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ON A CHOICE OF MICRO-SHAPE FUNCTIONS FOR A DYNAMIC BEHAVIOUR OF WAVY PLATES

1. Introduction. The subject of this paper is a dynamic analysis of the wavy-plate shown on Fig. 1. It is assumed that: $x_3 = z(\mathbf{x})$, $\mathbf{x} = (x_1, x_2) \in \Pi$, $\delta/l \ll 1$, $\delta/R \ll 1$, $z(\cdot)$ – Δ-periodic function, $l \equiv \sqrt{l_1^2 + l_2^2}$, $\Delta := (0,l_1)\times(0,l_2)$, $l \ll L_\Pi$, L_Π – smallest characteristic length dimension of Π.

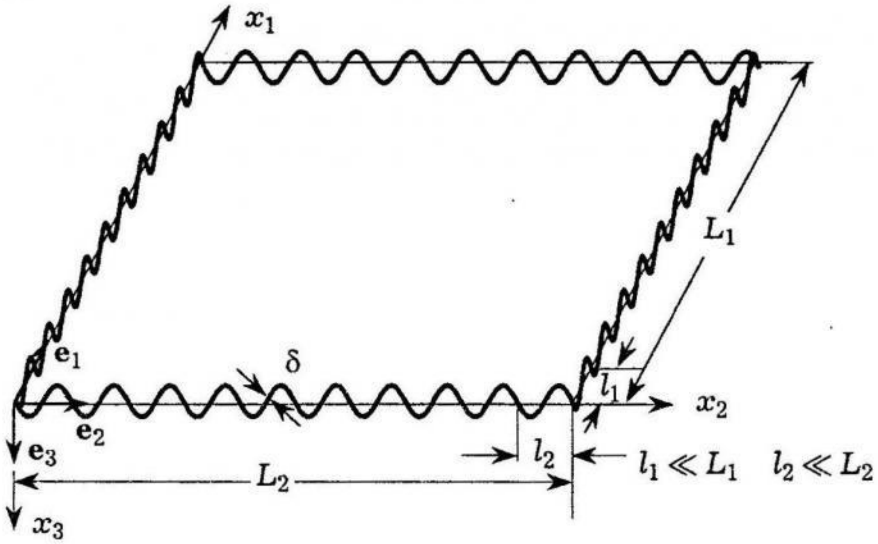

Fig. 1.

The paper is a continuation of previous investigations given in [1].

Objective:

(i) to obtain the governing equations of the averaged theory of a wavy-plates for the different form of the micro shape functions for in-plane and out of plane of the plate displacements;

(ii) to obtain the micro-shape functions from the solution of eigenvalue problem for a periodic cell Δ with the use of a finite element method;

(iii) to compare the obtained results from mezostructural theory to the results from the finite elements method, the homogenized theory and the orthotropic plates model.

Direct description of the wavy-plate is the same as the one given in [1].

2. Modelling approach.

1. *Averaging operator*. Let $f(\mathbf{z})$ be an integrable function defined on Π. In the sequel, we shall use the denotation

$$\langle f \rangle (\mathbf{x}) = \frac{1}{l_1 l_2} \int_{\Delta(\mathbf{x})} f(z)\, dz_1\, dz_2 \,. \tag{1}$$

2. *Long wave approximation.* The function $F(\mathbf{x},t)$, $\mathbf{x}\in\Pi$, will be called the regular macro function if for an arbitrary $z\in\Pi$ and every $\mathbf{x} \in \Delta(z)$, the following formula holds

$$(\forall f) \quad [\langle f\, F \rangle (\mathbf{x}) \cong \langle f \rangle F(\mathbf{x})] \tag{2}$$

for $F \in \{F,\ \nabla F,\ \dot{F},\ ....\}$.

3. *Basic-Kinematics Hypothesis*

We restrict consideration to the motion, the length of waves is large compared to the mezostructure length parameter l. The displacement field $u_i(\mathbf{x},t)$ of the wavy-plate can be approximated by

$$u_\alpha(\mathbf{x},t) = U_\alpha(\mathbf{x},t) + h(\mathbf{x}) V_\alpha(\mathbf{x},t), \qquad \mathbf{x} = (x^1, x^2) \in \Pi, \qquad t \geq 0,$$

$$u_3(\mathbf{x},t) = U_3(\mathbf{x},t) + g(\mathbf{x})\, V_3(\mathbf{x},t), \tag{3}$$

where : $U_i(\cdot\,, t)$, $V_i(\cdot\,, t)$ – regular Δ – macro functions (basic unknowns).

The macrodisplacements $U_i(\mathbf{x},t) = \langle u_i \rangle (\mathbf{x},t)$ describe the averaged motion of the wavy-plates.

The functions $h(\cdot) V_\alpha(\cdot, t)$ and $g(\cdot) V_3(\cdot, t)$ describe the local displacement oscillations, caused by mezostructure of the periodic plate.

3. A choice of micro-shape functions. Functions $h(\cdot)$ and. $g(\cdot)$ will be referred to as the micro-shape functions and are obtained as an approximate solution to the eigenvalue problem on a periodic cell Δ together with periodic boundary conditions. The choice of these functions will be determined by the analysis of free vibrations of a periodic cell Δ with the use of a finite element method.

The form of the micro-shape functions is obtained as an eigenvibrations form of a periodic cell Δ.

The forms of the eigenvibrations of the periodic cell Δ: $z = f \sin(2\pi x/l)$ (where $f/l = 0.1$ and $\delta/l = 0.1$) together with periodic boundary conditions can be approximated by functions:

Vibrations in plane $h = l^2 \sin(2\pi x/l)$, vibrations out of plane $g = l^2 \sin(4\pi x/l)$.

4. Averaged description: mezo-structural theory (MST). The aforementioned modelling hypotheses (1)–(3) lead from the direct description of the wavy-plate to a system of equations in macrodisplacements U_i and correctors V_i constituting the governing equations of the averaged theory of wavy-plates.

The equations of motion presented below in the coordinate form are

$$M^{i\alpha\beta}_{,\alpha\beta} - M^{i\alpha}_{,\alpha} - N^{i\alpha}_{,\alpha} + N^{i} + \langle\tilde{\rho}\rangle \ddot{U}^{i} = \tilde{p}^{i},$$

$$K^{\gamma} + L^{\gamma} + \langle\tilde{\rho}\, h\, h\rangle \ddot{V}^{\gamma} = \langle\tilde{p}^{\gamma} h\rangle, \qquad K^{3} + L^{3} + \langle\tilde{\rho}\, g\, g\rangle \ddot{V}^{3} = \langle\tilde{p}^{3} g\rangle. \tag{4}$$

Constitutive equations have a form

$$\begin{aligned}
N^{i\alpha} &= D^{i\alpha|j\beta} U_{j,\beta} + H^{i\alpha|\mu} V_{\mu} + H^{i\alpha|3} V_{3},\\
N^{i} &= D^{i|j\beta} U_{j,\beta} + C^{i|\mu} V_{\mu} + C^{i|3} V_{3},\\
K^{\alpha} &= H^{\alpha|j\beta} U_{j,\beta} + H^{\alpha|\mu} V_{\mu} + H^{\alpha|3} V_{3},\\
K^{3} &= H^{3|j\beta} U_{j,\beta} + H^{3|\mu} V_{\mu} + H^{3|3} V_{3},\\
M^{i\alpha\beta} &= B^{i\alpha\beta|j\gamma\delta} U_{j,\gamma\beta} - B^{i\alpha\beta|j\gamma} U_{j,\gamma} + B^{i\alpha\beta|\mu} V_{\mu} + B^{i\alpha\beta|3} V_{3},\\
M^{i|\alpha} &= -B^{i\alpha|j\gamma\delta} U_{j,\gamma\delta} + B^{i\alpha|j\gamma} U_{j,\gamma} - B^{i\alpha|\mu} V_{\mu} - B^{i\alpha|3} V_{3},\\
L^{\alpha} &= B^{\alpha|j\gamma\delta} U_{j,\gamma\beta} - B^{\alpha|j\tau} U_{j,\tau} + B^{\alpha|\mu} V_{\mu} + B^{\alpha|3} V_{3},\\
L^{3} &= B^{3|j\gamma\delta} U_{j,\gamma\delta} - B^{3|j\tau} U_{j,\tau} + B^{3|\mu} V_{\mu} + B^{3|3} V_{3},
\end{aligned} \tag{5}$$

where we have denoted

$$D^{i\alpha|j\beta} \equiv D\langle H^{\delta\alpha\gamma\beta} G^{i}_{\gamma} G^{j}{}_{\gamma} \sqrt{a}\rangle, \qquad H^{i\alpha|\mu} = H^{\mu|i\alpha} \equiv D\langle H^{\delta\alpha\gamma\beta} G^{i}_{\delta} G^{\mu}_{\gamma} h_{,\beta} \sqrt{a}\rangle, \tag{6}$$

$$H^{i\alpha|3} = H^{3|i\alpha} \equiv D\langle H^{\delta\alpha\gamma\beta} G^{i}_{\delta} G^{3}_{\gamma} g_{,\beta} \sqrt{a}\rangle,$$

$$D^{i|j\beta} \equiv D\left\langle H^{\alpha\delta\gamma\beta} \left\{ {\lambda \atop \alpha\delta} \right\} G^{i}_{\lambda} G^{j}_{\gamma} \sqrt{a}\right\rangle, \qquad C^{i|\mu} \equiv D\left\langle H^{\alpha\beta\gamma\delta} \left\{ {\lambda \atop \alpha\beta} \right\} G^{i}_{\lambda} G^{\mu}_{\gamma} h_{,\delta} \sqrt{a}\right\rangle,$$

$$C^{i|3} \equiv D\left\langle H^{\alpha\beta\gamma\delta} \left\{ {\lambda \atop \alpha\beta} \right\} G^{i}_{\lambda} G^{3}_{\gamma} g_{,\delta} \sqrt{a}\right\rangle, \qquad H^{\alpha|\mu} \equiv D\langle H^{\tau\beta\gamma\delta} G^{\alpha}_{\tau} G^{\mu}_{\gamma} h_{,\delta} h_{,\beta} \sqrt{a}\rangle,$$

$$H^{\alpha|3} = H^{3|\alpha} \equiv D\langle H^{\tau\beta\gamma\delta} G^{\alpha}_{\tau} G^{3}_{\gamma} g_{,\delta} h_{,\beta} \sqrt{a}\rangle,$$

$$B^{i\alpha\beta|j\gamma\delta} \equiv B\langle H^{\alpha\beta\gamma\delta} n^{i} n^{j} \sqrt{a}\rangle, \qquad B^{i\alpha\beta|\mu} = B^{\mu|i\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} h_{|\gamma\delta} n^{i} n^{\mu} \sqrt{a}\rangle,$$

$$B^{i\alpha\beta|3} = B^{3|i\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\gamma\delta} n^{i} n^{3} \sqrt{a}\rangle, \quad B^{i\alpha\beta|j\gamma} = B^{j\gamma|i\alpha\beta} \equiv B\left\langle H^{\alpha\beta\mu\delta} \left\{ {\gamma \atop \mu\delta} \right\} n^{i} n^{j} \sqrt{a}\right\rangle,$$

$$B^{i\alpha|j\gamma} \equiv B\left\langle H^{\mu\delta\tau\nu} \left\{ {\alpha \atop \mu\delta} \right\} \left\{ {\gamma \atop \tau\nu} \right\} n^{i} n^{j} \sqrt{a}\right\rangle, \qquad B^{i\alpha|\mu} = B^{\mu|i\alpha} \equiv B\left\langle H^{\beta\tau\gamma\delta} \left\{ {\alpha \atop \beta\tau} \right\} n^{i} n^{\mu} h_{|\gamma\delta} \sqrt{a}\right\rangle,$$

$$B^{i\alpha|3} = B^{3|i\alpha} \equiv B\left\langle H^{\beta\tau\gamma\delta} \left\{ {\alpha \atop \beta\tau} \right\} n^{i} n^{3} g_{|\gamma\delta} \sqrt{a}\right\rangle, \qquad B^{\alpha|\mu} \equiv B\langle H^{\gamma\beta\tau\delta} h_{|\gamma\beta} h_{|\tau\delta} n^{\alpha} n^{\mu} \sqrt{a}\rangle,$$

$$B^{\alpha|3} = B^{3|\alpha} \equiv B\langle H^{\gamma\beta\tau\delta} h_{|\gamma\beta} g_{|\tau\delta} n^{\alpha} n^{3} \sqrt{a}\rangle, \qquad B^{3|3} \equiv B\langle H^{\alpha\beta\gamma\delta} g_{|\alpha\beta} g_{|\gamma\delta} n^{3} n^{3} \sqrt{a}\rangle.$$

5. Applications. To compare the mezo-structural theory (MST), homogenized theory (HT), the orthotropic plate model and the finite element method we shall investigate the simple problem of the cylindrical bending of a rectangular wavy-plate. In this case the basic unknowns U_i and V_i depend only on arguments x_2 and t.

1. *Mezo-structural theory*

In this case, neglecting external loading, the system equations of motion will take the form

$$M^{122}_{,22} - M^{12}_{,2} - N^{12}_{,2} + N^1 + \langle\tilde{\rho}\rangle\ddot{U}^1 = 0,$$

$$M^{222}_{,22} - M^{22}_{,2} - N^{22}_{,2} + N^2 + \langle\tilde{\rho}\rangle\ddot{U}^2 = 0,$$

$$M^{322}_{,22} - M^{32}_{,2} - N^{32}_{,2} + N^3 + \langle\tilde{\rho}\rangle\ddot{U}^3 = 0, \tag{7}$$

$$K^1 + L^1\langle\tilde{\rho}hh\rangle\ddot{V}^1 = 0, \qquad K^2 + L^2 + \langle\tilde{\rho}hh\rangle\ddot{V}^2 = 0,$$

$$K^3 + L^3 + \langle\tilde{\rho}gg\rangle\ddot{V}^3 = 0.$$

After substituting the right-hand sides of Eqs. (5) into Eqs. (7), we obtain the system of equations for $U_i = U_i(\mathbf{x},t)$ and $V_i = V_i(\mathbf{x},t)$.

Let the wavy-plate midsurface be defined by $z = f\sin(2\pi x^2/l)$ and the micro-shape functions by $h = l^2\sin(2\pi x^2/l)$, $g = l^2\sin(4\pi x^2/l)$. Let us restrict these considerations to the analysis of free vibrations for the unbounded wavy-plate. In this case, we shall look for solutions of Eqs. (7) in the form

$$U_1 = 0, \qquad U_2 = A_2\sin(kx_2)\cos(\omega_2 t), \qquad U_3 = A_3\sin(kx_2)\cos(\omega t),$$

$$V_1 = 0, \qquad V_2 = C_2\cos(kx_2)\cos(\omega t), \qquad V_3 = C_3\cos(kx_2)\cos(\omega t), \tag{8}$$

$k := \pi/L$ is the wavenumber, L being the vibration wavelength ($L \gg l$). Substituting the right-hand of Eqs. (8) into Eqs. (7), we obtain non-trivial solutions only if

$$\begin{vmatrix} (\omega)^2\langle\tilde{\rho}\rangle - C_{33} & C_{35} & C_{36} \\ C_{53} & (\omega)^2\langle\tilde{\rho}hh\rangle - C_{55} & C_{56} \\ C_{63} & C_{65} & (\omega)^2\langle\tilde{\rho}gg\rangle - C_{66} \end{vmatrix} = 0, \tag{9}$$

where we have denoted

$$C_{33} \equiv B\langle H^{2222}(n^3)^2\sqrt{a}\rangle k^4 + \left[B\left\langle H^{2222}\left(\left\{ {2 \atop 22} \right\} n^3\right)^2\sqrt{a}\right\rangle + D\langle H^{2222}(G_2^3)^2\sqrt{a}\rangle\right] k^2,$$

$$C_{35} = C_{53} \equiv \left[B\left\langle H^{2222}\left\{ {2 \atop 22} \right\} n^2 n^3 h_{,22}\sqrt{a}\right\rangle - D\langle H^{2222}G_2^2 G_2^3 h_{,2}\sqrt{a}\rangle\right] k,$$

$$C_{36} = C_{63} \equiv \left[B\left\langle H^{2222}\left\{ {2 \atop 22} \right\} n^3 n^3 g_{,22}\sqrt{a}\right\rangle - D\langle H^{2222}G_2^3 G_2^3 g_{,2}\sqrt{a}\rangle\right] k, \tag{10}$$

$$C_{55} \equiv D\langle H^{2222}(G_2^2 h_{,2})^2 \sqrt{a}\rangle + B\langle H^{2222}(n^2 h_{,22})^2 \sqrt{a}\rangle ,$$

$$C_{56} = C_{65} \equiv -B\langle H^{2222} n^2 n^3 h_{,22}\, g_{,22} \sqrt{a}\rangle - D\langle H^{2222} G_2^2\, G_2^3\, h_{,2}\, g_{,2} \sqrt{a}\rangle ,$$

$$C_{66} \equiv D\langle H^{2222}(G_2^3\, g_{,2})^2 \sqrt{a}\rangle + B\langle H^{2222}(n^3\, g_{,22})^2 \sqrt{a}\rangle .$$

2. *Homogenized theory (HT).*

The homogenized model of dynamics of the wavy-plate can be derived from Eqs. (7)–(10) by the asymptotic approximation in which the mezostructure of the wavy-plate is scaled down $l \to 0$. Keeping in mind that $\delta/l = \text{const}$, we shall neglect mezoinertial terms $\langle \tilde{\rho} hh\rangle \to 0$, $\langle \tilde{\rho} gg\rangle \to 0$, and we can eliminate correctors V_i in Eqs. (7). Now, formula (9) leads to

$$\langle \tilde{\rho}\rangle (\omega)^2 = C_{33} - \frac{C_{66}(C_{35})^2 + C_{55}(C_{36})^2 + 2C_{35}C_{36}C_{56}}{C_{55}C_{66} - (C_{56})^2} . \tag{11}$$

3. *The orthotropic plate model*

Let us restrict these considerations to the analysis of transverse vibrations of orthotropic plates. In this case, equation of motion has a form [2]

$$B_{22}\, U_{3,2222} + \frac{\partial^2}{\partial t^2}(\tilde{\rho}\, U_3 - J_1\, U_{3,22}) = 0 , \tag{12}$$

where [2]

$$B_{22} = B \frac{1}{1 + (\pi f / l)^2} , \qquad J_1 = \frac{1}{l} \int_s (z^2 + x^2)\rho\, ds . \tag{13}$$

We shall look for a solution of Eqs. (12) in the form $U_3 = A_3 \sin(kx_2) \times$ $\times \cos(\omega t)$. For the free vibration frequency, we obtain the following expression

$$(\omega)^2 = \frac{B_{22}(k)^4}{\tilde{\rho} + J_1(k)^2} . \tag{14}$$

4. *The finite element method*

Now we shall look for a solution of the free vibrations for the simply supported wavy-plate with the use of the finite element method. The span of the wavy-plate is equal $10l$, where l – mezostructure length parameter (length of the periodic cell) (Fig. 2)

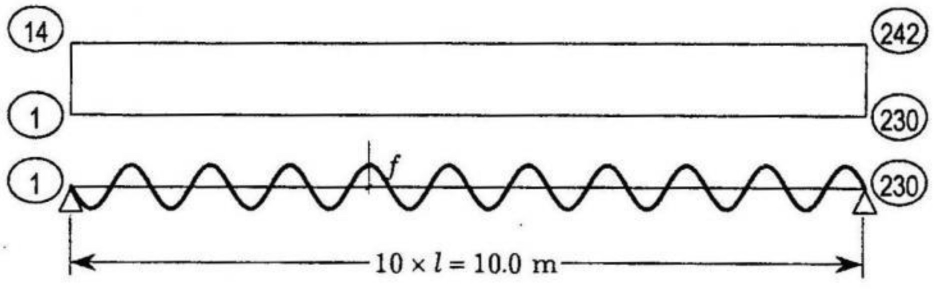

Fig. 2.

In Table for the mezostructural theory, the homogenized theory, the orthotropic plate model and the finite element method, free vibrations frequencies shown versus ratio δ/l, where $f/l = 1/10 = \text{const}$.

	δ/l	1/10	1/25	1/50	1/100
MST	ω_1	12.800	5.263	2.650	1.349
	ω_2	$21.221 \cdot 10^3$	$16.420 \cdot 10^3$	$15.467 \cdot 10^3$	$15.214 \cdot 10^3$
	ω_3	$34.954 \cdot 10^3$	$33.049 \cdot 10^3$	$32.837 \cdot 10^3$	$32.787 \cdot 10^3$
HT	ω	12,803	5.265	2.654	1.339
Orth.	ω	14.050	5.620	2.810	1.405
FEM	ω	13.925	5.570	2.785	1.392
MST/FEM	ω_1/ω	91.9%	94.4%	95.0%	96.9%

6. Conclusions.

(i) The finite element method can be successfully applied to determine the form of the micro-shape functions.

(ii) The assumed form of the micro-shape functions $h = l^2 \sin(2\pi x/l)$, $g = l^2 \sin(4\pi x/l)$ well describes a dynamic behaviour of wavy-plates for a different ratio δ/l and for the wavy amplitude $f \le l/10$.

(iii) Only mezo-structural model gives us lower and higher free vibrations frequencies for the assumed vibrations form of the wavy-plate.

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ПРО ВИБІР ФУНКЦІЙ МІКРОФОРМИ ДЛЯ ДИНАМІЧНОЇ ПОВЕДІНКИ ХВИЛЯСТИХ ПЛАСТИН

Обрано функцією мікроформи для хвилястих пластин за допомогою розв'язку задачі на власні коливання комірки періодичності, отриманого методом скінченних елементів. Вибрано також рівняння руху хвилястої пластини для різних типів функції мікроформи стосовно коливань у площині та з площини пластини. Обчислено та порівняно частоти власних коливань хвилястої пластини, одержані з мезоструктурної, згомогенізованої моделі, а також за теорією ортотропних пластин і методом скінченних елементів.

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