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LENGTH-SCALE VERSUS ASYMPTOTIC MODEL IN DYNAMICS OF THIN SUBSTRUCTURED CYLINDRICAL SHELLS

1. Introduction. The subject matter of this contribution is a thin linear-elastic cylindrical shell having a periodic structure (a periodically varying thickness and/or periodically varying elastic and inertial properties) along its midsurface. Structures like that are called the substructured shells, cf. [14].

Substructured shells and plates are usually described using homogenized models. These models from a formal point of view represent certain equivalent structures with constant or slowly varying stiffnesses and averaged mass densities.

The homogenized models of substructured shells and plates are usually derived by means of asymptotic methods. In the case of periodic plates, these asymptotic homogenization methods have been presented by Caillerie [2] (in this contribution two small parameters – thickness of a plate and the characteristic size of a periodicity cell – are used to investigate periodic plates), Kohn and Vogelius [6] (this paper deals with thin plates having a rapidly varying thickness), Lewiński [8] (in this contribution the homogenized stiffnesses are analysed) and others. The asymptotic approach to periodic shells has been proposed by Kalamkarov [5], Lutoborski [10], Lewiński and Telega [7]; the discussion of the above approach can be found in [14].

The formulation of mathematical models of shells by using the asymptotic expansions is rather complicated from the computational point of view. That is why the asymptotic procedures are restricted to the first approximation. Within this approximation we obtain models which neglect the effect of periodicity cell length dimensions on the global structure behaviour (the length-scale effect). This effect plays an important role mainly in the vibration and wave propagation analysis. To formulate the length-scale models in the framework of asymptotic homogenization we could find the higher-order terms of the asymptotic expansions, cf. [9]. Models of this kind have complicated analytical form and applied to the investigation of boundary-value problems often lead to the large number of boundary conditions which may be not well motivated from the physical viewpoint.

The alternative modelling procedure leading to the length-scale models of periodic structures which are plausible from the engineering standpoint and may constitute the basis for the numerical analysis, was proposed by Woźniak in [12] where the length-scale effect described by the extra unknown fields called internal variables is taken into account in the description of non-stationary processes. The results of [12] were generalized in [13] where this effect was taken into account also in the description of stationary processes. The Above approach has been applied to modelling and dynamic analysis of periodic plates in a series of papers, e.g. in [1] (this contribution deals with plates based on the Reissner-Hencky assumptions), in [3, 4] where Kirchhoff-type plates are studied and in [11] where wavy-plates are analysed.

A general modelling method based on the concept of internal variables and leading from 2D equations of thin shells with locally periodic structure to the averaged equations with slowly varying coefficients depending on the periodicity cell length dimensions has been proposed by Woźniak in [14]. In the present contribution this approach is applied to derive the length-scale model of thin linear-elastic cylindrical shells having the periodic structure along its midsurface. The length scales will be introduced to the global description of both inertial and constitutive properties of the shells under consideration.

The proposed length-scale model will be compared with a simplified one, in which the effect of the periodicity cell size on the overall shell behaviour is neglected.

We are to show that the introduced length-scale model will be plausible from the engineering standpoint being able to constitute the basis for numerical analysis of special problems.

2. Preliminaries. Denote by $\Omega \subset R^2$ a regular plane region of points $\Theta \equiv (\Theta^1, \Theta^2)$ and let E^3 be the physical space described by the Cartesian coordinate system $Ox^1x^2x^3$. Let us introduce the parametric representation of the undeformed smooth cylindrical shell midsurface $\mathcal{M}$ by means of: $\mathcal{M} := \{\mathbf{x} \equiv (x^1, x^2, x^3) \in E^3 : \mathbf{x} = \mathbf{x}\,(\Theta^1, \Theta^2), \Theta \in \Omega\ \}$, where $\mathbf{x}(\Theta^1, \Theta^2)$ is a position vector of an arbitrary point on $\mathcal{M}$.

Throughout the paper indices α, β, ... run over 1, 2 and are related to the midsurface parameters Θ^1, Θ^2; summation convention holds.

To every $\Theta \in \Omega$ we assign a covariant base vectors $\mathbf{a}_\alpha = \mathbf{x}_{,\alpha}$ and covariant midsurface first and second metric tensors denoted by $a_{\alpha\beta}$, $b_{\alpha\beta}$, respectively, which are given as follows:

$$a_{\alpha\beta} = \mathbf{a}_{,\alpha} \cdot \mathbf{a}_{,\beta}\,, \quad a_{\alpha\beta} = \text{const}, \qquad b_{\alpha\beta} = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta}\,, \quad b_{\alpha\beta} = \text{const}, \tag{2.1}$$

where $\mathbf{n}$ is a unit normal to $\mathcal{M}$.

Let $\delta(\Theta)$ stand for the shell thickness. We also define t as the time coordinate.

We shell denote by $\Delta := (0, l_1) \times (0, l_2)$ the region on Ω, where the length dimensions l_1, l_2 are assumed to be sufficiently large compared with the maximum shell thickness $\delta(\cdot)$ and sufficiently small as compared to the midsurface curvature radius R as well as the smallest characteristic length dimension L of the shell midsurface.

Let us assign to every $\Theta \in \Omega$ a periodicity cell $\Delta(\Theta)$ on $O\Theta^1\Theta^2$-plane by means of: $\Delta(\Theta) := \Theta + \Delta$, $\Theta \in \Omega_0$, $\Omega_0 := \{\Theta \in \Omega\ :\ \Delta(\Theta) \subset \Omega\}$, where the point $\Theta \in \Omega_0$ is a centre of cell $\Delta(\Theta)$ and the set Ω_0 is said to be the Δ-interior of Ω. Under given above assumptions for periods l_1, l_2, every shell element having midsurface $\mathbf{x}(\Delta(\Theta)) \subset \mathcal{M}$ constitutes a shallow shell.

A function $f(\Theta)$ defined on Ω_0 will be called Δ-periodic if it satisfies conditions of the form $f(\Theta^1, \Theta^2) = f(\Theta^1 \pm l_1, \Theta^2) = f(\Theta^1, \Theta^2 \pm l_2)$ in the whole domain of its definition. It is assumed that the shell thickness and its material properties are Δ-periodic functions of Θ.

A shell with Δ-periodic structure satisfying the aforementioned conditions will be referred to as *a shell with mesoperiodic structure* (a mesostructured shell, cf. [14].) The size of the mesostructure is described by *the mesostructure length parameter* l defined by $l := \sqrt{l_1^2 + l_2^2}$, where $\sup \delta(\cdot) \ll l \ll (R$ as well as $L)$.

Our considerations will be based on the simplified linear Kirchhoff-Love theory of thin elastic shells in which terms depending on the second metric tensor of $\mathcal{M}$ are neglected in the formulae for curvature changes.

Let $u_\alpha(\Theta, t)$, $w(\Theta, t)$ stand for the midsurface shell displacements in directions tangent and normal to $\mathcal{M}$, respectively. We denote by $\varepsilon_{\alpha\beta}(\Theta,t)$, $\kappa_{\alpha\beta}(\Theta,t)$ the membrane and curvature strain tensors and by $n^{\alpha\beta}(\Theta,t)$, $m^{\alpha\beta}(\Theta,t)$ the stress resultants and stress couples, respectively. The properties of shell are described 2D-shell stiffness tensors $D^{\alpha\beta\gamma\delta}(\Theta)$, $B^{\alpha\beta\gamma\delta}(\Theta)$ and let $\mu(\Theta)$ stand for a shell mass density per midsurface unit area. Let $f_\alpha(\Theta,t)$, $f(\Theta,t)$ be external force components per midsurface unit area, respectively tangent and normal to $\mathcal{M}$.

The equations of a shell theory under consideration consist of:

(i) the strain-displacement equations

$$\varepsilon_{\gamma\delta} = u_{(\gamma,\delta)} - b_{\gamma\delta}\, w, \qquad \kappa_{\gamma\delta} = -\, w_{,\gamma\delta}, \tag{2.2}$$

(ii) the stress-strain relations

$$n^{\alpha\beta} = D^{\alpha\beta\gamma\delta}\, \varepsilon_{\gamma\delta}, \qquad m^{\alpha\beta} = B^{\alpha\beta\gamma\delta}\, \kappa_{\gamma\delta}, \tag{2.3}$$

(iii) the equations of motion

$$n^{\alpha\beta}{}_{,\alpha} - \mu a^{\alpha\beta}\, \ddot{u}_\alpha + f^\beta = 0, \qquad m^{\alpha\beta}{}_{,\alpha\beta} + b_{\alpha\beta} n^{\alpha\beta} - \mu\, \ddot{w} + f = 0. \tag{2.4}$$

In the above equations the displacements $u_\alpha = u_\alpha(\Theta,t)$ and $w = w(\Theta,t)$, $\Theta \in \Omega$, are the basic unknowns.

For mesostructured shells, $\mu(\Theta)$, $D^{\alpha\beta\gamma\delta}(\Theta)$ and $B^{\alpha\beta\gamma\delta}(\Theta)$, $\Theta \in \Omega$, are highly oscillating Δ-periodic functions; that is why equations (2.2)–(2.4) cannot be directly applied to the numerical analysis of special problems. In order to derive from Eqs. (2.2)–(2.4) an averaged model of mesostructured cylindrical shells which has constant coefficients and describes the mesostructure size effect on the global dynamic shell behaviour the internal variable modelling approach to the thin shells with a locally periodic structure given by Woźniak in [14] will be applied. To make the analysis more clear, in the next section we shell outline the basic concepts of this approach, following the paper [14].

3. Basic concepts. Following [14] we outline below the basic concepts, which will be used in the course of modelling procedure:

(i) For an arbitrary integrable function $\varphi(\cdot) : \Omega \to \mathcal{R}$ we define *the averaging operation*:

$$\langle\varphi\rangle(\Theta) := \frac{1}{|\Delta(\mathbf{Q})|} \int_{\Delta(\mathbf{Q})} \varphi(\mathbf{Y}) \sqrt{a}\, d\Psi^1\, d\Psi^2, \quad \Theta \in \Omega_0, \quad \Psi \equiv (\Psi^1, \Psi^2) \in \Delta(\Theta), \tag{3.1}$$

where a is a determinant of the midsurface first metric tensor.

If $\varphi(\cdot)$ is a Δ-periodic function then $\langle\varphi(\cdot)\rangle(\Theta)$ is a constant and will be denoted by $\langle\varphi\rangle$.

(ii) A diferentiable function $F(\Theta,t)$ is called *slowly varying*, $F \in SV(\Delta)$, if for every integrable function $\varphi(\cdot)$ satisfies conditions of the form:

$$\langle\varphi F\rangle(\Theta,t) \cong \langle\varphi\rangle(\Theta)F(\Theta,t), \qquad \Theta \in \Omega_0, \tag{3.2}$$

and the similar conditions are also fulfilled by all derivatives of $F(\Theta,t)$. Roughly speaking, *the slowly varying function* can be treated as constant on an arbitrary periodicity cell Δ. The symbol «$\cong$» denotes a certain tolerance relation describing the accuracy of performed calculations.

(iii) By *a highly oscillating function*, $h \in HO(\Delta)$, we mean a differentiable Δ-periodic function $h(\cdot)$ such that for every $F \in SV(\Delta)$ conditions :

$$\langle\nabla(Fh)\rangle(\Theta,t) \cong \langle F\nabla h\rangle(\Theta,t)], \qquad \Theta \in \Omega_0, \tag{3.3}$$

are assumed to hold. Roughly speaking, in calculations of averages $\langle\cdot\rangle$, values of *a highly oscillating function* can be treated as negligibly small compared to the values of their derivatives.

For more detailed discussion of the internal variable modelling approach to periodic and locally-periodic structures the reader is referred to Woźniak [13, 14].

Using the governing 2D-equations of a shell theory (2.2)–(2.4) and auxiliary concepts outlined above as well as modelling hypotheses given in [14], the length-scale model of a mesostructured cylindrical shell will be derived in the subsequent section.

4. Governing equations. The idea of the internal variable approach is based on assumptions which restrict the class of unknown displacement fields $u_\alpha(\Theta,t)$, $w(\Theta,t)$ in (2.2)–(2.4) to a certain subclass, which includes arbitrary motions with wavelength of an order much larger then l on which there are superimposed disturbances of displacements caused by the highly oscillating character of the shell mesostructure.

Using the approach given in [14], we approximate the unknown midsurface shell displacements $u_\alpha(\Theta,t)$, $w(\Theta,t)$ in equations (2.2)–(2.4) by means of:

$$\begin{aligned} &u_\alpha(\Theta,t) \sim U_\alpha(\Theta,t) + h^A(\Theta)\, Q_\alpha^A(\Theta,t), \\ &w(\Theta,t) \sim W(\Theta,t) + g^A(\Theta)\, V^A(\Theta,t), \qquad A = 1, 2, ..., N, \qquad \Theta \in \Omega_0, \end{aligned} \tag{4.1}$$

(here and in the sequel summation convention over A holds), where:

(i) The unknown averaged displacement $U_\alpha(\Theta,t) = \langle\mu\rangle^{-1}\langle\mu\, u_\alpha\rangle(\Theta,t)$ and $W(\Theta,t) = \langle\mu\rangle^{-1}\langle\mu\, w\rangle(\Theta,t)$, respectively tangent and normal to $\mathcal{M}$, are slowly varying functions.

(ii) The unknown fields $Q_\alpha^A(\Theta,t)$, $V^A(\Theta,t)$, $A = 1, 2, ..., N$, describing from the quantitative point of view the displacement disturbances, are slowly varying functions called *internal variables*.

(iii) Fields $h^A(\Theta)$, $g^A(\Theta)$, $A = 1, 2, ..., N$, describing from the qualitative point of view the displacement disturbances, are assumed to be known in every problem; they are highly oscillating functions such that $\langle\mu\, h^A\rangle(\Theta) = \langle\mu\, g^A\rangle(\Theta) = 0$ and $h^A(\Theta) \in O(l)$, $h^A_{,\alpha}(\Theta) \in O(1)$, $g^A(\Theta) \in O(l^2)$, $g^A_{,\alpha}(\Theta) \in O(l)$, $g^A_{,\alpha\beta}(\Theta) \in O(1)$. They represent the expected shapes of disturbances and are obtained as approximate solutions to a special eigenvalue problem related to free vibrations on the cell $\Delta(\Theta)$ with periodic boundary conditions on $\partial\Delta(\Theta)$. This eigenvalue problem will be presented in the subsequent part of this section. Functions $h^A(\Theta)$, $g^A(\Theta)$ are referred to as the shape functions.

The finite sums $h^A(\Theta)\,Q^A_\alpha(\Theta,t)$, $g^A(\Theta)V^A(\Theta,t)$ in Eqs. (4.1) represent disturbances of displacements and are obtained as an approximate solution to a problem for vibrations $d_\alpha(\Psi,t) = u_\alpha - U_\alpha$ and $p(\Psi,t) = w - W$, $\Psi \equiv (\Psi^1,\Psi^2) \in \Delta(\Theta)$, formulated in the cell $\Delta(\Theta)$ under periodic boundary conditions on $\partial\Delta(\Theta)$. Denoting $\bar{f}^\beta = f^\beta + D^{\alpha\beta\gamma\delta}(U_{\gamma,\delta\alpha} - b_{\gamma\delta}\, W_{,\alpha}) - \mu\, a^{\alpha\beta}\,\ddot{U}_\alpha$, $\bar{f} = -B^{\alpha\beta\gamma\delta}\, W_{,\alpha\beta\gamma\delta} + b_{\alpha\beta}\, D^{\alpha\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) - \mu\,\ddot{W} + f$ and using (2.2)–(2.4), the aforementioned local problem states as follows:

$$D^{\alpha\beta\gamma\delta}(d_{\gamma,\delta\alpha} - b_{\gamma\delta}\, p_{,\alpha}) - \mu\, a^{\alpha\beta}\,\ddot{d}_\alpha + \bar{f}^\beta = 0,$$

$$-B^{\alpha\beta\gamma\delta} p_{,\alpha\beta\gamma\delta} + b_{\alpha\beta} D^{\alpha\beta\gamma\delta}(d_{\gamma,\delta} - b_{\gamma\delta} p) - \mu\,\ddot{p} + \bar{f} = 0, \tag{4.2}$$

and it is solved by applying the orthogonalization method known in structural dynamics. Using this method we have to formulate in $\Delta(\Theta)$ the eigenvalue problem for functions $h(\Psi)$, $g(\Psi)$, $\Psi \in \Delta(\Theta)$. Under the assumption that every shell element having midsurface $\mathbf{x}(\Delta(\Theta)) \subset \mathcal{M}$ constitutes a shallow shell, the aforementioned eigenvalue problem for the local problem (4.2) is formulated as follows:

$$D^{\alpha\beta\gamma\delta}\, h_{\gamma,\delta\alpha}(\Psi) + \mu\,\omega^2\, a^{\alpha\beta}\, h_\alpha(\Psi) = 0,$$

$$-B^{\alpha\beta\gamma\delta}\, g_{,\alpha\beta\gamma\delta}(\Psi) + \mu\,\omega^2 g(\Psi) = 0, \qquad \Psi \in \Delta(\Theta), \tag{4.3}$$

where $\langle\mu\, h_\alpha\rangle(\Theta) = \langle\mu\, g\rangle(\Theta) = 0$ and periodic boundary conditions for $h_\alpha(\cdot)$, $g(\cdot)$ on $\partial\Delta(\Theta)$ hold.

The eigenfunctions $h_\alpha(\Psi)$, $g(\Psi)$, $\Psi \in \Delta(\Theta)$, corresponding to the eingenvalue ω can be obtained, in most cases only in the approximate form represented by certain Δ-periodic functions $h^A_\alpha(\Psi)$, $g^A(\Psi)$, $\Psi \in \Delta(\Theta)$, $A = 1, 2, ..., N$.

The unknown functions $Q^A_\alpha(\Theta,t)$, $V^A(\Theta,t)$ in (4.1) are governed by the following orthogonality conditions:

$$\langle n^{\alpha\beta} h^A_{,\alpha} + (-\mu a^{\alpha\beta}\,\ddot{U}_\alpha - \mu a^{\alpha\beta}\, h^B\,\ddot{Q}^B_\alpha + f^\beta)\, h^A\rangle(\mathbf{Q}) = 0, \tag{4.4}$$

$$\langle m^{\alpha\beta} g^A_{,\alpha\beta} + (b_{\alpha\beta} n^{\alpha\beta} - \mu\ddot{W} - \mu g^B\,\ddot{V}^B + f)\, g^A\rangle(\mathbf{Q}) = 0, \quad A,B = 1,2,....,N, \quad \mathbf{Q} \in \Omega_0,$$

where

$$n^{\alpha\beta}(\Psi,t) = D^{\alpha\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}\, W) + D^{\alpha\beta\gamma\delta}(h^B_{,\delta}\, Q^B_\gamma - b_{\gamma\delta}\, g^B V^B),$$

$$m^{\alpha\beta}(\Psi,t) = -B^{\alpha\beta\gamma\delta}\, W_{,\gamma\delta} - B^{\alpha\beta\gamma\delta}\, g^{B}_{,\gamma\delta}\, V^{B}, \qquad \Psi \in \Delta(\Theta), \tag{4.5}$$

The unknown averaged displacement fields $U_\alpha(\Theta,t)$, $W(\Theta,t)$ in (4.1) are governed by the following averaging conditions implied by Eqs. (4.2):

$$\langle n^{\alpha\beta}_{,\alpha} - \mu\, a^{\alpha\beta}\ddot{U}_\alpha - \mu\, a^{\alpha\beta} h^{B}\, \ddot{Q}^{B}_{\alpha} + f^{\beta}\rangle(\mathbf{Q}) = 0,$$

$$\langle m^{\alpha\beta}_{,\alpha\beta} + b_{\alpha\beta} n^{\alpha\beta} - \mu \ddot{W} - \mu g^{B} \ddot{V}^{B} + f\rangle(\mathbf{Q}) = 0, \quad A,B = 1,2,....,N, \quad \mathbf{Q} \in \Omega_0, \tag{4.6}$$

where $n^{\alpha\beta}$, $m^{\alpha\beta}$ are given by (4.5).

The averaging conditions (4.6) have to be considered together with the ortogonality conditions (4.4).

Setting $N^{\alpha\beta} = \langle n^{\alpha\beta}\rangle$, $M^{\alpha\beta} = \langle m^{\alpha\beta}\rangle$, from Eqs. (4.4), (4.6) we obtain the length-scale model of mesostructured cylindrical shells. This model is represented by:

(i) *the constitutive equations*

$$\begin{aligned}
N^{\alpha\beta} &= \langle D^{\alpha\beta\gamma\delta}\rangle (U_{\gamma,\delta} - b_{\gamma\delta}\, W) + \langle D^{\alpha\beta\gamma\delta}\, h^{B}_{,\delta}\rangle Q^{B}_{\gamma} - \underline{\langle D^{\alpha\beta\gamma\delta} g^{B}\rangle} b_{\gamma\delta}\, V^{B},\\
M^{\alpha\beta} &= -\langle B^{\alpha\beta\gamma\delta}\rangle W_{\gamma,\delta} - \langle B^{\alpha\beta\gamma\delta}\, g^{B}_{,\gamma\delta}\rangle V^{B},\\
H^{A\beta} &= \langle D^{\alpha\beta\gamma\delta} h^{A}_{,\alpha}\rangle (U_{\gamma,\delta} - b_{\gamma\delta} W) + \langle D^{\alpha\beta\gamma\delta}\, h^{A}_{,\alpha} h^{B}_{,\delta}\rangle Q^{B}_{\gamma} - b_{\gamma\delta} \underline{\langle D^{\alpha\beta\gamma\delta} h^{A}_{,\alpha} g^{B}\rangle} V^{B},\\
G^{A} &= -b_{\alpha\beta} \underline{\langle D^{\alpha\beta\gamma\delta} g^{A}\rangle} (U_{\gamma,\delta} - b_{\gamma\delta} W) + \langle B^{\alpha\beta\gamma\delta} g^{A}_{,\gamma\delta}\rangle W_{,\alpha\beta} - b_{\alpha\beta} \underline{\langle D^{\alpha\beta\gamma\delta} g^{A} h^{B}_{,\delta}\rangle} Q^{B}_{\gamma} +\\
&\quad + (\langle B^{\alpha\beta\gamma\delta} g^{A}_{,\alpha\beta} g^{B}_{,\gamma\delta}\rangle + b_{\alpha\beta} \underline{\langle D^{\alpha\beta\gamma\delta} g^{A} g^{B}\rangle} b_{\gamma\delta})\, V^{B}, \quad A,B = 1,2,....,N,
\end{aligned} \tag{4.7}$$

(ii) *the system of three averaged partial differential equations of motion for averaged displacements* $U_\alpha(\Theta, t), W(\Theta, t)$

$$\begin{cases} N^{\alpha\beta}_{,\alpha} - \langle\mu\rangle a^{\alpha\beta}\ddot{U}_{\alpha} + \langle f^{\beta}\rangle = 0,\\ M^{\alpha\beta}_{,\alpha\beta} + b_{\alpha\beta} N^{\alpha\beta} - \langle\mu\rangle \ddot{W} + \langle f\rangle = 0, \end{cases} \tag{4.8}$$

(iii) *the system of* $3N$ *ordinary differential equations for the internal variables* $Q^{B}_{\alpha}(\Theta,t)$, $V^{B}(\Theta,t)$ *called the dynamic evolution equations*

$$\begin{cases} \underline{\langle\mu\, h^{A}\, h^{B}\rangle} a^{\gamma\beta}\, \ddot{Q}^{B}_{\gamma} + H^{A\beta} + \underline{\langle f^{\beta} h^{A}\rangle} = 0,\\ \underline{\langle\mu\, g^{A}\, g^{B}\rangle} \ddot{V}^{B} + G^{A} + \underline{\langle f g^{A}\rangle} = 0. \end{cases} \tag{4.9}$$

The underlined coefficients in Eqs. (4.7), (4.9) depend on the mesostructure length parameter l and hence describe the effect of the mesostructure size on the shell overall behaviour.

The internal variables do not enter the displacement boundary conditions and hence the number and form of these conditions are similar to those of the well known 2D-theory which is governed by equations (2.2)−(2.4).

The characteristic features of equations (4.7)−(4.9) are :

(i) All aforesaid equations have constant coefficients.

(ii) Terms involving $\langle \mu\, h^A h^B \rangle$, $\langle \mu\, g^A g^B \rangle$ in dynamic evolution equations (4.9) are of an order $O(l^2)$, $O(l^4)$, respectively, and *describe the effect of mesostructure size on the dynamic shell behaviour*.

(iii) Terms with $\langle D^{\alpha\beta\gamma\delta} g^A \rangle, \langle D^{\alpha\beta\gamma\delta} g^A h^B_{,\delta} \rangle$ and $\langle D^{\alpha\beta\gamma\delta} g^A g^B \rangle$ in constitutive equations (4.7) are of an order $O(l^2)$, $O(l^2)$, $O(l^4)$, respectively, and describe this effect on the shell response also in the quasi-stationary problems.

(iv) Solutions to initial-boundary value problems for above equations have a physical sense only if they are represented by sufficiently regular slowly varying functions $U_\alpha(\Theta,t)$, $W(\Theta,t)$, $Q^B_\alpha(\Theta,t)$, $V^B(\Theta,t)$.

Substituting the constitutive equations (4.7) into the equation of motion (4.8) and the dynamic evolution equations (4.9), under extra denotations

$$\tilde{D}^{\alpha\beta\gamma\delta} \equiv \langle D^{\alpha\beta\gamma\delta} \rangle, \quad D^{A\alpha\beta\gamma} \equiv \langle D^{\alpha\beta\gamma\delta}\, h^A_{,\delta} \rangle, \quad D^{AB\beta\gamma} \equiv \langle D^{\alpha\beta\gamma\delta}\, h^A_{,\alpha}\, h^B_{,\delta} \rangle,$$

$$F^{AB\beta\gamma\delta} \equiv \underline{\langle D^{\alpha\beta\gamma\delta} h^A_{,\alpha} g^B \rangle}, \quad L^{A\alpha\beta\gamma\delta} \equiv \underline{\langle D^{\alpha\beta\gamma\delta}\, g^A \rangle},$$

$$\tilde{B}^{\alpha\beta\gamma\delta} \equiv \langle B^{\alpha\beta\gamma\delta} \rangle, \quad K^{A\alpha\beta} \equiv \langle B^{\alpha\beta\gamma\delta}\, g^A_{,\gamma\delta} \rangle,$$

$$L^{AB} \equiv (\langle B^{\alpha\beta\gamma\delta} g^A_{,\alpha\beta}\, g^B_{,\gamma\delta} \rangle + b_{\alpha\beta}\, \underline{\langle D^{\alpha\beta\gamma\delta}\, g^A\, g^B \rangle}\, b_{\gamma\delta}), \tag{4.10}$$

we obtain the following governing relations for $U_\alpha(\Theta,t)$, $W(\Theta,t)$, $Q^B_\alpha(\Theta,t)$, $V^B(\Theta,t)$ as the basic kinematic unknowns:

$$\tilde{D}^{\alpha\beta\gamma\delta}(U_{\gamma,\delta\alpha} - b_{\gamma\delta}\, W_{,\alpha}) + D^{B\alpha\beta\gamma}\, Q^B_{\gamma,\alpha} - \underline{L^{B\alpha\beta\gamma\delta}} b_{\gamma\delta} V^B_{,\alpha} - \langle \mu \rangle\, a^{\alpha\beta} \ddot{U}_\alpha + \langle f^\beta \rangle = 0,$$

$$\tilde{B}^{\alpha\beta\gamma\delta}\, W_{,\alpha\beta\gamma\delta} + K^{B\alpha\beta}\, V^B_{,\alpha\beta} - b_{\alpha\beta}\, \tilde{D}^{\alpha\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta} W) - b_{\alpha\beta}\, D^{B\alpha\beta\gamma}\, Q^B_\gamma +$$

$$+ b_{\alpha\beta}\, \underline{L^{B\alpha\beta\gamma\delta}}\, b_{\gamma\delta} V^B + \langle \mu \rangle \ddot{W} - \langle f \rangle = 0, \tag{4.11}$$

$$D^{A\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta} W) + D^{AB\beta\gamma}\, Q^B_\gamma - \underline{F^{AB\beta\gamma\delta}} b_{\gamma\delta} V^B + \underline{\langle \mu h^A h^B \rangle}\, a^{\gamma\beta} \ddot{Q}^B_\gamma + \underline{\langle f^\beta h^A \rangle} = 0,$$

$$-b_{\alpha\beta} \underline{L^{A\alpha\beta\gamma\delta}}(U_{\gamma,\delta} - b_{\gamma\delta} W) + K^{A\alpha\beta} W_{,\alpha\beta} - b_{\alpha\beta} \underline{F^{AB\beta\alpha\gamma}} Q^B_\gamma + \underline{L^{AB}} V^B + \underline{\langle \mu g^A g^B \rangle} \ddot{V}^B + \underline{\langle f g^A \rangle} = 0$$

Thus, the class of length-scale models of mesostructured cylindrical shells has been obtained the form of which is determined by the choice of the shape functions $h^A(\cdot)$, $g^A(\cdot)$, $A = 1, ..., N$, describing from the qualitative point of view the expected shapes of oscillations.

The underlined coefficients in Eqs. (4.11) describe the effect of the mesostructure size on the shell overall behaviour.

It can be shown that for homogeneous structures with constant thickness and for homogeneous initial conditions for internal variables, Eqs. (4.9) have only trivial solution $Q^A_\alpha = V^A = 0$ and Eqs. (4.7), (4.8) reduce to the well-known linear elastodynamic relations for cylindrical shells. Thus we conclude that the internal variables describe the effect of heterogeneity on the shell global behaviour.

5. Asymptotic equations. The simplified model of the mesostructured cylindrical shells can be derived directly from the length-scale model by a limit passage $l \to 0$, i.e. by neglecting the underlined terms which depend on the mesostructure length parameter l. Hence, Eqs. $(4.11)_{3,4}$ yield :

$$D^{AB\beta\gamma} Q^B_\gamma = -D^{A\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W), \quad L^{AB}V^A = -K^{B\gamma\delta} W_{,\gamma\delta}. \tag{5.1}$$

From the positive definitenees of the strain energy it follows that $N \times N$ matrix L^{AB} is non-singular as well as the linear transformation determined by components $D^{AB\beta\gamma}$ is always invertible. Hence a solution to equations (5.1) can be written in the form:

$$Q^B_\gamma = -G^{BC}_{\gamma\eta} D^{C\eta\mu\vartheta}(U_{\mu,\vartheta} - b_{\mu\vartheta}W), \quad V^A = -E^{AB}K^{B\gamma\delta}W_{,\gamma\delta}. \tag{5.2}$$

where $G^{AB}_{\alpha\beta}$ and E^{AB} are defined by

$$G^{AB}_{\alpha\beta} D^{BC\beta\gamma} = \delta^\gamma_\alpha \delta^{AC}, \quad E^{AB}L^{BC} = \delta^{AC}. \tag{5.3}$$

Setting

$$D^{\alpha\beta\gamma\delta}_{eff} \equiv \tilde{D}^{\alpha\beta\gamma\delta} - D^{A\alpha\beta\eta}\, G^{AB}_{\eta\xi}\, D^{B\xi\gamma\delta}, \quad B^{\alpha\beta\gamma\delta}_{eff} \equiv \tilde{B}^{\alpha\beta\gamma\delta} - K^{A\alpha\beta}\, E^{AB}\, K^{B\gamma\delta}. \tag{5.4}$$

and substituting the expression (5.2) into Eqs. $(4.11)_{1,2}$ and Eqs. $(4.7)_{1,2}$, in which the underlined terms are neglected, we arrive at the asymptotic shell model governed by:

(i) *equations of motion*

$$B^{\alpha\beta\gamma\delta}_{eff} W_{,\alpha\beta\gamma\delta} - b_{\alpha\beta} D^{\alpha\beta\gamma\delta}_{eff}(U_{\gamma,\delta} - b_{\gamma\delta}W) + \langle\mu\rangle\ddot{W} - \langle f\rangle = 0,$$

$$D^{\alpha\beta\gamma\delta}_{eff}(U_{\gamma,\delta\alpha} - b_{\gamma\delta}W_{,\alpha}) - \langle\mu\rangle a^{\alpha\beta}\ddot{U}_\alpha + \langle f^\beta\rangle = 0, \tag{5.5}$$

(ii) *constitutive equations*

$$N^{\alpha\beta} = D^{\alpha\beta\gamma\delta}_{eff}(U_{\gamma,\delta} - b_{\gamma\delta}W), \quad M^{\alpha\beta} = -B^{\alpha\beta\gamma\delta}_{eff} W_{,\gamma\delta}, \tag{5.6}$$

where $D^{\alpha\beta\gamma\delta}_{eff}$, $B^{\alpha\beta\gamma\delta}_{eff}$ are called the effective stiffnesses.

The obtained above asymptotic shell model governed by Eqs. (5.5), (5.6) is not able to describe the length-scale effect on the overall dynamic shell behaviour being independent of the mesostructure length parameter l.

6. Conclusions. In this paper an averaged 2D-model of thin mesostructured cylindrical shells, which describes the effect of periodicity cell length dimensions on the global dynamic shell behaviour (the length-scale effect) has been derived. In order to derive it the internal variable modelling approach to the thin shells with a locally periodic structure proposed by Woźniak in [14] has been applied. The resulting length-scale model is governed by Eqs. (4.7)–(4.9) or by Eqs. (4.7) and (4.11) with denotations (4.10). From this model by a limit passage $l \to 0$, i.e. by neglecting the underlined terms which depend on the mesostructure length parameter l, the asymptotic model of the shell under consideration has been obtained. This simplified model is governed by Eqs. (5.5), (5.6) with denotations (5.4). Contrary to the length-scale model, the simplified one is not able to describe the effect of mesostructure size being independent of the mesostructure length parameter l.

Using the aforementioned internal variable approach, the unknown midsurface shell displacements $u_\alpha = u_\alpha(\Theta,t)$ and $w = w(\Theta,t)$, $\Theta \in \Omega$, in governing equations of linear Kirchhoff-Love shell theory (2.2)–(2.4) are assumed to be obtained by a superimposition of displacement disturbances caused by the highly oscillating character of the shell mesostructure on arbitrary motions with wavelength of an order much larger then l. In the resulting length-scale and asymptotic models, the overall shell motions are described by unknown averaged over Δ slowly varying displacement fields $U_\alpha(\Theta,t)$, $W(\Theta,t)$, $\Theta \in \Omega_0$ The cell oscillating parts of displacements are described by highly oscillating functions $h^A(\cdot)$, $g^A(\cdot)$, $A = 1, ..., N$, representing the expected shapes of cell oscillations and obtained as approximate solutions to a certain eigenvalue problem for free vibrations on the cell Δ under periodicity boundary conditions and by unknown slowly varying fields $Q^B_\alpha(\Theta,t)$, $V^B(\Theta,t)$, $B = 1, ..., N$, $\Theta \in \Omega_0$, called internal variables. The internal variables do not enter the displacement boundary conditions. This fact is essential for the applications of these models, since for the boundary-value problems formulated within a framework of both length-scale and simplified models, we deal with boundary conditions imposed only on averaged displacements $U_\alpha(\cdot,t)$, $W(\cdot,t)$. The number and physical sense of these conditions are similar to those of the well known 2D-theory which is governed by equations (2.2)–(2.4). Moreover, in the framework of asymptotic approximation approach, the internal variables are governed by *a system of linear algebraic equations* and hence they can be easily eliminated (which is always possible) from the asymptotic model.

The main features of the resulting length-scale model governed by Eqs. (4.7)–(4.9) are:

(i) The form of it is relatively simple; it is represented by constitutive equations (4.7) and by a system of three partial differential equations (4.8) for averaged displacements $U_\alpha(\Theta,t)$, $W(\Theta,t)$, $\Theta \in \Omega_0$, coupled with ordinary differential equations (4.9) for internal variables $Q^B_\alpha(\Theta,t)$, $V^B(\Theta,t)$, $B = 1, ..., N$, $\Theta \in \Omega_0$, involving only time derivatives. *All aforementioned equations have constant coefficients, which can be easily determined by calculations the integrals over Δ*. Hence, they can be effectly applied to engineering problems.

(ii) The inertial properties of this model are described not only by an averaged mass density $\langle\mu\rangle$ but also by averages $\langle h^A h^B\rangle$, $\langle\mu\, g^A g^B\rangle$, $A, B = 1, ..., N$, which depend on the mesostructure length parameter l and hence describe the effect of the mesostructure size on the global dynamic shell behaviour. The elastic properties of the shell under consideration also depend on the periodicity cell length dimensions and hence the length-scale effect on the shell response is also described in quasi-stationary problems.

The main features of the resulting asymptotic model governed by Eqs. (5.5), (5.6) are:

(i) The form of it is very simple; the internal variables $Q_\alpha^B(\cdot,t)$, $V^B(\cdot,t)$, $B=1,\ldots,N$, are governed by a system of 3N linear algebraic equations (5.1) and after eliminating them by means of Eqs. (5.2), we arrive at the governing equations expressed only in terms of averaged displacements U_α, W. These equations consist of a system of three partial differential equations (5.5) for averaged displacements $U_\alpha(\Theta,t)$, $W(\Theta,t)$, $\Theta \in \Omega_0$ and constitutive relations (5.6).

(ii) The constant coefficients $D_{eff}^{\alpha\beta\gamma\delta}$, $B_{eff}^{\alpha\beta\gamma\delta}$, which are found in the Eqs. (5.5), (5.6), are called the effective stiffnesses and we calculate them from Eqs. (5.4).

(iii) It does not take into account the length-scale effect being independent of the mesostructure length parameter l.

Solutions to problems formulated for length-scale and asymptotic models have a physical sense only if they are represented by sufficiently regular slowly varying functions $U_\alpha(\Theta,t)$, $W(\Theta,t)$, $Q_\alpha^B(\Theta,t)$, $V^B(\Theta,t)$, $B=1,\ldots,N$, $\Theta \in \Omega_0$. This requirement imposes certain restrictions on the class of problems described by the models under consideration.

The comparison of solutions to special problems, obtained within the framework of both length-scale and asymptotic models, will make it possible to evaluate the effect of the mesostructure size on the global dynamic shell behaviour. Carrying out this analysis we have to determine the length-scale and simplified models, using the same shape functions $h^A(\cdot)$, $g^A(\cdot)$, $A=1,\ldots,N$.

Problems related to various applications of Eqs. (4.7)–(4.9) and Eqs. (5.5), (5.6) to dynamics of mesostructured cylindrical shells are reserved for separate papers.

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МАСШТАБНИЙ ЕФЕКТ ЗА ВІБРАЦІЇ МЕЗОСТРУКТУРНИХ ЦИЛІНДРИЧНИХ ОБОЛОНОК

Досліджено тонку лінійно-пружну циліндричну оболонку, яка має періодичну структуру в напрямах, дотичних до серединної поверхні. Сформульовано двовимірну усереднену модель такої оболонки, що враховує вплив розмірів комірки періодичності на глобальну динамічну поведінку оболонки (ефекти масштабу). Згаданим ефектом нехтують у відомих асимптотичних теоріях плит та оболонок. Модель побудовано за допомогою запропонованого Возняком [14] *методу моделювання тонких оболонок з локальною періодичною структурою. Одержану модель порівняно з асимптотичною моделлю, в якій ефект масштабу відсутній.*

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