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A DISCRETE MODEL FOR WAVE PROPAGATION PROBLEMS IN PERIODIC COMPOSITE MEDIA

In most cases the exact analysis of problems on wave propagation in material continua with periodic structure is not possible even using computer methods. That is why different approximate mathematical models for wave propagation in heterogeneous periodic media have been proposed, cf. [1–5]. In this contribution we are to show that above problems can be modelled by certain discrete plane periodic mass-point systems with a complex structure and ternary interactions. For the sake of simplicity considerations will be restricted to the linear-elastic material structures and plane problems for an unbounded medium. At the end of the lecture we are to show an example of applications of the general theory.

Notations. The superscripts a, b, c run over $1,\ldots,n$ and the superscript k takes the values $1,\ldots,m$. Indices A, B, C run over $0,1,\ldots,N$ except in denotations Δ_A, $\overline{\Delta}_A$ where $A = 1,\ldots,N$ unless otherwise stated. Summation convention holds for all twice-repeated indices. Points on E^2 are denoted by $\mathbf{p}$, $\mathbf{x}$ and points belonging to a subset Λ of E^2 by $\mathbf{z}$. Symbol t stands for a time coordinate.

In order to describe a periodic structure it is convenient to introduce the Bravais lattice $\Lambda \equiv \{\mathbf{z} \in E^2 : \mathbf{z} = \nu_1\,\mathbf{d}^1 + \nu_2\,\mathbf{d}^2,\ \nu_\alpha = 0, \pm 1, \pm 2, \ldots, \alpha = 1, 2\}$, where $\mathbf{d}^1$ and $\mathbf{d}^2$ are basis vectors on E^2. For an arbitrary subset Ξ of E^2 and for any $\mathbf{p} \in E^2$ define $\Xi(\mathbf{z}) \equiv \mathbf{z} + \Xi$ and $\mathbf{p}(\mathbf{z}) \equiv \mathbf{p} + \mathbf{z}, \mathbf{z} \in \Lambda$. Let Δ be a regular region on E^2 such that $E^2 = \bigcup \overline{\Delta}(\mathbf{z})$, $\mathbf{z} \in \Lambda$ and $\Delta(\mathbf{z}_1) \cap \Delta(\mathbf{z}_2) = \varnothing$ for every $\mathbf{z}_1, \mathbf{z}_2 \in \Lambda$ and $\mathbf{z}_1 \neq \mathbf{z}_2$. We shall also assume that there exist a simplicial subdivision of Δ into m simplexes T^k, $k = 1,\ldots,m$, which implies the simplicial subdivision of E^2 into simplexes $T^k(\mathbf{z})$, $\mathbf{z} \in \Lambda$. Hence $\bigcup \overline{T}^k = \overline{\Delta}$, and

$$T \equiv \{T^k(\mathbf{z}) : \mathbf{z} \in \Lambda, k = 1, \ldots, m\} \tag{1}$$

constitutes a set of all simplexes for the subdivision of E^2. It can be seen that for the aforementioned simplicial subdivision of Δ there exist a set of vertices $\mathbf{p}^a \in \overline{\Delta}, a = 1,\ldots,n$, such that

$$S \equiv \{\mathbf{p}^a(\mathbf{z}) : \mathbf{z} \in \Lambda, \quad a = 1,\ldots,n\} \tag{2}$$

is a set of all vertices for the related subdivision of E^2. In the sequel we shall assume that n is the smallest number of vertices $\mathbf{p}^a \in \overline{\Delta}$ for which S is a set

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of all vertices. Under this requirement the decomposition of $\mathbf{p}^a(\mathbf{z})$ in the form $\mathbf{p}^a(\mathbf{z}) = \mathbf{p}^a + \mathbf{z}$, $\mathbf{z} \in \Lambda$, $a = 1, ..., n$ is unique. In the subsequent considerations both simplicial subdivision of Δ and a set of n vertices $\mathbf{p}^a$, $a = 1, ..., n$, are assumed to be known. Let $\mathbf{d}^A \in \Lambda$, $A = 0,1, ..., N$, be a system of vectors where $\mathbf{d}^0 = \mathbf{0}$ and $\mathbf{p}^a$, $a = 1, ..., n$ a set of vertices such that all simplex vertices belonging to $\bar{\Delta}$ can be uniquely represented in the form $\mathbf{p}^a_A = \mathbf{p}^a + \mathbf{d}^A$. Hence every T^k can be represented as $T^k = \mathbf{p}^a_A\, \mathbf{p}^b_B\, \mathbf{p}^c_C$. For an arbitrary function $f(\cdot)$ defined on S with values in a certain linear space we shall introduce the finite differences

$$\Delta_A f(\mathbf{z}) \equiv f(\mathbf{z} + \mathbf{d}^A) - f(\mathbf{z}), \qquad \bar{\Delta}_A f(\mathbf{z}) \equiv f(\mathbf{z}) - f(\mathbf{z} - \mathbf{d}^A),$$

which for $A = 0$ reduce to identities. Hence in the sequel all finite difference operators $\Delta_A, \bar{\Delta}_A$ will be defined only for $A = 1, ..., N$. Following the notation introduced above we also denote $T^k(\mathbf{z}) \equiv T^k + \mathbf{z}$ and $\mathbf{p}^a_A(\mathbf{z}) \equiv \mathbf{p}^a_A + \mathbf{z}$ for every $\mathbf{z} \in \Lambda$. The aforementioned concepts and definitions will be used in the subsequent considerations in order to describe wave propagation problems in periodic composite media.

Let E^2 represents a two dimensional linear-elastic continuum with a periodic piecewise homogeneous material structure and $\Delta \subset E^2$ stands for the representative element of this structure. Let the aforementioned simplicial subdivision of Δ be treated as a decomposition of Δ into m finite triangle elements T^k. It has to be assumed that every element T^k with a sufficient accuracy will be treated as homogeneous. Bearing in mind that $T^k = \mathbf{p}^a_A\, \mathbf{p}^b_B\, \mathbf{p}^c_C$ and denoting by $\lambda^\alpha, \alpha = 1,2,3$, the barycentric coordinates of an arbitrary point $\mathbf{x} \in T^k$ we have $\lambda^\alpha > 0$, $\lambda^1 + \lambda^2 + \lambda^3 = 1$ and $\lambda^\alpha = \mathbf{a}^\alpha \cdot \mathbf{x} + b^\alpha$ where $\mathbf{a}^\alpha \in E^2$ are the known vectors and b^α are the known scalars, [7]. Consequently, the displacement $\mathbf{u}(\mathbf{x},t)$ of an arbitrary point $\mathbf{x} \in T^k$ at a time instant t will be taken in the form $\mathbf{u}(\mathbf{x},t) = \lambda^1\, \mathbf{u}^a_A(t) + \lambda^2\, \mathbf{u}^b_B(t) + \lambda^3\, \mathbf{u}^c_C(t)$ where $\mathbf{u}^a_A(t)$, $\mathbf{u}^b_B(t)$, $\mathbf{u}^c_C(t)$ are displacements of vertices $\mathbf{p}^a_A$, $\mathbf{p}^b_B$, $\mathbf{p}^c_C$, respectively, and $\lambda^\alpha = \mathbf{a}^\alpha \cdot \mathbf{x} + b^\alpha$, $\alpha = 1, 2, 3$, are the shape functions. Hence the displacement gradient $\nabla\mathbf{u}$ in every $T^k = \mathbf{p}^a_A\, \mathbf{p}^b_B\, \mathbf{p}^c_C$ is equal to

$$\nabla\mathbf{u}(\mathbf{x},t) = \mathbf{a}^1 \otimes \mathbf{u}^a_A(t) + \mathbf{a}^2 \otimes \mathbf{u}^b_B(t) + \mathbf{a}^3 \otimes \mathbf{u}^c_C(t), \qquad \mathbf{x} \in T^k, \tag{3}$$

and constant for every time t. Substituting into (3) decompositions of the form

$$\mathbf{u}^a_A(t) = \mathbf{u}^a(t) + \Delta_A\, \mathbf{u}^a(t), \quad A = 0, 1, ..., N \tag{4}$$

where $\mathbf{u}^a(t)$ is a displacement of mass point $\mathbf{p}^a \in T^k$, and introducing the linearized strain tensor

$$\varepsilon = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \tag{5}$$

we obtain the strain energy function assigned to T^k defined by

$$\Phi^k = \frac{1}{2} F^k \varepsilon : \mathbf{C}_k : \varepsilon \tag{6}$$

where F^k is the area of T^k and $\mathbf{C}_k$ is the elastic modulae tensor related to the material of finite element T^k. Combining (3)–(6) it can be seen that formula (6) makes it possible to derive function

$$\Phi^k = \Phi^k(\Delta_A \mathbf{u}^a, \mathbf{u}^b - \mathbf{u}^c), \tag{7}$$

depending on differences $\Delta_A \mathbf{u}^a$ and $\mathbf{u}^b - \mathbf{u}^c$. Hence strain energy function Φ assigned to the representative element Δ has the form

$$\Phi = \sum \Phi^k(\Delta_A \mathbf{u}^a, \mathbf{u}^b - \mathbf{u}^c). \tag{8}$$

At the same time a constant mass distribution in every $T^k(\mathbf{z})$, $\mathbf{z} \in \Lambda$, will be replaced by three concentrated masses at vertices $\mathbf{p}_A^a(\mathbf{z})$, $\mathbf{p}_B^b(\mathbf{z})$, $\mathbf{p}_C^c(\mathbf{z})$, where $T^k = \mathbf{p}_A^a\, \mathbf{p}_B^b\, \mathbf{p}_C^c$. Hence the kinetic energy function K assigned to the representative element Δ of the periodic structure is given by

$$K = \frac{1}{2} M^{ab}\, \dot{\mathbf{u}}^a \cdot \dot{\mathbf{u}}^b, \tag{9}$$

where $M^{ab} = \delta^{ab} m^b$ (no summation over b) and m^b is a total concentrated mass assigned to the vertex $\mathbf{p}^a$. Formulae (8), (9) hold for an arbitrary element $\Delta(\mathbf{z}) = \Delta + \mathbf{z}$, $\mathbf{z} \in \Lambda$ provided that arguments $\mathbf{u}^a$ are replaced by $\mathbf{u}^a(\mathbf{z}, t)$.

The aforementioned simplicial subdivision and concentration of masses m^a at points $\mathbf{p}^a$ makes it possible to apply the discrete plane periodic mass-point system with a complex structure and ternary interactions as a model for the analysis of wave propagation in composite materials under consideration. The above approach requires the use of a certain parametrization of applied mass-point system. We shall assume that the position of mass point system in its reference equilibrium state coincides with a set S defined by (2). Hence every mass-point will be identified with its reference position $\mathbf{p}^a(\mathbf{z})$, $\mathbf{z} \in \Lambda$, $a = 1, \ldots, n$ and it will be assumed that to every point $\mathbf{p}^a(\mathbf{z}) \in S$ there is assigned mass m^a which due to the periodicity of system is independent of $\mathbf{z} \in \Lambda$. The system of ternary interactions will be parametrized by a set T of all simplexes assuming that points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in S$ can interact if and only if $\mathbf{x}_1 = \mathbf{p}_A^a(\mathbf{z})$, $\mathbf{x}_2 = \mathbf{p}_B^b(\mathbf{z})$, $\mathbf{x}_3 = \mathbf{p}_C^c(\mathbf{z})$ for some $\mathbf{z} \in \Lambda$, where $\mathbf{p}_A^a(\mathbf{z})\, \mathbf{p}_B^b(\mathbf{z})\, \mathbf{p}_C^c(\mathbf{z}) = T^k(\mathbf{z})$ for some $k \in \{1, \ldots, m\}$. Hence every simplex $T^k(\mathbf{z})$, $\mathbf{z} \in \Lambda$, $k = 1, \ldots, m$, will be identified with a certain ternary interaction. To every interaction $T^k(\mathbf{z}) \in T$ there is assigned a strain energy function Φ^k which by means of the periodicity of system is independent of $\mathbf{z} \in \Lambda$. Bearing in mind that $\mathbf{u}_A^a(\mathbf{z}, t)$ is a displacement vector of point $\mathbf{p}^a_A(\mathbf{z})$ at time t we obtain that arguments of Φ^k are $\left|\mathbf{u}^a_A(\mathbf{z},t)-\mathbf{u}^b_B(\mathbf{z},t)\right|$. Because of $\mathbf{u}^a_A(\mathbf{z},t)=\Delta_A\,\mathbf{u}^a(\mathbf{z},t)+\mathbf{u}^a(\mathbf{z},t)$ we obtain $\Phi^k=\Phi^k\big(\Delta_A\,\mathbf{u}^a(\mathbf{z},t),\ \mathbf{u}^b(\mathbf{z},t)-\mathbf{u}^c(\mathbf{z},t)\big)$ bearing in mind that $\Phi^k(\cdot)$ are hemitropic functions of arguments which are specified by simplex $T^k=\mathbf{p}^a_A\,\mathbf{p}^b_B\,\mathbf{p}^c_C$.

It is evident that to every $\mathbf{z}\in\Lambda$ there is assigned a certain repetitive element of the periodic mass-point system comprising n mass points $\mathbf{p}^a(\mathbf{z})$, $a=1,\ldots,n$, and m ternary interactions $T^k(\mathbf{z})$, $k=1,\ldots,m$. The strain and kinetic energy functions assigned to an arbitrary repetitive element are respectively given by

$$\Phi=\sum\Phi^k\big(\Delta_A\mathbf{u}^a(\mathbf{z},t)\,,\ \mathbf{u}^b(\mathbf{z},t)-\mathbf{u}^c(\mathbf{z},t)\big),\tag{10}$$

$$K=\frac{1}{2}M^{ab}\,\dot{\mathbf{u}}^a(\mathbf{z},t)\cdot\dot{\mathbf{u}}^b(\mathbf{z},t).\tag{11}$$

Using the approach detailed in [6] it can be shown that the above formulae lead to the following equations of motion

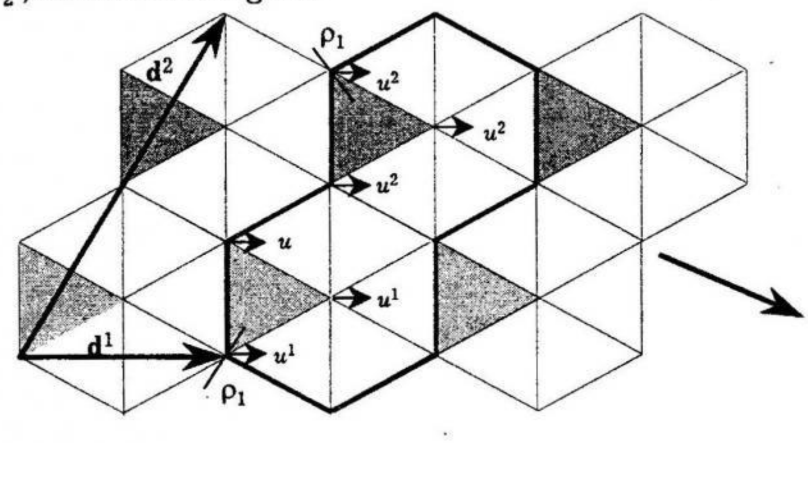
$$\bar{\Delta}_A\,\mathbf{S}^a_A(\mathbf{z},t)+\mathbf{h}^a(\mathbf{z},t)-M^{ab}\,\ddot{\mathbf{u}}^b(\mathbf{z},t)+\mathbf{f}^a(\mathbf{z},t)=\mathbf{0}\tag{12}$$

where $\mathbf{f}^a(\mathbf{z},t)$ is an external force acting at $\mathbf{p}^a(\mathbf{z})$ at time t and $\mathbf{S}^a_A$, $\mathbf{h}^a$ are generalized internal forces defined by the constitutive equations

$$\mathbf{S}^a_A(\mathbf{z},t)=\frac{\partial\Phi}{\partial\Delta_A\mathbf{u}^a(\mathbf{z},t)},\qquad \mathbf{h}^a(\mathbf{z},t)=-\frac{\partial\Phi}{\partial\mathbf{u}^a(\mathbf{z},t)}.\tag{13}$$

Equations (12), (13) have to hold for an arbitrary time instant t and every $\mathbf{z}\in\Lambda$. Taking into account the aforementioned considerations we jump to the conclusion that equations (12), (13) are the governing equations of linear-elastic composite materials in the finite element approximation. Hence, we have obtained a discrete model for the wave propagation problems in periodic composite media.

General considerations will be illustrated by the example of dispersion analysis for the unbounded, linear-elastic, isotropic, homogeneous medium with periodically distributed two kinds of rigid inclusions having mass densites ρ_1, ρ_2, as shown in figure.



In the analysis we shall restrict ourselves to displacements parallel to vector $\mathbf{d}^1$ and to the plane wave propagation in direction shown in figure. Dynamics of this structure will be modelled by the Bravais lattice with ternary interactions and basis vectors $\mathbf{d}^1$, $\mathbf{d}^2$. A representative element of this structure is bounded by the bold line in figure.

Let a simplicial subdivision of this element will be given by its decomposition into m equi-angular triangle elements. In the subsequent considerations $m = 1, \ldots, 10$, because two inclusions in the repeated element are rigid. The strain energy function assigned to T^m is $\Phi^m = \frac{1}{2} F^m \sigma^m_{\alpha\beta} \varepsilon^m_{\alpha\beta}$ where F^m is the area of T^m and $\sigma^m_{\alpha\beta}$, $\varepsilon^m_{\alpha\beta}$ are stress and strain tensors, respectively. Relationship between stress tensor and strain tensor in an elastic isotropic medium in the small strain range can be written as $\sigma^m_{\alpha\beta} = \lambda \delta_{\alpha\beta} \varepsilon^m_{\gamma\gamma} + 2\mu \varepsilon^m_{\alpha\beta}$, where λ and μ are Lame constants. Let h be the height of triangle elements T^m, and define $d = 2h$, $\kappa_1 = \frac{\sqrt{3}}{6}(\lambda + 5\mu)$, $\kappa_2 = \frac{2\sqrt{3}}{3}(\lambda + 2\mu)$, $m_\alpha = \frac{\sqrt{3}}{3} h^2 (5\rho + \rho_\alpha)$, where ρ and ρ_α, $\alpha = 1, 2$, are mass densities related to the deformable and rigid simplexes, respectively. Then using formulae (10)–(13) we obtain the following equations of motion:

$$m_1 \ddot{u}^1(x,t) = \kappa_1 (u^2(x-d,t) + 2u^2(x,t) + u^2(x+d,t) - 4u^1(x,t)) + \\ + \kappa_2 (u^1(x-d,t) - 2u^1(x,t) + u^1(x+d,t)),$$

$$m_2 \ddot{u}^2(x,t) = \kappa_1 (u^1(x-d,t) + 2u^1(x,t) + u^1(x+d,t) - 4u^2(x,t)) + \\ + \kappa_2 (u^2(x-d,t) - 2u^2(x,t) + u^2(x+d,t)). \tag{14}$$

Let us look for the wave solution to these equations in the form

$$u^1(x,t) = A_1 e^{i(kx-\omega t)}, \qquad u^2(x,t) = A_2 e^{i(kx-\omega t)}. \tag{15}$$

The substitution of equations (15) into (14) yields the system of two equations for A_1 and A_2

$$A_1 (2\kappa_2 (\cos dk - 1) - 4\kappa_1 + m_1 \omega^2) + A_2 2\kappa_1 (\cos dk + 1) = 0,$$

$$A_1 2\kappa_1 (\cos dk + 1) + A_2 (2\kappa_2 (\cos dk - 1) - 4\kappa_1 + m_2 \omega^2) = 0.$$

Nontrivial solutions A_1, A_2 exist if

$$\begin{vmatrix} 2\kappa_2 (\cos dk - 1) - 4\kappa_1 + m_1 \omega^2 & 2\kappa_1 (\cos dk + 1) \\ 2\kappa_1 (\cos dk + 1) & 2\kappa_2 (\cos dk - 1) - 4\kappa_1 + m_2 \omega^2 \end{vmatrix} = 0.$$

This condition yields a relation between ω and k in the form of two dispersion branches

$$\omega^2 = \frac{1}{m_1 m_2} \Big[(m_1 + m_2)(2\kappa_1 - \kappa_2(\cos dk - 1)) \pm \\ \pm \sqrt{[2\kappa_1 - \kappa_2(\cos dk - 1)]^2 (m_1 - m_2)^2 + 4\kappa_1^2 m_1 m_2 (\cos dk + 1)^2} \Big]. \tag{16}$$

Now, we shall discuss the dispersion given by two branches of the ω vs. k curve with the particular attention to the cases $k = 0$ and $k = \pm\pi/d$. For the large wavelengths $\frac{1}{2}dk \ll 1$ and $\cos dk \approx 1 - \frac{1}{2}d^2 k^2$. Hence

$$\sqrt{(2\kappa_1 + \tfrac{1}{2}\kappa_2 d^2 k^2)^2(m_1 - m_2)^2 + 4\kappa_1^2 m_1 m_2 (2 - \tfrac{1}{2}d^2k^2)^2} \approx$$

$$\approx 2\kappa_1(m_1 + m_2) + \frac{\frac{1}{2}(m_1 - m_2)^2 \kappa_2 - 2\kappa_1 m_1 m_2}{m_1 + m_2} d^2 k^2$$

and formula (16) reduces to

$$\omega_-^2 = \frac{2}{m_1 + m_2}(\kappa_1 + \kappa_2) d^2 k^2,$$

$$\omega_+^2 = 4\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\kappa_1 + \left[\kappa_2\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - \frac{2}{m_1 + m_2}(\kappa_1 + \kappa_2)\right] d^2 k^2. \qquad (17)$$

For the wavelengths of an order of $2d$ we assume $dk = \pi - \varepsilon$ with $\varepsilon \to 0^+$. Then $\cos dk \approx -1 + \varepsilon^2/2$, the radical in (16) becomes

$$\sqrt{\left[2\kappa_1 + (2 - \tfrac{1}{2}\varepsilon^2)\kappa_2\right]^2 (m_1 - m_2)^2 + \kappa_1^2 m_1 m_2 \varepsilon^4} \approx 2(\kappa_1 + \kappa_2)(m_1 - m_2) - \kappa_2 (m_1 - m_2)\varepsilon^2$$

and

$$\omega_-^2 = \frac{4}{m_1}(\kappa_1 + \kappa_2) + \frac{1}{2}\kappa_2\left(\frac{1}{m_2} - \frac{3}{m_1}\right)\varepsilon^2,$$

$$\omega_+^2 = \frac{4}{m_2}(\kappa_1 + \kappa_2) - \frac{1}{2}\kappa_2\left(\frac{3}{m_2} - \frac{1}{m_1}\right)\varepsilon^2. \qquad (18)$$

It has to be emphasized that formulae (17) are physically reliable while (18) have only a formal meaning. This statement is due to the fact that the uniform strains in finite elements (simplexes) shown in figure take place only for the large wavelengths. However, both formulae (17) and (18) as well as the dispersion branches given by (16) describe a certain mass point system with a periodic structure and ternary interactions and serve as an illustration of general equations (12), (13).

Summarizing the results of this contribution we shall formulate the following conclusions:

1. The periodic mass-point systems with ternary interactions can be applied as a discrete model for the wave propagation problems in periodic composites.
2. The governing equations (12), (13) have simple finite-difference form which is identical for every $\mathbf{z} \in \Lambda$.
3. Equations (12), (13) can describe wave propagation problems in periodic composite media with a desired degree of accuracy, because they hold for an arbitrary decomposition of the representative composite element into finite elements provided that it implies the periodic simplicial subdivision of E^2.

More detailed discussion of the proposed approach and its applications to the formulation of continuum and asymptotic models for the long wave deformations will be studied in the forthcoming papers.

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Іовіта Рихлевська, Іоланта Шимчик, Чеслав Возняк

ДИСКРЕТНА МОДЕЛЬ ПОШИРЕННЯ ХВИЛЬ У ПЕРІОДИЧНИХ КОМПОЗИЦІЙНИХ МАТЕРІАЛАХ

Запропоновано дискретну модель задач динаміки лінійно-пружних композитних матеріалів. Показано, що ця модель є класом систем матеріальних точок складної структури. Отримані результати становлять новий математичний апарат для аналізу задач поширення хвиль у неоднорідних періодичних середовищах.

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