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MACRO-DYNAMICS OF ELASTIC MICRO-NONPERIODIC COMPOSITES

1. Introduction. In modern engineering systems we also encounder composite structural elements in which macro-properties are functions of position in the body. They are structures that are not made of one standardized composite material but possess different desired macro-properties in different parts [1–3].

The aim of this contribution is to propose a general method of mathematical modelling for macro-heterogeneous elastic composite structures. All considerations are carried out in the context of the linear elasticity theory, under assumption of the perfect bonding between material constituents of the composite and for the deterministic description of the spatial distributions of constituents. The obtained results are transformed to the form of an engineering theory, which is a basis for calculations and design of macro-heterogeneous composites.

Notations. Throughout the paper subscripts i, j, k, l take the values 1, 2, 3 being related to the curvilinear coordinate system. The sub and superscripts a, b run over 1, ..., S if not stated otherwise. The summation convention holds for all aforementioned indices.

2. Basic assumptions. The subject of the analysis is a linear non-periodic composite body, which in its initial natural state occupies a region Ω in a 3-space parametrized by the curvilinear coordinates $(\mathbf{x} = x_1, x_2, x_3)$. The properties of these bodies are determined by a mass density $\rho(\cdot)$ and the tensor of elastic modulae $A^{ijkl}(\cdot)$. We restrict ourselves to composites for which there exist a decomposition of Ω into a very large number of small mutually disjointed cells $\Delta(\mathbf{x})$. We assume that the adjacent elements have almost identical distributions of material constituents but the romote elements can be distinctly different. For every $\Delta(\mathbf{x})$ we shall introduce the averaging operator

$$\langle f \rangle_A = \frac{1}{vol(V^A)} \int_{V^A} f(\mathbf{x})\, dV\,, \qquad A = 1, 2, 3, \ldots, S\,, \tag{2.1}$$

where $f(\cdot)$ is an arbitrary integrable function defined (almost everywhere) on Δ.

3. Analysis. The governing equation of the proposed micro-macro elastodynamics we obtained of the well known actions functional.

$$A = \int_{\Omega} \left(\tfrac{1}{2}\dot{u}_i \dot{u}^i - \tfrac{1}{2} A^{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \rho u_i b^i\right) dV \tag{3.1}$$

By applaying assumptions that

$$\varepsilon_{ij} \cong \frac{1}{2}(U_{i/j} + U_{j/i}) + \frac{1}{2}(h_{a/i}\, Q_j^a + h_{a/j}\, Q_i^a) \tag{3.2}$$

(see paper [1], $U_i(\cdot,\tau)$, $Q_i^a(\cdot,\tau)$ are arbitrary independent regular functions, $h_a(\cdot)$ is the known system of functions postulated a priori in every problem. After some manipulations we obtain the approximation A ($A \cong A_0$), where

$$A_0 \cong \int_\Omega \Big[\frac{1}{2}\langle\rho\rangle_0 \dot U_i \dot U^i + \frac{1}{2}\langle\rho h_a h_b\rangle_0 \dot Q_i^a \dot Q^{bi} - \frac{1}{2}\left\langle A^{ijkl}\right\rangle_0 U_{(i/j)} U_{(k/l)} -$$

$$+ \left\langle A^{ijkl} h_{a(j}\right\rangle_0 U_{(k/l)} Q_{i)}^a - \frac{1}{2}\left\langle A^{ijkl} h_{a/j} h_{b/l}\right\rangle_0 Q_i^a Q_k^b + \langle\rho\rangle_0 b^i U_i \Big] dV. \tag{3.3}$$

Lagrange equations for A_0 read

$$S^{ij}{}_{/j} + \langle\rho\rangle_0 b^i = \langle\rho\rangle_0 \ddot U^i, \qquad H_a^i + \langle\rho h_a h_b\rangle_0 \ddot Q^{bi} = 0, \tag{3.4}$$

where

$$S^{ij} = \left\langle A^{ijkl}\right\rangle_0 U_{(k/l)} + \left\langle A^{ijkl} h_{a/(k}\right\rangle_0 Q_{l)}^a,$$

$$H_a^i = \left\langle A^{ijkl} h_{a/j}\right\rangle_0 U_{(k/l)} + \left\langle A^{ijkl} h_{a/j} h_{b/l}\right\rangle_0 Q_k^b. \tag{3.5}$$

Denoting by σ^{ij} components of a stress tensor, by means $\sigma^{ij} = A^{ijkl}(U_{(k/l)} + h_{a/(k} Q_{l)}^a)$, we obtain

$$S^{ij}(\cdot,\tau) = \left\langle\sigma^{ij}\right\rangle(\cdot,\tau), \qquad H_a^i(\cdot,\tau) = \left\langle\sigma^{ij} h_{a/j}\right\rangle(\cdot,\tau), \tag{3.6}$$

4. Axisymmetric case. In this section we shall consider problems corresponding to equations (3.4) for the axisymmetric case. After calculating the Christoffel symbols in the cylindrical coordinates (r,φ,z) with axisymmetric loads, we obtain following equations of motion

$$\frac{\partial S^{rr}}{\partial r} + \frac{\partial S^{rz}}{\partial z} + \frac{S^{rr} - S^{\varphi\varphi}}{r} = \langle\rho\rangle_0 \ddot U_r,$$

$$\frac{\partial S^{rz}}{\partial r} + \frac{\partial S^{zz}}{\partial z} + \frac{1}{r} S^{rz} = \langle\rho\rangle_0 \ddot U_z. \tag{4.1}$$

Assuming that $\langle\rho h_1 h_1\rangle_0 = 0$ and setting $h_1 = \sin(2\pi r/l)\sin(2\pi\varphi/\alpha)$, equations (3.5) can be written as

$$S^{rr} = \gamma_1(r)\frac{\partial U_r}{\partial r} + \gamma_2(r)\frac{1}{r}U_r + \gamma_3(r)\frac{\partial U_z}{\partial z},$$

$$S^{\varphi\varphi} = \gamma_2(r)\frac{\partial U_r}{\partial r} + \gamma_4(r)\frac{1}{r}U_r + \gamma_2(r)\frac{\partial U_z}{\partial z},$$

$$S^{zz} = \gamma_3(r)\frac{\partial U_r}{\partial r} + \gamma_2(r)\frac{1}{r}U_r + \gamma_1(r)\frac{\partial U_z}{\partial z},$$

$$S^{rz} = c_1\left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}\right). \tag{4.2}$$

The functions $\gamma_1(r)$, $\gamma_2(r)$, $\gamma_3(r)$, $\gamma_4(r)$ are given below

$$\gamma_1(r) = a_1 - b_1\frac{1}{r^2}, \qquad \gamma_2(r) = a_2 - b_2\frac{1}{r^2},$$

$$\gamma_3(r) = a_2 - b_1\frac{1}{r^2}, \qquad \gamma_4(r) = a_1 - b_3\frac{1}{r^2}, \tag{4.3}$$

where

$$a_1 = \lambda_m + 2\mu_m + \varsigma[\lambda_z + 2\mu_z - (\lambda_m + 2\mu_m)], \qquad a_2 = \lambda_m + \varsigma(\lambda_z - \lambda_m),$$

$$b_1 = \frac{\alpha_1^2(\lambda_z - \lambda_m)^2}{m}, \qquad b_2 = \frac{\alpha_1^2(\lambda_m - \lambda_z)[\lambda_m + 2\mu_m - (\lambda_z + 2\mu_z)]}{m},$$

$$b_3 = \frac{\alpha_1^2[\lambda_m + 2\mu_m - (\lambda_z + 2\mu_z)]^2}{m}, \qquad c_1 = \mu_m + \varsigma(\mu_z - \mu_m),$$

$$m = \frac{\pi^2}{l^2}[\mu_m + (\mu_z - \mu_m)\alpha_2] + \frac{\pi^2}{\alpha^2}\{\lambda_m + 2\mu_m + [\lambda_z + 2\mu_z - (\lambda_m + 2\mu_m)]\alpha_3\},$$

$$\alpha_1 = \frac{1}{\alpha l}\Big[\frac{ll'}{\pi}\cos(\pi l'/l) - \frac{l^2}{\pi^2}\sin(\pi l'/l)\Big]\sin(\pi\beta/\alpha),$$

$$\alpha_2 = \Big[\frac{2l'}{l} + \frac{1}{\pi}\sin(2\pi l'/l)\Big]\Big[\frac{\beta}{2\alpha} - \frac{1}{4\pi}\sin(2\pi\beta/\alpha)\Big],$$

$$\alpha_3 = \Big[\frac{2l'}{l} - \frac{1}{\pi}\sin(2\pi l'/l)\Big]\Big[\frac{\beta}{2\alpha} + \frac{1}{4\pi}\sin(2\pi\beta/\alpha)\Big],$$

$$\varsigma = \frac{F_z}{F_m} = \text{const}. \tag{4.4}$$

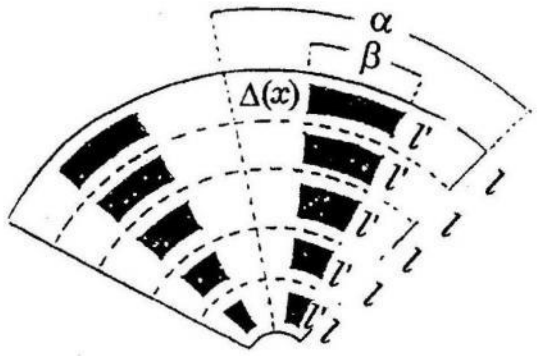
5. Example. As an illustrative example of the application of the general considerations given in Sec. 4 we shall now consder the wave shear propagating along z-axis. To this end we assume that $U_r(\cdot) = 0$, $U_z(\cdot) = 0$, $U_\varphi = U_\varphi(r,z,\tau)$, the matrix and the reinforcement are made of homogeneous isotropic linear – elastic materials and the perfect bonding between constituents. On this assumption we may obtain the explicit of equation of motion (see equations (4.1), (4.2)).

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\langle\mu\rangle_0}{\mu^{eff}}\frac{\partial^2}{\partial z^2} - \frac{1}{c_2^2}\frac{\partial^2}{\partial\tau^2}\Big)U_\varphi(\cdot) = 0, \tag{5.1}$$

where

$$\langle\mu\rangle_0 = (1 - l'/l)\mu_m + l'\mu_z/l, \qquad \mu^{eff} = \langle\mu\rangle_0 - \frac{(\langle\mu h_{,r}\rangle_0)^2}{\langle\mu h^2{}_{,r}\rangle_0},$$

$$(\langle\mu h_{,r}\rangle_0)^2 = 4(\mu_m - \mu_z)^2, \qquad \langle\mu h_{,r}^2\rangle_0 = 4\Big(\frac{l}{l - l'}\mu_m + \frac{l}{l'}\mu_z\Big). \tag{5.2}$$



The scheme of the laminate ($\alpha = \beta$).

In our case we introduce only one shape function $h(r)$, which is piecewise linear and takes the values $h(0) = h(l/2) = h(l) = 0$, $h((l - l')/2) = 1$, $h((l + l')/2) = -1$, (l, l' shown in Figure).

Looking for solution to eqs. (5.1) in the form $U_\varphi(\cdot) = U'_\varphi(r)e^{i(kz-\omega\tau)}$, we obtain following formulae for the wave velocity

$$c = c_2\sqrt{\frac{\langle\mu\rangle_0}{\mu^{eff}} + \frac{\lambda^2\beta^2}{4\pi^2}}, \quad c_2 = \sqrt{\frac{\mu^{eff}}{\langle\rho\rangle_0}}, \qquad (5.3)$$

where $\lambda = 2\pi/k$ is the wavelength of a wave propagation and coefficient β we calculation of the following equation $\beta l J_0(\beta l) = 2J_1(\beta l)$ where $J_0(\cdot)$, $J_1(\cdot)$ are Bessel functions.

More detailed investigations related to the applications and verification of the proposed engineering approach to macro – heterogeneous composite structures will be presented separately.

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МАКРОДИНАМІКА ПРУЖНИХ МІКРО-НЕПЕРІОДИЧНИХ КОМПОЗИТІВ

Розглянуто динамічну задачу теорії пружності для циліндра, яка містить кругові сектори з іншого матеріалу. Рівняння гомогенізованої моделі такого композиту отримано за допомогою процедури гомогенізації з мікролокальними параметрами.

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