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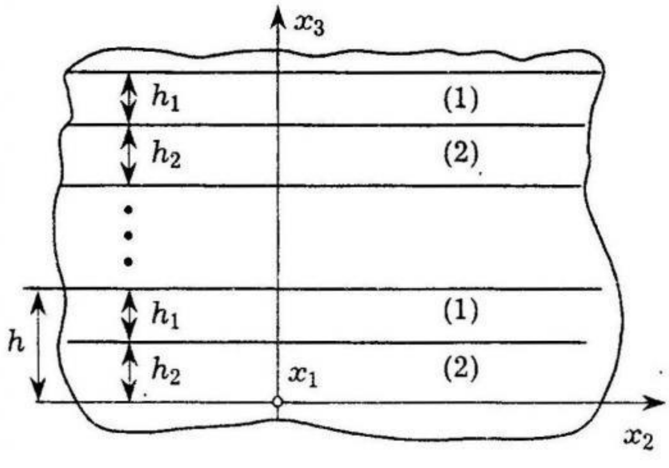
# DIFFUSOELASTICITY OF PERIODICALLY STRATIFIED BODIES

The diffusion phenomen is often observed in nature as well as it is utilized in material technology (for example the thin – film technique). The construction of waste dumps have to take into consideration the diffusion processes. The penetration of diffused substance in solids can lead to strains and stresses. The problem of modelling of interactions between diffusion and deformation of homogeneous bodies were considered in many papers (see, for instance [1–3]). In this paper we consider an elastic nonhomogeneous body which in a natural (underformed) configuration is composed of periodically repeated two different isotropic homogeneous layers (see Figure). Let $h_1$, $h_2$ be the layer thicknesses, and $h = h_1 + h_2$ be the thickness of each basic unit of the body. Let the axis $x_3$ be normal to the layering. Let $\lambda^{(r)}$, $\mu^{(r)}$, $r = 1, 2$, denote Lame' constants, $D^{(r)}$ denote the diffusion coefficients, $\chi^{(r)}$ and $\gamma_c{}^{(r)}$ denote the coupling coefficients of diffusion and stresses of the subsequent layers, respectively. By $\sigma_{ij}{}^{(r)}$, $r = 1, 2$; $i, j = 1, 2, 3$, we denote the components of stress tensor and let $\eta_i{}^{(r)}$ be the components of diffusion fluxes in the layer of $r$-th kind.



The scheme of periodically elastic dody.

As a basic of consideration the following constitutive relations in the layer of $r$-th kind are taken into account [5]:

$$\sigma_{ij}{}^{(r)} = 2\mu^{(r)}\varepsilon_{ij} + \left(\lambda^{(r)}\varepsilon_{kk} - \gamma_c{}^{(r)}c\right)\delta_{ij},$$

$$\eta_i{}^{(r)} = -D^{(r)}c,_i + \chi^{(r)}u_{k,ki},$$

$$\delta_{ij} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j, \end{cases}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{1}$$

where $\mathbf{u}(\mathbf{x}, t) \equiv (u_1, u_2, u_3)$ $(\mathbf{x}, t)$ denotes the displacement vector, $c = c(\mathbf{x}, t)$ denotes the concentration of the diffusing substance, and t denotes time. The equations of coupled diffusoelasticity take the form:

$$\sigma_{ij'j}^{(r)} + \rho^{(r)} X_i = \rho^{(r)} \ddot{u}_i \,,$$

$$\eta_{i'i}^{(r)} + \sigma = \dot{c} \,, \tag{2}$$

where $\rho^{(r)}$ denotes the mass densities, $\mathbf{X} = (X_1, X_2, X_3)$ is the body force vector, $\sigma = \sigma(\mathbf{x}, t)$ is the internal source of substance, $B_r$ is the part of the region occupied by the material of the $r$-th kind. The equations (2) should be considered together with adequate initial and boundary conditions.

If the body is composed of sufficiently large number of repeated layers then it seemes to be useful to apply of homogenized procedure with microlocal parameters [4]. Let an approximate solution of diffusoelasticity for microperiodic two-layered bodies be given in the form:

$$u_i(\mathbf{x}, t) = U_i(\mathbf{x}, t) + \underline{\alpha(x_3)\, W_j(\mathbf{x}, t)},$$

$$c(\mathbf{x}, t) = C(\mathbf{x}, t) + \underline{\alpha(x_3)\, G(\mathbf{x}, t)}, \tag{3}$$

where $\alpha(\cdot)$: $R \to R$ is the known a priori shape function satisfying the conditions

$$\alpha(x_3 + h) = \alpha(x_3),$$

$$\int_{x_3}^{x_3+h} \alpha(y)\, dy = 0, \qquad |\alpha(x_3)| < h, \qquad \forall\ x_3 \in R \tag{4}$$

and it is given as follows:

$$\alpha(x_3) = \begin{cases} x_3 - 0.5\, h_1, & \text{for} \quad 0 \le x_3 \le h_1, \\ \dfrac{-\gamma\, x_3}{1-\gamma} - 0.5\, h_1 + \dfrac{h_1}{1-\gamma}, & \text{for} \quad h_1 \le x_3 \le h, \end{cases} \tag{5}$$

where

$$\gamma = \frac{h_1}{h}. \tag{6}$$

The functions $U_i(\cdot)$, $C(\cdot)$ are unknown functions interpreted as components of macrodisplacements and macroconcentration, respectively. The functions $W_i(\cdot)$, $G(\cdot)$ are extra unknowns called the kinematical and diffusional microlocal parameters. They are related with the microperiodic structure of the body. Since $|\alpha(x_3)| < h$ for every $x_3 \in R$, then for small h the underlined terms in Eqs. (3) are small and will be neglected. However, the derivative $\alpha'(\cdot)$ is not small and the terms involving $\alpha'(\cdot)$ cannot be neglected. After some calculations [5], which here will be omitted, the following system of equations is obtained

$$\tilde{\mu}\, U_{i'ij} + \left(\tilde{\lambda} + \tilde{\mu}\right) U_{j'ji} + [\mu]\, W_{j'j}\, \delta_{i3} + [\mu]\, W_{i'3} + [\lambda]\, W_{3'i} - \tilde{\gamma}_c C_{'i} + \tilde{\rho}\, X_i - \tilde{\rho} \ddot{U}_i = 0 \,, \tag{7}$$

$$\tilde{D}\, C_{,ii} + [D]\, G_{,3} - \dot{C} + \sigma - \tilde{\chi}\, U_{k,kii} - [\chi]\, W_{3,ii} = 0$$

and

$$-\{[u]\,(U_{i,3} + U_{3,i}) + [\lambda]\, U_{k,k}\delta_{i3}\} + [\gamma_c]\, C\,\delta_{i3} = \hat{\mu}\, W_i + (\hat{\lambda} + \hat{\mu})\, W_3\,\delta_{i3}\,,$$

$$[D]\, C_{,3} + \hat{D}\, G - [\chi]\, U_{k,k3} - \hat{\chi} W_{3,3} = 0\,, \tag{8}$$

where the material parameters in Eqs. (7), (8) can be expressed in the form:

$$\tilde{f} = f^{(1)}\gamma - (1-\gamma)\, f^{(2)}\,,$$

$$[f] = \gamma\left(f^{(1)} - f^{(2)}\right),$$

$$\hat{f} = f^{(1)}\gamma + \frac{\gamma^2}{1-\gamma} f^{(2)} \tag{9}$$

for an arbitrary h-periodic function $f(\cdot)$ taking a constant valve $f^{(r)}$ in a layer of $r$-th kind (the material parameters can be obtained by substituting for a functions $f(\cdot)$ the $h$-periodic functions $\lambda$, $\mu$, $\chi$, $\rho$, $\gamma_c$, $D$). The equations (8) constitute a system of four linear algebraic equations for the microlocal parameters $W_i$ and $G$. The microlocal parameters $W_i$, $G$ can be elimated from Eqs. (7).

The stresses $\sigma_{ij}{}^{(r)}$ and the diffusion fluxes $\eta_i{}^{(r)}$ in a layer of $r$-th kind are expressed in the form

$$\eta_\beta^{(r)} = -D^{(r)}\, C_{,\beta} + \chi^{(r)}\left(U_{k,k\beta} + \alpha' W_{3,\beta}\right),$$

$$\eta_3^{(r)} = -D^{(r)}(C_{,3} + \alpha' G) + \chi^{(r)}(U_{k,k3} + \alpha' W_{3,3}),$$

$$\sigma_{\beta\gamma}^{(r)} = \mu^{(r)}\left(U_{\beta,\gamma} + U_{\gamma,\beta}\right) + \left[\lambda^{(r)}\left(U_{k,k} + \alpha' W_3\right) - \gamma_c^{(r)}\, C\right]\delta_{\beta\gamma}\,,$$

$$\sigma_{\beta 3}^{(r)} = \mu^{(r)}\left(U_{\beta,3} + U_{3,\beta} + \alpha' W_{,\beta}\right),$$

$$\sigma_{33}^{(r)} = 2\,\mu^{(r)}\left(U_{3,3} + \alpha' W_3\right) + \lambda^{(r)}\left(U_{k,k} + \alpha' W_3\right) - \gamma_c^{(r)} C\,, \tag{10}$$

$$r = 1,\ 2, \qquad \beta, \gamma = 1,\ 2,$$

where

$$\alpha' = \begin{cases} 1, & \text{for} \quad r = 1, \\ -\dfrac{\gamma}{1-\gamma}, & \text{for} \quad r = 2. \end{cases}$$

It can be shown that the obtained components of diffusion fluxes $\eta_3{}^{(r)}$ and stress tensors $\sigma_{3i}{}^{(r)}$, $r = 1, 2$; $i = 1, 2, 3$, satisfy conditions of continuity on interfaces.

The presented model of diffusoelasticity in periodic two-layered bodies can be treated as a basis of theory and a starting point for applications in environmental engineering, composite materials.

Assuming that the considered body is homogeneous, so $\rho^{(1)} = \rho^{(2)}$, $\lambda^{(1)} = \lambda^{(2)}$, $\mu^{(1)} = \mu^{(2)}$, $\chi^{(1)} = \chi^{(2)}$, $D^{(1)} = D^{(2)}$, $\gamma_c{}^{(1)} = \gamma_c{}^{(2)}$, the presented equations of homogenized model reduce to the relations for homogenized bodies [3].

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**Станіслав Матисяк, Радослав Мєшковський**

**ДИФУЗІЙНІ ПРОЦЕСИ В ПЕРІОДИЧНО ШАРУВАТИХ ПРУЖНИХ ТІЛАХ**

*У роботі представлено гомогенізовану модель дифузії в пружних мікроперіодичних шаруватих середовищах. Розв'язки побудовані в рамках лінійної зв'язаної теорії механодифузії* [1–3] *на основі процедури гомогенізації з мікролокальними параметрами* [4]. *Рівняння гомогенізованої моделі виражаються за допомогою невідомих макропереміщень, макроконцентрації та мікролокальних параметрів.*

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