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ON MESOSHAPE FUNCTIONS IN STRUCTURAL DYNAMICS OF THIN PERIODIC PLATES

1. Introduction. Main objects of this contribution are thin plates with a periodic structure along one direction in planes parallel to the plate midplane and with constant material and geometrical properties along the perpendicular direction (see Fig. 1). Plates of this kind are composed of many identical elements which are periodically distributed along one direction. Every element will be treated as a thin plate.

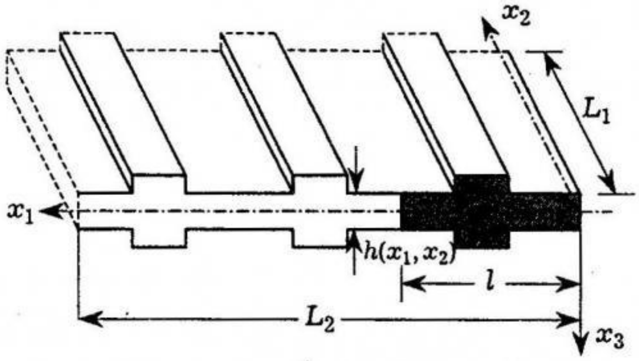

Fig. 1. An example of plate with one-directional periodic structure.

The exact analysis of periodic plates within solid mechanics is too complicated to constitute the basis for investigations of most engineering problems because obtained governing equations comprise highly oscillating functional coefficients. Thus, problems of such plates are investigated in the framework of different approximated methods. So called effective rigidities plate theories were presented e.g. in [3, 5, 8, 10] where periodic plates are described by governing equations of certain homogeneous plates with constant homogenised rigidities and averaged mass densities. An analysis of the periodic plate behaviour in the framework of asymptotic homogenisation methods is rather complicated from the computational point of view and hence it is restricted to the first approximation. Using asymptotic procedures we obtain averaged models neglecting the effect of the element length in the periodicity direction, called *the length-scale effect*. These models were restricted to the static problems.

In order to investigate non-stationary problems certain models (e.g. based on the concept of the continuum with the extra local degrees of freedom, [11]) were proposed. Short wave propagation problems were investigated in [1] and some refined models describing long wave problems for the periodic bodies were presented in [17]. These models take into account *the length-scale effect* on dynamic response of a body and are physically reasonable and simple enough to be applied in the analysis of engineering problems. The models of this kind were applied to selected dynamic problems of periodic structures, e.g. for the Hencky-Reissner periodic plates [2], for the Kirchhoff periodic plates [6–7], for the periodic wavy-plates [14], for composite lattice-type structures [4] and others [9, 12–13, 15–16]. As results of such modelling we obtain governing equations with constant coefficients for averaged displacements and also additional unknowns. These unknowns together with highly oscillating periodic functions make it possible to take into account the length-scale effect on the dynamic body behaviour. The aforementioned periodic functions describe a form of oscillations inside periodicity elements and they are called *mesoshape functions*. These functions should be obtained as solutions to eigenvalue problems on the periodicity element.

The aim of this contribution is to show that for many cases of periodicity element eigenfunctions being solutions to the aforementioned eigenvalue problem can be assumed in an approximated form what is sufficient from the computational point of view. For this purpose free vibrations of special kinds of periodic plate bands will be analysed.

2. Preliminary concepts. In this paper thin linear-elastic plates with a periodic structure along one direction and constant properties along the perpendicular direction in planes parallel to the plate midplane. An example of a such plate is presented in Fig. 1. These plates will be called *uniperiodic plates*. Introducing $Ox_1x_2x_3$ the orthogonal carthesian coordinate system in the physical space and setting $\mathbf{x} \equiv (x_1, x_2)$, $z \equiv x_3$, the region of a plate is denoted by $\Omega := \{(\mathbf{x}, z) : -h(\mathbf{x})/2 < z < h(\mathbf{x})/2,\ \mathbf{x} \in \Pi\}$, where Π is the rectangular plate midplane with length dimensions L_1, L_2 and $h(\mathbf{x})$ is the plate thickness at the point $\mathbf{x} \in \Pi$. We also define t as the time coordinate. Let l stand for the period of plate structure in the x_1-axis direction. Hence $(-l/2, l/2)$ is the interval in the plate midplane along this direction. We assume that the plate has the l-periodic heterogeneous structure and that l is sufficiently small compared to the minimum characteristic length dimension of the plate midplane, $l \ll L = \min(L_1, L_2)$, and sufficiently large compared to the maximum plate thickness $h \ll l$. This parameter will be called *the mesostructure length parameter*. We shall assume that $h(\cdot)$ is the l-periodic function of x_1 and independent of x_2 and all material and inertial properties of the plate are also l-periodic functions of x_1, independent of x_2 and even functions of z.

Let us denote periodicity intervals $I(x_1)$ $(I(x_1) \equiv x_1 + (-l/2, l/2)$, $x_1 \in \Lambda_0$, where $\Lambda_0 = \{x_1\colon x_1 \in [0, L_1],\ I(x_1) \subset [0, L_1]\}$. For an arbitrary integrable function f we define the averaging operator on $I(x_1)$ given by

$$\langle f \rangle (x_1, x_2) \equiv l^{-1} \int_{I(x_1)} f(y_1, x_2)\, dy_1, \qquad x_1 \in \Lambda_0. \tag{2.1}$$

For a l-periodic function f in x_1 formula (2.1) leads to $\langle f \rangle (x_2)$.

Throughout the paper subscripts α, β, ... (i, j, ...) run over 1, 2 (1, 2, 3) and indices A, B, ... run over 1, ..., N. Summation convention holds for all aforementioned indices. Let u_i, e_{ij}, s_{ij} stand for displacements, strains and stresses, $w(x_1, x_2, t)$ be a plate deflection, and p^+, p^- be loads in the z-axis direction on the upper and lower boundaries of the plate, respectively, b be the constant body force in the z-axis direction and ρ be the plate mass density. We shall define $c_{\alpha\beta\gamma\delta} := a_{\alpha\beta\gamma\delta} - a_{\alpha\beta 33}\, a_{\gamma\delta 33}(a_{3333})^{-1}$, where a_{ijkl} are components of the elastic modulae tensor and assume that $z = \text{const}$ are material symmetry planes, hence $c_{3\alpha\beta\gamma} = c_{333\gamma} = 0$.

3. Modelling procedure. In this section it will be presented the modelling procedure which is similar to that shown in papers [7, 6]. This procedure will be used to obtain governing equations for uniperiodic plates.

The starting point of considerations are the well known Kirchhoff plate theory assumptions: *the kinematic relations, the strain–displacement equations, the stress-strain relations* (under the plane stress assumption $s_{33} = 0$), *the virtual work principle*. From these relations we shall obtain the partial differential equation of the fourth order. However, for periodic plates this equation involves highly oscillating periodic coefficients. In order to pass to the equations with constant or slowly varying coefficients but retaining the effect of mesostructure length parameter l the additional kinematic assumptions have to be introduced.

Introduce functions, which are l-periodic in x_1 and independent of x_2

$$\mu := \int_{-h/2}^{h/2} \rho\, dz, \quad \vartheta := \int_{-h/2}^{h/2} \rho z^2 \rho\, dz, \quad d_{\alpha\beta\gamma\delta} := \int_{-h/2}^{h/2} z^2 c_{\alpha\beta\gamma\delta}\, dz,$$

describing the following plate properties as a mean mass density per an unit area, a rotational inertia and a bending stiffness, respectively.

The modelling procedure of the presented model is based on the following kinematic assumption.

The main assumption is that the averaged plate deflection $W(\mathbf{x}, t) \equiv = \langle\mu\rangle^{-1}(x_1)\, \langle\mu w\rangle\, (\mathbf{x}, t)$, $\mathbf{x} = (x_1, x_2)$ together with its all derivatives are *slowly varying functions* in x_1, i.e., satisfy conditions of the form $\langle fW\rangle(\mathbf{x}) \cong \langle f\rangle(\mathbf{x})\, W(\mathbf{x})$ for every integrable function f defined on Π. In the sequel W will be referred to as *the plate macrodeflection*. Moreover, it is assumed that the deflection disturbances $v \equiv w - W$ caused by the periodic plate structure are highly oscillating functions in x_1, i. e., satisfy the following conditions: (i) in calculations of averages their values can be neglected comparing to the values of their derivatives, i. e., $\langle (vF)_{,1}\rangle(\mathbf{x}, t) \cong \langle v_{,1}F\rangle(\mathbf{x}, t)$ for every $\mathbf{x} \in \Pi$ and for every slowly varying function F defined on Π; (ii) for every $x_1 \in \Lambda_0$ there exist a periodic function v_{x_1} being a restriction of disturbances v to the closure $\bar{I}(x_1)$ of an interval $I(x_1)$, such that $v(\bar{I}(x_1) \cong v_{x_1}$; (iii) $v(\mathbf{x}, t) \in O(l^2)$, $l\nabla v(\mathbf{x}, t) \in O(l^2)$, $l^2\nabla\nabla v(\mathbf{x}, t) \in O(l^2)$, $\mathbf{x} \in \Pi$, where l is the mesostructure length parameter, here $O(l^2) \to 0$ and $|l^{-2}O(l^2)| \geq c$ for some $c > 0$ with $l \to 0$.

Now in modelling the proper class of disturbances is specified. Denote a plate bending stiffness and a plate mass density per midplane unit area by $B(x_1)$ and $\mu(x_1)$, $x_1 \in \Lambda$, respectively, which are l-periodic functions in x_1 and independent of x_2. we shall formulate the eigenvalue problem for a function $\hat{g}(y_1)$ given by the equation

$$[B(y_1)\,\hat{g}\,(y_1)_{,11}]_{,11} - \mu(y_1)\,\varpi^2\,\hat{g}(y_1) = 0, \qquad y_1 \in I(x_1),\ x_1 \in \Lambda_0, \tag{3.1}$$

and by the periodic boundary conditions on the boundary of an interval $I(x_1)$ together with the continuity conditions inside $I(x_1)$ and the condition $\langle\mu\hat{g}\rangle = 0$. Hence, $\hat{g}$ is a sufficiently smooth solution to this problem. Let $\hat{g}^A$, $A = 1, 2, \ldots$, be a sequence of eigenfunctions defined on $I(x_1)$ and related to the sequence of eigenvalues ϖ_A. Every $\hat{g}^A(y_1)$ is the principal mode of free vibrations of every plate segment in the periodicity interval $I(x_1)$. In the modelling procedure we restrict this sequence to the $N \geq 1$ first eigenfunctions which can be approximated by so called *mesoshape functions* g^A, i. e., $g^A(y_1) \approx \hat{g}^A(y_1)$, $A = 1, \dots, N$, $y_1 \in I(x_1)$. It is assumed that the *mesoshape functions* g^A, $A = 1, \dots, N$, constitute a sequence of sufficiently smooth functions which are linear independent, l-periodic in x_1 and independent of x_2, such that $g^A(y_1) \in O(l^2)$, $l\nabla g^A(y_1) \in O(l^2)$, $l^2\nabla\nabla g^A(y_1) \in O(l^2)$.

In the modelling we assume that the disturbances v can be approximated by $v(y_1, x_2, t) = g^A(y_1)V^A(x_1, x_2, t)$, $y_1 \in I(x_1)$, $x_1 \in \Lambda_0$, $x_2 \in [0, L_2]$, $A = 1, \dots, N$, where $g^A(y_1)$ are the known *mesoshape functions* and $V^A(x_1, x_2, t)$ are extra unknowns being slowly varying functions in x_1 and x_2 which are called *macrointernal variables.*

In most problems analysis will be restricted to the simplest case $N = 1$ in which we take into account only the lowest natural vibration mode related to equation (3.1).

In the sequel it will be shown that for the interval with a non-complicated structure we can assume the mesoshape function in an approximated form of the solution to the eigenvalue problem.

4. Governing equations. After some manipulations from the Kirchhoff plate theory relations and the additional kinematic assumption the governing equations of *the length-scale model* will be derived:

- *Equations of motion*

$$M_{\alpha\beta,\alpha\beta} + \langle\mu\rangle \ddot{W} - \langle\vartheta\rangle \ddot{W}_{,\alpha\alpha} - \underline{\langle\vartheta g^B_{,\alpha}\rangle}\ddot{V}^B_{,\alpha} = \langle p\rangle + b\langle\mu\rangle, \tag{4.1}$$

- *Constitutive equations*

$$\begin{aligned} M_{\alpha\beta} &= \langle d_{\alpha\beta\gamma\delta}\rangle W_{,\gamma\delta} + \langle d_{\alpha\beta\gamma\delta}\, g^B_{,\gamma\delta}\rangle V^B, \\ M^A &= \langle d_{\alpha\beta\gamma\delta}\, g^A_{,\alpha\beta}\rangle W_{,\gamma\delta} + \langle d_{\alpha\beta\gamma\delta}\, g^A_{,\alpha\beta}\, g^B_{,\gamma\delta}\rangle V^B, \end{aligned} \tag{4.2}$$

- *Evolution equations*

$$M^A + \underline{\langle\mu\, g^A g^B\rangle}\ddot{V}^B + \underline{\langle\vartheta g^A_{,\alpha}\rangle}\ddot{W}_{,\alpha} + \underline{\langle\vartheta\, g^A_{,\alpha}\, g^B_{,\alpha}\rangle}\ddot{V}^B = \underline{\langle p g^A\rangle}, \tag{4.3}$$

where mesoshape functions g^A, $A = 1, \dots, N$, are functions only in x_1.

The above equations are the basis for investigations of overall behaviour of uniperiodic plates. The underlined terms depend on the mesostructure length parameter l. Moreover, these equations involve averaged coefficients (in brackets $\langle\rangle$) which are constant, except terms $\langle p\rangle$, $\langle pg^A\rangle$ which can be slowly varying functions in x_1 and x_2. Functions $W(x_1, x_2, t)$, $V^A(x_1, x_2, t)$, $A = 1, \dots, N$, are the basic unknowns; which have to be slowly varying functions both in x_1 and x_2. The function W is called a macrodeflection and the functions V^A are called macrointernal variables, because boundary conditions for these functions should not be defined.

It is easy to see that to derive the governing equations (4.1)−(4.3) we have previously to obtain the mesoshape functions g^A, $A = 1, \dots, N$, for every periodic plate under consideration as approximated solutions to the eigenvalue problem for equation of the form (3.1). In most cases we restrict considerations to a small number N of mode shapes and in this contribution we assume that $N = 1$; hence, denote $g \equiv g^1$.

5. Analysis: an example of a periodic plate band.

5.1. Free vibrations of a periodic plate band. In order to evaluate differences between applying of an exact $\hat{g}$ or approximated g form of mesoshape functions free vibrations of a periodic plate band will be considered. For this purpose equations (4.1)–(4.3) will be applied. It will be assumed that body forces b and loads p are neglected. Let us consider a simply supported on the opposite edges plate band. It is made of an isotropic periodically varying piece-wise homogeneous material. The plate thickness h is a periodic function. For assumed mesoshape function g it can be proved that $\langle \vartheta g_{,1}\rangle = 0$. Denote $x \equiv x_1$, $L \equiv L_1$ and $V \equiv V^1$ as well as

$$B \equiv \frac{Eh^3}{12(1-\nu^2)}, \qquad D^1_{11} \equiv \langle d_{1111}\, g_{,11}\rangle, \qquad D^{11} \equiv \left\langle d_{1111}(g_{,11})^2\right\rangle,$$

$$m \equiv \langle \mu\rangle, \qquad m^{11} \equiv l^{-4}\left\langle \mu(g)^2\right\rangle, \qquad j \equiv \langle \vartheta\rangle, \qquad j^{11} \equiv l^{-2}\left\langle \vartheta(g_{,1})^2\right\rangle,$$

where E is the plate Young modulus, ν is the plate Poisson ratio, h is the periodic plate thickness. Substituting (4.2) to (4.1) and (4.3) we arrive at

$$\langle B\rangle W_{,1111} + m\ddot{W} - j\,\ddot{W}_{,11} + D^1_{11}V_{,11} = 0,$$

$$D^1_{11}\, W_{,11} + D^{11}V + l^2(l^2 m^{11} + j^{11})\ddot{V} = 0. \tag{5.1}$$

Introduce the wave number $k \equiv 2\pi/L$. Solutions to equations (5.1) will be assumed in the form satisfying boundary conditions for a simply supported plate band:

$$W(x,t) = A_W \sin(kx)\cos(\omega t), \quad V(x,t) = A_V \sin(kx)\cos(\omega t), \tag{5.2}$$

where A_W, A_V are amplitudes. Substituting these solutions to (5.1) we obtain the system of linear algebraic equations for amplitudes A_W, A_V

$$\begin{bmatrix} \langle B\rangle k^4 - \omega^2(m + jk^2) & -D^1_{11}k^2 \\ -D^1_{11}k^2 & D^{11} - \omega^2 l^2(l^2 m^{11} + j^{11}) \end{bmatrix} \begin{Bmatrix} A_W \\ A_V \end{Bmatrix} = \{0\}.$$

After some manipulations we arrive at formulae for resonance frequencies ω_1 and ω_2 called *a macro-resonance* and *a meso-resonance frequency*, respectively:

$$(\omega_{1,2})^2 \equiv \tfrac{1}{2}[(m + jk^2)l^2(l^2m^{11} + j^{11})]^{-1}\{\langle B\rangle k^4 l^2(l^2 m^{11} + j^{11}) + (m + jk^2)D^{11} \mp$$

$$\mp\sqrt{[\langle B\rangle k^4 l^2(l^2m^{11} + j^{11}) - (m + jk^2)D^{11}]^2 + 4(m + jk^2)l^2(l^2m^{11} + j^{11})(D^1_{11}k^2)^2}\}, \tag{5.3}$$

which take into account the effect of the mesostructure length parameter l.

5.2. Analysis of the eigenvalue problem for a plate band having a periodically varying piece-wise constant thickness

Let us consider a plate band having a periodically varying piece-wise constant thickness h, simply supported on the opposite edges and made of an isotropic homogeneous material. The periodicity interval $I_l \equiv [-l/2, l/2]$ is

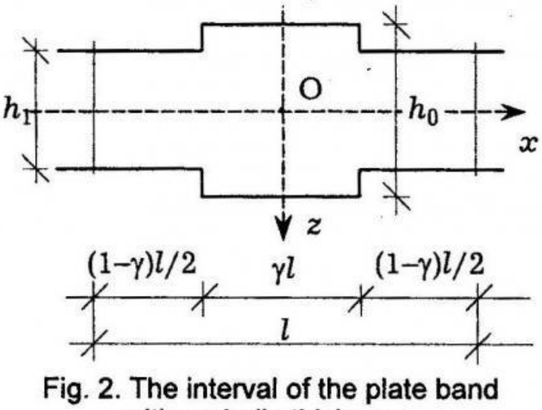

Fig. 2. The interval of the plate band with periodic thickness.

shown in Fig. 2. It can be treated as a plate band with the span l. The plate thickness h is assumed as

$$h(x)=\begin{cases}h_0 & \text{if } x\in(-\gamma l/2,\gamma l/2),\\ h_1=\eta h_0 & \text{if } x\in[-l/2,-\gamma l/2]\cup[\gamma l/2,l/2], \quad \eta\in[0,1].\end{cases} \tag{5.4}$$

We introduce the wave number α and denote $\lambda\equiv\alpha l$. In order to obtain eigenfunctions for the interval $I(x_1)$ we have to solve the eigenvalue problem (3.1) which takes the form $[B(x)\hat{g}(x)_{,11}]_{,11}-\mu(x)\lambda^2\hat{g}(x)=0$, with conditions: $\hat{g}$ are l-periodic functions, $\langle\mu\hat{g}\rangle=0$. Using the Krylov – Prager functions

$$S(\alpha x)=\tfrac{1}{2}[\cosh(\alpha x)+\cos(\alpha x)],\qquad U(\alpha x)=\tfrac{1}{2}[\cosh(\alpha x)-\cos(\alpha x)],$$

$$T(\alpha x)=\tfrac{1}{2}[\sinh(\alpha x)+\sin(\alpha x)],\qquad Q(\alpha x)=\tfrac{1}{2}[\sinh(\alpha x)-\sin(\alpha x)],$$

and denoting

$$\Gamma_1(\lambda)\equiv T(\tfrac{1}{2}\gamma\lambda)+\eta^{\frac{3}{2}}\{S(\tfrac{1}{2}\gamma\lambda)T[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+\eta^{\frac{1}{2}}Q(\tfrac{1}{2}\gamma\lambda)U[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+$$
$$+\eta^{-2}U(\tfrac{1}{2}\gamma\lambda)Q[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+\eta^{-\frac{3}{2}}T(\tfrac{1}{2}\gamma\lambda)[S[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]-1]\},$$

$$\Xi_1(\lambda)\equiv Q(\tfrac{1}{2}\gamma\lambda)+\eta^{\frac{3}{2}}\{U(\tfrac{1}{2}\gamma\lambda)T[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+\eta^{\frac{1}{2}}T(\tfrac{1}{2}\gamma\lambda)U[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+$$
$$+\eta^{-2}S(\tfrac{1}{2}\gamma\lambda)Q[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+\eta^{-\frac{3}{2}}Q(\tfrac{1}{2}\gamma\lambda)[S[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]-1]\},$$

$$\Gamma_2(\lambda)\equiv S(\tfrac{1}{2}\gamma\lambda)Q[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+\eta^{\frac{1}{2}}Q(\tfrac{1}{2}\gamma\lambda)S[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+$$
$$+\eta^{-2}U(\tfrac{1}{2}\gamma\lambda)T[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+\eta^{-\frac{3}{2}}T(\tfrac{1}{2}\gamma\lambda)U[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}],$$

$$\Xi_2(\lambda)\equiv U(\tfrac{1}{2}\gamma\lambda)Q[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+\eta^{\frac{1}{2}}T(\tfrac{1}{2}\gamma\lambda)S[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+$$
$$+\eta^{-2}S(\tfrac{1}{2}\gamma\lambda)T[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}]+\eta^{-\frac{3}{2}}Q(\tfrac{1}{2}\gamma\lambda)U[\tfrac{1}{2}(1-\gamma)\lambda\eta^{-\frac{1}{2}}],$$

the solution to eigenvalue problem is looked for in the form

$$\hat{g}(x)=l^2A(\lambda)S(\lambda|x|/l)+U(\lambda|x|/l),\qquad \text{if } |x|\le\tfrac{1}{2}\gamma l,$$

$$\hat{g}(x)=l^2\{A(\lambda)\{S(\tfrac{1}{2}\gamma\lambda)S[\lambda(|x|/l-\tfrac{1}{2}\gamma)\eta^{-\frac{1}{2}}]+\eta^{\frac{1}{2}}Q(\tfrac{1}{2}\gamma\lambda)T[\lambda(|x|/l-\tfrac{1}{2}\gamma)\eta^{-\frac{1}{2}}]+$$
$$+\eta^{-2}U(\tfrac{1}{2}\gamma\lambda)U[\lambda(|x|/l-\tfrac{1}{2}\gamma)\eta^{-\frac{1}{2}}]+\eta^{-\frac{3}{2}}T(\tfrac{1}{2}\gamma\lambda)Q[\lambda(|x|/l-\tfrac{1}{2}\gamma)\eta^{-\frac{1}{2}}]\}+$$
$$+U(\tfrac{1}{2}\gamma\lambda)S[\lambda(|x|/l-\tfrac{1}{2}\gamma)\eta^{-\frac{1}{2}}]+\eta^{\frac{1}{2}}T(\tfrac{1}{2}\gamma\lambda)T[\lambda(|x|/l-\tfrac{1}{2}\gamma)\eta^{-\frac{1}{2}}]+ \tag{5.5}$$
$$+\eta^{-2}S(\tfrac{1}{2}\gamma\lambda)U[\lambda(|x|/l-\tfrac{1}{2}\gamma)\eta^{-\frac{1}{2}}]+\eta^{-\frac{3}{2}}Q(\tfrac{1}{2}\gamma\lambda)Q[\lambda(|x|/l-\tfrac{1}{2}\gamma)\eta^{-\frac{1}{2}}]\},\ \text{if } |x|\in(\tfrac{1}{2}\gamma l,\tfrac{1}{2}l],$$

where $A(\lambda)\equiv-C\Xi_1(\lambda)\Gamma_1(\lambda)^{-1}$, C is a constant. Restricting our considerations to symmetric vibrations the equation for the eigenvalue λ takes the form

$$\Gamma_1(\lambda)\ \Xi_2(\lambda)-\Gamma_2(\lambda)\ \Xi_1(\lambda)=0.$$

From the above equation we can derive eigenvalues λ_A, $A=1,2,\ldots$. Our analysis is restricted to $A=1$. Hence, we obtain the smallest eigenvalue λ dependent on the parameter η and the exact form of the mesoshape functions $\hat{g}$ related to this eigenvalue is defined by (5.6).

6. Numerical results. In this section it will be shown that the mesoshape function g for the interval in Fig. 2 related to the eigenvalue λ (for $A = 1$) can be assumed in the approximated form

$$g(x) = l^2 [\cos(2\pi x/l) + c], \qquad x \in [-l/2,\ l/2], \tag{6.1}$$

where the constant c derived from the condition $\langle \mu g \rangle = 0$ is $c = \langle \mu \rangle^{-1} \times \langle \mu \cos(2\pi x/l) \rangle$.

In order to evaluate differences between results obtained by using exact (5.5) and approximated form (6.1) of mesoshape function resonance frequencies of the plate band with periodic thickness h given by (5.4) will be analysed. Taking into account (5.3) we denote resonance frequencies obtained by using the mesoshape function in the approximated form (6.1) by $\tilde{\omega}$ and those obtained by using the mesoshape function in the exact form (5.5) by ω. Introduce dimensionless coefficients:

$$\eta_0 \equiv h_0/l, \quad q \equiv l/L, \quad \Omega_1 \equiv \tilde{\omega}_1/\omega_1, \quad \Omega_2 \equiv \tilde{\omega}_2/\omega_2,$$

where η_0 describes the plate thickness, q is called the dimensionless mesostructure parameter and Ω_1, Ω_2 are the ratios of macro- and meso-resonance frequency, respectively. Diagram of Ω_2, Ω_2 versus the parameter $\eta \in [0.7, 1.0]$ is presented in Fig. 3. These diagrams are made for $\eta_0 = 0.1$, $q = 0.01$.

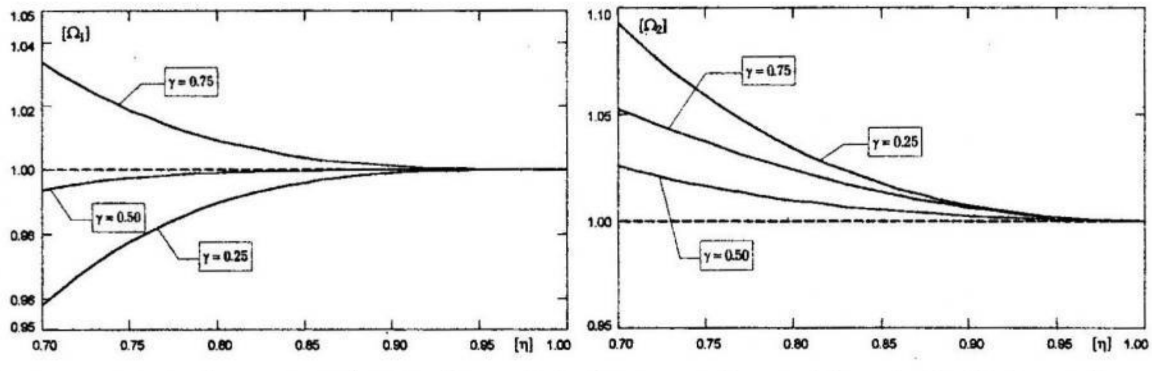

Fig. 3. Diagrams of ratios Ω_1 and Ω_2 for a plate band with periodic thickness h.

Analysing obtained results the following conclusions can be formulated:

- Differences between macro-resonance frequencies obtained by using the exact (5.5) or the approximated (6.1) form of mesoshape function are small and they are smaller than 5% for plates in which differences of thickness inside the periodicity interval are smaller than 30% (i. e. $h_0 - h_1 \leq 0.3h_0$).
- For the above plates differences between meso-resonance frequencies obtained by using the exact or the approximated form of mesoshape function are smaller than 10%.

7. Conclusions. The length-scale model for plates with periodic structure along one direction and constant properties along the perpendicular direction have been presented in the contribution. These refined models, formulated in the framework of the averaged length-scale theory of periodic bodies [17], were applied to many dynamic problems of periodic structures in series of papers [2, 4, 6–7, 9, 12–16] aforementioned in Sec. 1. These models are physically reasonable and simple enough to be applied in the analysis of engineering problems.

The presented modelling procedure for periodic plates leads to a system of differential equations with constant coefficients for the macrodeflection W and the macrointernal variables V^A. The governing equations ((4.1)–(4.3)) describe the length-scale effect on the plate behaviour by some coefficients (underlined in these equations) dependent on the mesostructure length parameter l. The length-scale effect on the dynamic body behaviour is described by the extra unknowns V^A, $A = 1, \dots, N$, called macrointernal variables and additional mode-shape functions g^A, which are called mesoshape functions. The macrointernal variables V^A are governed by ordinary differential equations involving time derivatives. Moreover, mesoshape functions g^A describe oscillations of displacements inside periodicity intervals. These functions should be obtained as properly chosen approximations of solutions to eigenvalue problems for natural vibrations of separated periodicity intervals what was shown in Sec. 3.

Mesoshape functions obtained in Sec. 5 for the special case of plate band were used to analyse free vibrations for such plate. Presented there results make it possible to evaluate differences between resonance frequencies obtained by using the approximated and the exact form of mesoshape function. Analysing these results we can formulate the following general conclusions:

1° For many special problems the mesoshape functions g^A, $A = 1, \dots, N$, can be assumed as approximated solutions to the eigenvalue problem for periodicity interval, what is sufficient from the computational point of view.

2° Plates with a periodic structure caused only by periodically varying thickness can be investigated by using the approximated form of mesoshape function if differences between values of the thickness inside the periodicity interval are relatively small, i. e., for the parameter $\eta \geq 0.7$; for $\eta < 0.7$ it should be used the exact form of mesoshape function.

Summarising, the analytical exact solutions $\hat{g}^A$ to the eigenvalue problems (3.1) can be obtained only for plates with intervals whose structure is rather not too complicated. In most cases instead of the exact solutions to eigenvalue problems we have to look for an approximate form of mesoshape functions.

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1. *Bakhvalov N. S., Panasenko G. P.* Osrednienie processov v periodiceskich sredach. (in Russian) – Moscow: Nauka, 1984.
2. *Baron E., Woźniak C.* On the micro-dynamics of composite plates // Arch. Appl. Mech. – 1995. – **66**. – P. 126–133.

3. *Caillerie D.* Thin elastic and periodic plates // Math. Meth. in the Appl. Sci. – 1984. – **6**. – P. 159–191.
4. *Cielecka I.* On the continuum the dynamic behaviour of certain composite lattice-type structures // J. Theor. Appl. Mech. – 1995. – **33**. – P. 351–360.
5. *Duvaut G., Metellus A. M.* Homogéneisation d'une plaque mince en flexion des structure périodique et symmetrique. (in French) // Paris C.R. Acad. Sci. – 1976. – **283(A)**. – P. 947–950.
6. *Jędrysiak J.* On dynamics of thin plates with a periodic structure // Eng. Trans. – 1998. – **46**. – P. 73–87.
7. *Jędrysiak J., Woźniak C.* On the elastodynamics of thin microperiodic plates // J. Theor. Appl. Mech. – 1995. – **33**. – P. 337–349.
8. *Kohn R.V., Vogelius M.* A new model for thin plates with rapidly varying thickness // Int. J. Solids Structures – 1984. – **20**. – P. 333–350.
9. *Konieczny S., Woźniak M.* On the wave propagation in fibre-reinforced composites // J. Theor. Appl. Mech. – 1995. – **33**. – P. 375–384.
10. *Lewicski T.* Homogenizing stiffnesses of plates with periodic structure // Int. J. Solids Structures – 1992. – **21**. – P. 309–326.
11. *Maewal A.* Construction of models of dispersive elastodynamic behaviour of periodic composites a computational approach // Comp. Meth. Appl. Mech. Engng. – 1986. – **57**. – P. 191–205.
12. *Matysiak S. J., Nagórko W.* On the wave propagation in periodically laminated composites // Bull. Polon. Acad. Sci., Tech. Sci. – 1995. – **43**. – P. 1–12.
13. *Mazur-Śniady K.* Macro-dynamics of micro-periodic elastic beams // J. Theor. Appl. Mech. – 1993. – **31**. – P. 781–793.
14. *Michalak B., Woźniak C., Woźniak M.* The dynamic modelling of elastic wavy-plates // Arch. Appl. Mech. – 1995. – **66**. – P. 177–186.
15. *Wągrowska M., Woźniak C.* Macro-modelling of dynamic problems for visco-elastic composite materials // Int. J. Engng. Sci. – 1996. – **34**. – P. 923–932.
16. *Wierzbicki E.* Nonlinear macro-micro dynamics of laminated structures // J. Theor. Appl. Mech. – 1995. – **33**. – P. 291–307.
17. *Woźniak C.* Internal variables in dynamics of composite solids with periodic microstructure // Arch. Mech. – 1997. – **49**. – P. 421–441.

Ярослав Єндрисяк

ПРО ФУНКЦІЇ МЕЗОФОРМИ В СТРУКТУРНІЙ ДИНАМІЦІ ТОНКИХ ПЕРІОДИЧНИХ ПЛАСТИН

Розглянуто проблему вибору мезоформної функції в структурній динаміці тонких пластин з періодичною структурою у площинах паралельних до серединної поверхні. Запропонована модель з ефектом масштабу дає змогу врахувати вплив величини комірки періодичності на поведінку пластини. У процесі моделювання використано поняття функцій мезоформи. Ці функції описують осциляцію всередині комірки періодичності і є розв'язками задачі про власні коливання у випадку природних коливань окремо взятої комірки періодичності з урахуванням періодичних крайових умов. На прикладі показано, що здебільшого можна використовувати наближений власний розв'язок як функцію мезоформи.

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