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Iwona Cielecka

Łódź University of Technology

CONTINUUM MODELLING THE DYNAMIC PROBLEMS FOR LATTICE-TYPE PLATES

Introduction. In this paper we deal with the formulation and application of a continuum model to study linearized elastodynamics for lattice-type plates having an arbitrary complex periodic lay-out in Ox_1x_2-plane; two examples of this lay-out are shown in Fig. 1. It is assumed that the length dimensions of a representative cell of the periodic structure are small compared to the minimum characteristic length dimension of the whole latticed plate and that the mass distribution in this plate can be approximated by assigning concentrated masses and inertia moments to every nodal joint of a lattice. Hence the lattice-type plate under consideration is represented by a certain plane periodic system of mutually interacting rigid joints.

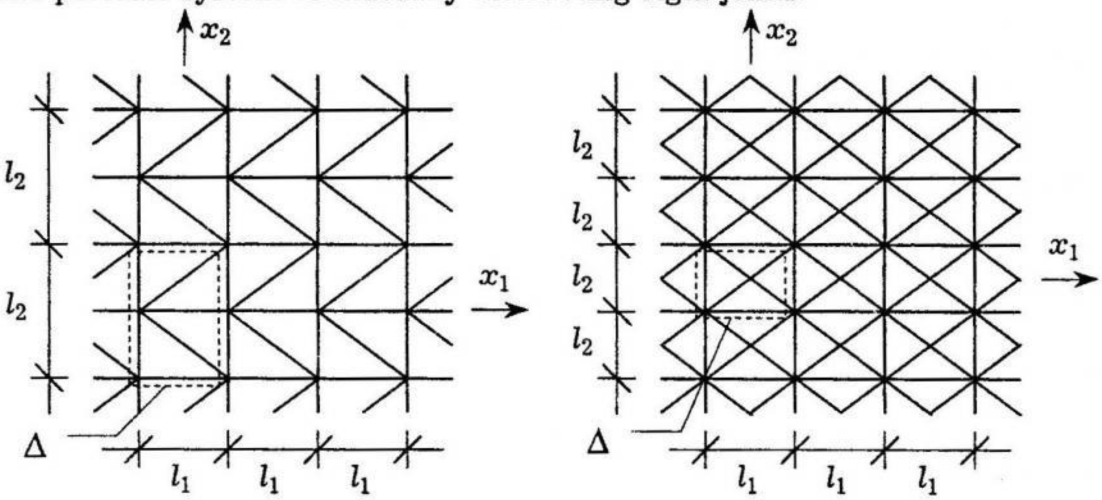

Fig. 1. Examples of periodic lattice-type plates and their representative elements.

It is known that a direct approach to dynamics of periodic systems with a very large number of interacting rigid bodies leads to computational difficulties due to a large number of ordinary differential equations describing the problem under consideration. That is why different averaged continuum models have been proposed in order to reduce the number of basic unknowns and to simplify the analysis of particular problems. From many results obtained in this manner, let us mention those related to frame-type lattice structures, summarized in [2], where the analysis was restricted to static problems. More sophisticated modelling approach, based on the asymptotic procedures of the homogenization theory, leads to the formulation of continuum models for periodic structures but neglects the effect of the unit cell size on the global behaviour of discrete system.

The model proposed is based on the concept of internal variables, [3], being able to describe dynamics of lattice-type plates of an arbitrary complex lay-out. It is assumed that the length dimension in $0x_1x_2$-plane of every rigid nodal joint are negligibly small as compared with the spans of interconnecting beams. The obtained model involves the effect of size of the representative periodicity cell on the global behaviour of plate under consideration. The governing equations of the model are applied to the analysis of free vibrations in the rectangular lattice structure.

Denotations. Subscripts i, j, k, l run over 1, 2 and are related to Cartesian orthogonal coordinates x_1, x_2 in the Ox_1x_2-plane. Indices a and A run over 1, ..., n and 1, ..., N, respectively; indices α, β take the values 1, ..., $n-1$. Summation convention holds for all aforementioned indices unless otherwise stated. Points on the Ox_1x_2-plane are denoted by $\mathbf{x} = (x_1, x_2)$ and t is the time coordinate.

Preliminaries and modelling assumptions. Let $\Delta = (-0.5l_1,\ 0.5l_1) \times (-0.5l_2, 0.5l_2)$ represent a cell on Ox_1x_2-plane which is assumed to be representative for a whole periodic lattice, cf. Fig. 1. It means that Δ contains the representative structural element for the lattice-type plate. It has to be emphasized that the choice of this element is not unique and depends on the class of motions we are investigate. It is assumed that the underformed representative element is made of N prismatic linear-elastic beams B^A, $A = 1, \ ..., N$ axes of which are situated on the plane Ox_1x_2. The beams B^A in the representative cell are interconnected by n rigid joints j^a, $a = 1, ..., n$. It is assumed that Ox_1x_2 is a symmetry plane, both for every beam and every rigid joint treated as certain spatial (3-dimensional) elements. The beams are subjected to bending and torsion in the planes perpendicular to Ox_1x_2-plane and the rigid joints rotate in the aforementioned planes and their centers displace in the direction normal to Ox_1x_2-plane. By Ω we define a region on Ox_1x_2-plane obtained as an interior of a union of all closures of repeated cells. It has to be remembered that the periodic structure of the whole lattice-type plate can be disturbed in the structural elements situated near the boundary $\partial\Omega$ of Ω.

Denoting by L the smallest characteristic length dimension of Ω and setting $l := \sqrt{l_1^2 + l_2^2}$ it will be assumed that $l/L \ll 1$. This is why l will be referred to as the microstructure length parameter of the lattice-type plate.

Significant properties of a beam B^A will be given by the flexural stiffness EI^A, the torsional stiffness GI_0^A and the span l^A. The concentrated mass assigned to a joint j^a will be denoted by M^a . The rotational moment of inertia of a joint j^a will be represented by the second order tensor J_{ij}^a. To every beam B^A we shall assign unit vectors $\mathbf{t}^A$, $\mathbf{n}^A$ shown in Fig. 2.

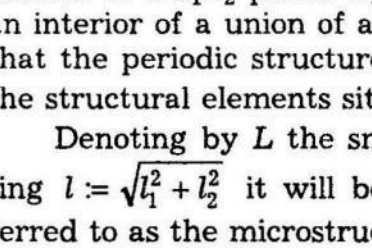

Fig. 2. Orientation of beam B^A.

Let us denote by w^a a displacement (deflection) of the joint j^a in the direction of x_3-axis and by φ_n^a and φ_t^a rotations of j^a in the planes normal to $\mathbf{t}^A$, $\mathbf{n}^A$, respectively. Assuming that joints j^a and j^b are interconnected by a beam B^A denote

$$\Delta_A w := (w^b - w^a)/l^A, \quad \varphi_{An} := 0.5(\varphi_n^a + \varphi_n^b), \quad \Delta_A \varphi_n := \varphi_n^b - \varphi_n^a, \quad \Delta_A \varphi_t := \varphi_t^b - \varphi_t^a, \tag{1}$$

Let us also assume that every beam B^A can be considered in the framework of the Euler-Bernoulli beam theory. Then the strain components related to B^A can be taken in the form (no summation over A in formulae (2)–(4))

$$\tilde{\varepsilon}^A := \Delta_A w + \varphi_{An}, \quad \kappa^A := \Delta_A \varphi_n, \quad \tilde{\kappa}^A := \Delta_A \varphi_t. \tag{2}$$

Hence, using additional notations

$$\tilde{\Lambda}^A := 12E\,I^A\,(l^A)^{-1}, \quad \mathrm{K}^A := E\,I^A\,(l^A)^{-1}, \quad \tilde{\mathrm{K}}^A := G\,I_0^A\,(l^A)^{-1}, \tag{3}$$

the strain energy σ^A assigned to a beam B^A is equal to

$$\sigma^A = \frac{1}{2}\tilde{\Lambda}^A(\tilde{\varepsilon}^A)^2 + \frac{1}{2}\mathrm{K}^A(\kappa^A)^2 + \frac{1}{2}\tilde{\mathrm{K}}^A(\tilde{\kappa}^A)^2. \tag{4}$$

It has to be remembered that all aforementioned denotations and formulae are related to an arbitrary but fixed repeated element of the periodic lattice-type plate under consideration (possibly except some elements situated near boundary $\partial\Omega$ of Ω).

Let us denote by $\mathcal{E}$ set of all points on the plane Ox_1x_2 which are centers of all mutually disjoined cells constituting the region Ω. Then the deflection and rotation vector of the joint j^a belonging to a cell with center $\mathbf{z}$, $\mathbf{z} \in \mathcal{E}$, at an arbitrary instant t, will be denoted by $w^a(\mathbf{z},t), \varphi^a(\mathbf{z},t)$, respectively. All external loads acting on the medium are assumed to be applied exclusively to the centers of rigid joints. The resultant external force and external couples applied to the joint j^a in a cell with a center $\mathbf{z} \in \mathcal{E}$ will be denoted by $f^a(\mathbf{z},t)$ and $\mathbf{m}^a(\mathbf{z},t)$, respectively. Introducing the action functional $\mathcal{A} = \mathcal{I} - \mathcal{K} - \mathcal{W}$ where

$$\mathcal{I} = \frac{1}{2}\sum_{\mathbf{z}\in\mathcal{E}}\sum_{A=1}^{N}\left[\tilde{\Lambda}^A\left(\tilde{\varepsilon}^A(\mathbf{z},t)\right)^2 + \mathrm{K}^A\left(\kappa^A(\mathbf{z},t)\right)^2 + \tilde{\mathrm{K}}^A\left(\tilde{\kappa}^A(\mathbf{z},t)\right)^2\right],$$

$$\mathcal{K} = \frac{1}{2}\sum_{\mathbf{z}\in\mathcal{E}}\sum_{a=1}^{n}\left[M^a\left(\dot{w}^a(\mathbf{z},t)\right)^2 + J_{ij}^a\;\dot{\varphi}_i^a(\mathbf{z},t)\,\dot{\varphi}_j^a(\mathbf{z},t)\right],$$

$$\mathcal{W} = \sum_{\mathbf{z}\in\mathcal{E}}\sum_{a=1}^{n}\left[f^a(\mathbf{z},t)\,w^a(\mathbf{z},t) + m_i^a(\mathbf{z},t)\,\varphi_i^a(\mathbf{z},t)\right], \tag{5}$$

and taking into account formulae (1), (2), from the principle of stationary action we derive equations of motion for $w^a(\mathbf{z},t)$, $\varphi_i^a(\mathbf{z},t)$, $\mathbf{z} \in \mathcal{E}$, $a = 1, ..., n$, $i = 1, 2$. These equations represent a discrete model of a periodic lattice-type plate but are not convenient in investigations of its global dynamic behaviour since the number of points $\mathcal{E}$ is very large. That is why relations (1), (2), (5) together with assumptions formulated below will be treated only as a basis for deriving a continuum model of the lattice-type plate under consideration.

In order to pass from the discrete model of the periodic lattice-type plate under consideration to a certain refined continuum model we have to recall two auxiliary concepts of the theory of periodic materials and structures introduced in [1,3]. The first from them is the concept of a slowly varying function. Let $F(\cdot,t)$ be a sufficiently regular real-valued function defined on Ω and depending on time t, the values of which for every t and every $\mathbf{x},\mathbf{y} \in \Omega$ such that $\|\mathbf{x}-\mathbf{y}\| < l$ satisfy conditions $|F(\mathbf{x},t) - F(\mathbf{y},t)| < \varepsilon_F$, where ε_F is a positive number determining the accuracy of calculations of F. If similar conditions hold also for all derivatives of F (including time-derivatives) then $F(\cdot,t)$ will be called a regular slowly varying function (related to the microstructure length parameter l and to certain accuracy parameters $\varepsilon_F, \varepsilon_{\nabla F}, \varepsilon_{\dot{F}}$, ...).

The second auxiliary concept is that of an oscillation-shape matrix. Define $\nu := n-1$ and let $h^{a\alpha}$, $g^{a\alpha}$, $\alpha = 1, ..., \nu$ be the real numbers constituting $n \times \nu$ matrices of a rank ν and satisfying conditions

$$\sum_{a=1}^{n} M^a h^{a\alpha} = 0, \quad \sum_{a=1}^{n} J_{ij}^a g^{a\alpha} = 0, \quad \alpha = 1,...,\nu, \quad i,j = 1, 2. \tag{6}$$

The physical meaning of theses concepts will be explained below.

The first modelling hypothesis interrelates the deflection $w^a(\mathbf{z},t)$ and the rotations $\varphi_i^a(\mathbf{z},t)$ of the joint j^a in a cell with the center $\mathbf{z}$, $\mathbf{z} \in \mathcal{L}$, with certain regular slowly varying functions $W(\cdot,t)$, $Q^\alpha(\cdot,t)$, $\Phi_i(\cdot,t)$, $R_i^\alpha(\cdot,t)$ which will be treated as basic kinematic unknowns. This hypothesis will be assumed in the form

$$\begin{aligned} &w^a(\mathbf{z},t) = W(\mathbf{x},t) + l\, h^{a\alpha} Q^\alpha(\mathbf{x},t), \\ &\varphi_i^a(\mathbf{z},t) = \Phi_i(\mathbf{x},t) + l\, g^{a\alpha} R_i^\alpha(\mathbf{x},t), \quad \mathbf{z} \in \mathcal{L}, \quad \mathbf{x} \in \Omega, \end{aligned} \tag{7}$$

where $\mathbf{x}$ is a position vector of the joint j^a. Bearing in mind conditions (6) imposed on $h^{a\alpha}$, $g^{a\alpha}$ and because of $|W(\mathbf{x},t) - W(\mathbf{z},t)| < \varepsilon_W$, $|\Phi_i(\mathbf{x},t) - \Phi_i(\mathbf{z},t)| < \varepsilon_{\Phi_i}$, etc., under denotations $M = \sum_{a=1}^{n} M^a$, $J_{ik} = \sum_{a=1}^{n} J_{ik}^a$ we obtain

$$\begin{aligned} &W(\mathbf{z},t) = M^{-1} \sum_{a=1}^{n} M^a\, w^a(\mathbf{z},t) + O(\varepsilon_W) + O(\varepsilon_Q), \\ &\Phi_i(\mathbf{z},t) = J_{ik}^{-1} \sum_{a=1}^{n} J_{kl}^a\, \varphi_l^a(\mathbf{z},t) + O(\varepsilon_\Phi) + O(\varepsilon_R), \quad \mathbf{z} \in \mathcal{L}, \end{aligned} \tag{8}$$

where $\mathbf{J}^{-1}$ is the inverse of the matrix $\mathbf{J}$ with components J_{ik}. It can be seen that the fields $W(\mathbf{z},t), \Phi_i(\mathbf{z},t)$ represent respectively weighted averaged deflections and rotations of repeated elements of the structure, while $Q^\alpha(\mathbf{z},t)$, $R_i^\alpha(\mathbf{z},t)$ describe respectively the disturbances in deflections and rotations at a time t within these elements caused by the complex lay-out of the lattice-type plate under consideration. Fields Q^α, R_i^α will be called internal variables; the meaning of this term will be explained below.

The second modelling hypothesis is related to the concept of slowly varying function on the basis of which we shall approximate finite differences of these functions by the values of appropriate derivatives and we shall neglect increments of introduced functions inside an arbitrary cell in calculation of averages over this cell. From this hypothesis we obtain the formulae for strain components in an arbitrary beam B^A belonging to a cell with the center $\mathbf{z}$. To this end, under assumption that joints j^a, j^b are interconnected by a beam B^A, define $h^{A\alpha} := h^{b\alpha} - h^{a\alpha}$, $g_A^\alpha := 0.5\,(g^{a\alpha} + g^{b\alpha})$, $g^{A\alpha} := g^{b\alpha} - g^{a\alpha}$ and $\lambda^A = l/l^A$. Also define

$$\Gamma_i(\mathbf{x},t) := W_{,i}(\mathbf{x},t) + \varepsilon_{ij}\Phi_j(\mathbf{x},t), \qquad \mathbf{x} \in \Omega, \tag{9}$$

where ε_{ij} stand for the Ricci symbol. After simple calculations from (2), (7) and (9) we obtain (no summation over A!)

$$\tilde{\varepsilon}^A(\mathbf{z},t) = t_i^A\,\Gamma_i(\mathbf{z},t) + \lambda^A\,h^{A\alpha}\,Q^\alpha(\mathbf{z},t) + l\,n_i^A\,g_A^\alpha\,R_i^\alpha(\mathbf{z},t),$$

$$\kappa^A(\mathbf{z},t) = l^A\,n_i^A\,t_j^A\,\Phi_{i,j}(\mathbf{z},t) + l\,n_i^A\,g^{A\alpha}\,R_i^\alpha(\mathbf{z},t),$$

$$\tilde{\kappa}^A(\mathbf{z},t) = l^A\,t_i^A\,t_j^A\,\Phi_{i,j}(\mathbf{z},t) + l\,t_i^A\,g^{A\alpha}\,R_i^\alpha(\mathbf{z},t), \quad \mathbf{z} \in \mathcal{L}. \tag{10}$$

It has to be emphasized that restrictions imposed on the class of motions under consideration reduce to the requirement that the basic unknown fields $W(\cdot,t)$, $Q^\alpha(\cdot,t)$, $\Phi_i(\cdot,t)$, $R_i^\alpha(\cdot,t)$ have to be regular slowly varying functions for every t. Let us also observe that the oscillation-shape matrices $h^{a\alpha}$, $g^{a\alpha}$, $a = 1, \ldots, n$, $\nu = 1, \ldots, n-1$, are not uniquely determined but their choice is irrelevant.

Governing equations. The governing equations for the deflection W, rotations Φ_i and the extra unknowns Q^α, R_i^α will be obtained from the principle of stationary action under the assumptions formulated above. It can be seen that the finite sums over $\mathcal{L}$ in formulae (5) can be approximated by integrals over Ω. Setting $|\Delta| = l_1 l_2$ let us introduce the notations

$$A_{ij} := |\Delta|^{-1}\sum_{A=1}^{N}\tilde{\Lambda}^A\,t_i^A\,t_j^A, \qquad C_{ijkl} := |\Delta|^{-1}\sum_{A=1}^{N}\left(l^A\right)^2\left(\mathrm{K}^A n_i^A n_k^A + \tilde{\mathrm{K}}^A t_i^A t_k^A\right)t_j^A t_l^A,$$

$$A^{\alpha\beta} := |\Delta|^{-1}\sum_{A=1}^{N}\left(\lambda^A\right)^2\tilde{\Lambda}^A\,h^{A\alpha}\,h^{A\beta},$$

$$A_{ij}^{\alpha\beta} := |\Delta|^{-1}\sum_{A=1}^{N}\left[\tilde{\Lambda}^A n_i^A\,n_j^A\,g_A^\alpha\,g_A^\beta + \left(\mathrm{K}^A n_i^A\,n_j^A + \tilde{\mathrm{K}}^A\,t_i^A\,t_j^A\right)g^{A\alpha}g^{A\beta}\right],$$

$$B_{ijk}^\alpha := |\Delta|^{-1}\sum_{A=1}^{N}\left(\lambda^A\right)^{-1}\left(\mathrm{K}^A n_i^A n_j^A + \tilde{\mathrm{K}}^A t_i^A t_j^A\right)t_k^A g^{A\alpha},$$

$$D_i^{\alpha} := |\Delta|^{-1}\sum_{A=1}^{N}\lambda^A\,\tilde{\Lambda}^A\,t_i^A\,h^{A\alpha}, \qquad D_{ij}^{\alpha} := |\Delta|^{-1}\sum_{A=1}^{N}\tilde{\Lambda}^A\,t_i^A\,n_j^A\,g_A^{\alpha},$$

$$D_i^{\alpha\beta} := |\Delta|^{-1}\sum_{A=1}^{N}\lambda^A\,\tilde{\Lambda}^A\,n_i^A\,h^{A\alpha}\,g_A^{\beta}, \qquad \rho := |\Delta|^{-1}\sum_{a=1}^{n}M^a, \qquad \chi_{ij} := h^{-2}|\Delta|^{-1}\sum_{a=1}^{n}J_{ij}^a,$$

$$\rho^{\alpha\beta} := |\Delta|^{-1}\sum_{a=1}^{n}M^a\,h^{a\alpha}\,h^{a\beta}, \qquad \chi_{ij}^{\alpha\beta} := h^{-2}|\Delta|^{-1}\sum_{a=1}^{n}J_{ij}^a\,g^{a\alpha}\,g^{a\beta}, \tag{11}$$

where h stands for mean height of the beams in the direction normal to Ox_1x_2-plane.

Moreover, let us assume that there exist continuous functions $f(\cdot,t)$, $f^{\alpha}(\cdot,t)$, $m_i(\cdot,t)$, $m_i^{\alpha}(\cdot,t)$ defined on Ω for every t being the slowly varying functions, such that the conditions

$$f(\mathbf{z},t) = |\Delta|^{-1}\sum_{a=1}^{n}f^a(\mathbf{z},t) + O(\varepsilon_f), \qquad f^{\alpha}(\mathbf{z},t) = |\Delta|^{-1}\sum_{a=1}^{n}f^a(\mathbf{z},t)\,h^{a\alpha} + O(\varepsilon_f), \tag{12}$$

$$m_i(\mathbf{z},t) = h^{-1}|\Delta|^{-1}\sum_{a=1}^{n}m_i^a(\mathbf{z},t) + O(\varepsilon_m), \quad m_i^{\alpha}(\mathbf{z},t) = h^{-1}|\Delta|^{-1}\sum_{a=1}^{n}m_i^a(\mathbf{z},t)\,g^{a\alpha} + O(\varepsilon_m),$$

hold for every $\mathbf{z} \in \mathcal{E}$. After substituting to (5) the right-hand sides of equations (7), (10), (12) and taking into account the approximation hypothesis (related to calculations of averages), as well as the conditions (6) and the notations (11), we arrive at the integral form of the action functional $\mathcal{A} = \mathcal{F} - \mathcal{K} - \mathcal{W}$, where now

$$\mathcal{F} = \int_{\Omega}\Big(\frac{1}{2}A_{ij}\,\Gamma_i\,\Gamma_j + \frac{1}{2}C_{ijkl}\,\Phi_{i,j}\,\Phi_{k,l} + \frac{1}{2}A^{\alpha\beta}\,Q^{\alpha}\,Q^{\beta} + \frac{1}{2}l^2A_{ij}^{\alpha\beta}\,R_i^{\alpha}\,R_j^{\beta} +$$
$$+ l^2B_{ijk}^{\alpha}\,R_i^{\alpha}\,\Phi_{j,k} + D_i^{\alpha}\,\Gamma_i\,Q^{\alpha} + lD_{ij}^{\alpha}\,\Gamma_i\,R_j^{\alpha} + lD_i^{\alpha\beta}\,Q^{\alpha}\,R_i^{\beta}\Big)dx_1dx_2,$$

$$\mathcal{K} = \int_{\Omega}\Big(\frac{1}{2}\rho\,\dot{W}\,\dot{W} + \frac{1}{2}l^2\,\rho^{\alpha\beta}\,\dot{Q}^{\alpha}\,\dot{Q}^{\beta} + \frac{1}{2}h^2\chi_{ij}\,\dot{\Phi}_i\,\dot{\Phi}_j + \frac{1}{2}h^2\,l^2\chi_{ij}^{\alpha\beta}\,\dot{R}_i^{\alpha}\,\dot{R}_j^{\beta}\Big)dx_1dx_2,$$

$$\mathcal{W} = \int_{\Omega}\big(fW + lf^{\alpha}\,Q^{\alpha} + h\,m_i\,\Phi_i + h\,l\,m_i^{\alpha}\,R_i^{\alpha}\big)dx_1dx_2. \tag{13}$$

From the principle of stationary action we obtain the following equations for a deflection W and rotations Φ_i

$$\big(A_{ij}\,\Gamma_j + D_i^{\alpha}\,Q^{\alpha} + l\,D_{ij}^{\alpha}\,R_j^{\alpha}\big)_{,i} - \rho\,\ddot{W} + f = 0, \tag{14}$$

$$\big(C_{kijl}\,\Phi_{j,l} + l^2B_{kji}^{\alpha}\,R_j^{\alpha}\big)_{,i} + \varepsilon_{ki}\,\big(A_{ij}\Gamma_j + D_i^{\alpha}Q^{\alpha} + lD_{ij}^{\alpha}R_j^{\alpha}\big) - h^2\,\chi_{ki}\,\ddot{\Phi}_i + hm_k = 0,$$

which are coupled with equations for extra unknowns Q^{α}, R_i^{α}:

$$l^2\rho^{\alpha\beta}\,\ddot{Q}^{\beta} + A^{\alpha\beta}Q^{\beta} + D_i^{\alpha}\Gamma_i + lD_i^{\alpha\beta}R_i^{\beta} = lf^{\alpha},$$

$$h^2l^2\,\chi_{ij}^{\alpha\beta}\ddot{R}_j^{\beta} + l^2A_{ij}^{\alpha\beta}R_j^{\beta} + lD_{ji}^{\alpha}\Gamma_j + l^2B_{ijl}^{\alpha}\Phi_{j,l} + lD_i^{\beta\alpha}Q^{\beta} = hlm_i^{\alpha}, \tag{15}$$

where Γ_j is defined by Eq. (9). The obtained equations have to be satisfied for every t in the region Ω of Ox_1x_2 and represent a continuum model of the periodic lattice-type plate under consideration.

It can be seen that the extra unknowns Q^α, R_i^α are governed by the ordinary differential equations (15). Hence in general Q^α, R_i^α do not enter boundary conditions and that is why they have been called internal variables. Similarly, the obtained continuum model will be referred to as the internal variable model (*IV*-model).

The governing equations (14), (15) can be also written in the alternative form given by:

(i) Equations of motion

$$P_{i,i} - \rho\ddot{W} + f = 0, \qquad M_{ki,i} + \varepsilon_{ki}P_i - h^2\chi_{ki}\ddot{\Phi}_i + hm_k = 0. \tag{16}$$

(ii) Dynamic evolution equations

$$l^2\rho^{\alpha\beta}\ddot{Q}^\beta + S^\alpha = lf^\alpha, \qquad h^2l^2\chi_{ij}^{\alpha\beta}\ddot{R}_j^\beta + H_i^\alpha = hlm_i^\alpha. \tag{17}$$

(iii) Constitutive equations

$$\begin{bmatrix} P_i \\ M_{ki} \\ S^\alpha \\ H_i^\alpha \end{bmatrix} = \begin{bmatrix} A_{ij} & 0 & D_i^\beta & lD_{ij}^\beta \\ 0 & C_{kijl} & 0 & l^2B_{kji}^\beta \\ D_i^\alpha & 0 & A^{\alpha\beta} & lD_j^{\alpha\beta} \\ lD_{ji}^\alpha & l^2B_{ijl}^\alpha & lD_i^{\beta\alpha} & l^2A_{ij}^{\alpha\beta} \end{bmatrix} \begin{bmatrix} \Gamma_j \\ \Phi_{j,l} \\ Q^\beta \\ R_j^\beta \end{bmatrix}, \tag{18}$$

which have to be considered together with $\Gamma_i = W_{,i} + \varepsilon_{ij}\Phi_j$. From a formal viewpoint equations (16) are similar to the known plate-type Cosserat continuum equations, [2]. However, contrary to the Cosserat media, we also deal here with dynamic evolution equations (17) which are coupled with Cosserat equations (16) *via* the constitutive equations (18).

Let us observe that the nonasymptotic modelling procedure applied above leads to the occurrence of the microstructure length parameter l in equations (16)-(18); that is why the effect of cell size on the dynamic behaviour of structure can be described in the framework of proposed model.

The governing equations of internal variable model describe dynamics of lattice-type plate of an arbitrary complex periodic lay-out in the Ox_1x_2-plane. If we deal with the latticed plate of a simple lay-out, i.e., having only one rigid joint in every repeated element (in this case $n = 1$) then Q^α, R_i^α drop out from all equations and we pass to the Cosserat model of lattice-type plate which coincides with that discussed in [2].

Example. The governing equations of internal variable model will be now applied to the analysis of free vibrations of the lattice-type plate strip simply supported on the opposite edges $x_1 = \pm 0.5L$; the lay-out of this latticed plate is shown in Fig. 3. We shall consider the simplest continuum model of this structure; that is why the cell Δ will be assumed in the form given in Fig. 3. This cell has two rigid joints; in this case $n = 2$ and $\nu = n - 1 = 1$, i.e., the oscilla-

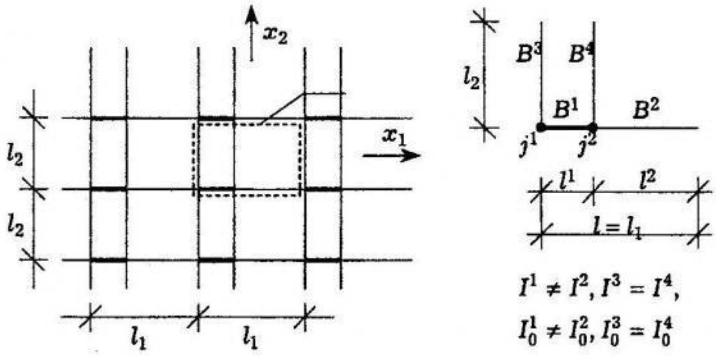

Fig. 3. Scheme of the lattice-type plate and its representative element Δ.

tion-shape matrices reduce to vectors with components h^{11}, h^{21} and g^{11}, g^{21}. Moreover, let the axes of all beams be parallel to the pertinent coordinate axes x_1, x_2. Let us consider the case in which the material of every beam is characterized by the Young modulus E and the Kirchhoff modulus G; the inertia moments I^A, I_0^A, $A = 1, ..., 4$ are interrelated as shown in Fig. 3. In this special case masses assigned to all nodal joints are equal and the rotational inertia moments assigned to all joints satisfy conditions $J_{12}^a = J_{21}^a = 0$, $a = 1, 2$. Assuming that all unknown functions depend only on x_1 and time t, bearing in mind definitions (9), (11) and neglecting external loadings, from (14), (15) we obtain three independent systems of equations. The first of them is related to unknowns W, Φ_2 and Q^1:

$$
\begin{aligned}
&A_{11}\left(W_{,1} + \Phi_2\right)_{,1} + D_1^1 Q_{,1}^1 - \rho\ddot{W} = 0, \\
&C_{2121}\,\Phi_{2,11} - A_{11}\left(W_{,1} + \Phi_2\right) - D_1^1 Q^1 - h^2 \chi_{22}\ddot{\Phi}_2 = 0, \\
&l^2 \rho^{11} \ddot{Q}^1 + A^{11} Q^1 + D_1^1\left(W_{,1} + \Phi_2\right) = 0,
\end{aligned}
\tag{19}
$$

the second one is related to unknowns Φ_1, R_1^1:

$$
\begin{aligned}
&C_{1111}\,\Phi_{1,11} + l^2 B_{111}^1 R_{1,1}^1 - A_{22}\,\Phi_1 - h^2 \chi_{11}\ddot{\Phi}_1 = 0, \\
&h^2 l^2 \chi_{11}^{11} \ddot{R}_1^1 + l^2 A_{11}^{11} R_1^1 + l^2 B_{111}^1 \Phi_{1,1} = 0,
\end{aligned}
\tag{20}
$$

and we also obtain an independent equation for R_2^1:

$$
h^2 l^2 \chi_{22}^{11} \ddot{R}_2^1 + l^2 A_{22}^{11} R_2^1 = 0,
\tag{21}
$$

where $l = l_1$ is the microstructure length parameter shown in Fig. 3. In the above systems of equations we also deal with another length parameter h, where $h \ll l$. That is why we shall look for the free vibration frequencies of the plate neglecting terms involving h^2. Under this assumption, setting $W = a_W \exp i(\omega t - kx_1)$, $\Phi_2 = a_\Phi \exp i(\omega t - kx_1)$, $Q^1 = a_Q \exp i(\omega t - kx_1)$, for the free vibration frequency ω we obtain the dispersion relation

$$
\begin{aligned}
&\rho^2\left(A_{11} l^2 + C_{2121}\varepsilon^2\right)\omega^4 - \rho\left[A_{11}A^{11} - \left(D_1^1\right)^2 + \left(A_{11}\varepsilon^2 + A^{11}\right)C_{2121}k^2\right]\omega^2 + \\
&\qquad + \left[A_{11}A^{11} - \left(D_1^1\right)^2\right]C_{2121}\,k^4 = 0,
\end{aligned}
\tag{22}
$$

where $\varepsilon := kl = 2\pi l/L$, L being the wavelength, is a nondimensional parameter. Because W, Φ_2, Q^1 have to be regular slowly varying functions hence

$l/L \ll 1$ and parameter ε is sufficiently small compared to 1. From (22) we obtain the asymptotic formulae for free vibration frequencies which can be expressed in the following form

$$(\omega_1)^2 = \frac{C_{2121}}{\rho} k^4 + O(\varepsilon^6), \qquad (\omega_2)^2 = \frac{A_{11} A^{11} - (D_1^1)^2}{l^2 \rho (A_{11} + C_{2121} k^2)} + o(\varepsilon^{-2}), \tag{23}$$

where coefficients $C_{2121}, A_{11}, A^{11}, D_1^1, \rho$ calculated from formulae (11) are equal to

$$C_{2121} = E\,(I^1 l^1 + I^2 l^2)/(l_1 l_2), \qquad A_{11} = 12E\,(I^1/l^1 + I^2/l^2)/(l_1 l_2),$$

$$A^{11} = 48E l_1 \left[I^1/(l^1)^3 + I^2/(l^2)^3\right]/l_2, \qquad D_1^1 = 24\,E\left[I^2/(l^2)^2 - I^1/(l^1)^2\right]/l_2,$$

$$\rho = 2M/(l_1 l_2). \tag{24}$$

It can be seen that the dispersion effect as well as higher vibration frequency ω_2 described by the above formulae are caused by the presence in equations (14), (15) terms involving the microstructure parameter l. It has to be emphasized that using the known homogenized continuum model of latticed plates we can obtain only lower vibration frequency $\omega^2 = \rho^{-1} C_{2121} k^4$.

More detailed approach to the problems investigated in this contribution will be given in a forthcoming paper.

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Івона Цілецька

КОНТИНУАЛЬНЕ МОДЕЛЮВАННЯ ДИНАМІЧНИХ ЗАДАЧ ДЛЯ СІТКОВИХ ПЛАСТИН

Запропоновано континуальну модель періодичних лінійно-пружних сіткових пластин, яка враховує вплив розміру мікроструктури на динамічну поведінку пластини. Одержані рівняння використано для аналізу поширення хвиль у частинному випадку сіткової пластини. Доведено, що масштабний ефект відіграє важливу роль, і ним не можна нехтувати під час аналізу.

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