model are:

1° the isotropic form of constitutive equations (13) which have constant coefficients,

2° the dispersive form of the dynamic evolution equations (12) which depends explicitly on the microstructure length parameter l.

The basis kinematic unknowns are: the averaged displacement vector field v_α and the tensor field $W_{\alpha\beta}$. It has to be emphasized that the the above model has a physical meaning only under assumption that $v_\alpha(\cdot, t)$, $W_{\alpha\beta}(\cdot, t)$ are slowly varying functions, i.e. their wavelengths are sufficiently large compared to the microstructure length parameter l.

Applications of the above averaged model to the dispersive analysis will be given in a separate paper, [11].

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ДИСПЕРСІЙНІ МОДЕЛІ СТІЛЬНИКОВИХ КОМПОЗИТІВ

Сформульовано усереднену математичну модель лінійно-пружних композитів, які мають однорідну ізотропну основу, зміцнену регулярною шестикутною сіткою волокон або регулярним шестикутним гнучким кістяком. Основний результат праці полягає у доведенні ізотропності моделі.

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