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PLANE CONTACT PROBLEMS FOR A LAYER RESTING ON THE COMBINED FOUNDATION

1. Introduction. The plane contact problem for an elastic layer which rests on the rigid planar base was studied in [4]. Analogous problem for a layer supported by the Winkler foundation was solved in [2]. In this contribution we consider the contact of a rigid punch with an elastic isotropic layer resting on the *combined foundation*, which is a combination of the rigid base and the Winkler medium. Two cases of the punch geometry are considered: the parabolic cylinder (fig. 1,a) and flat strip (fig. 1,b).

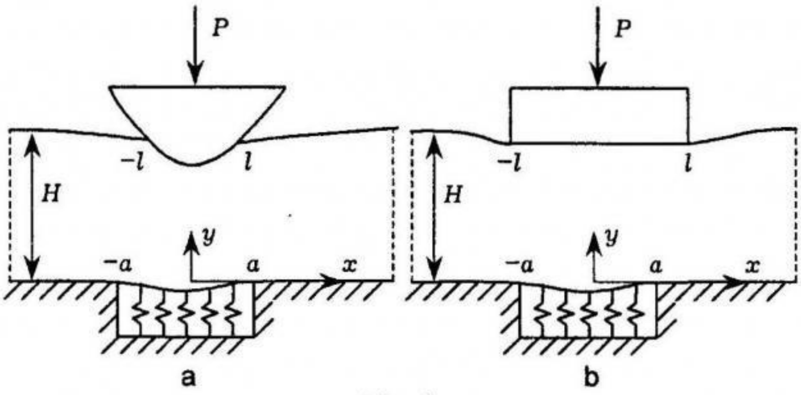

Fig. 1.

The problems involving an underground excavation have the great practical importance in geotechnics and mining engineering. Previously [5], the similar axially symmetric contact problems for a layer supported by the combined foundation was studied.

2. System of integral equations of contact problems. Mathematically, the problems formulated above are reduced to the solution of Lame equations

$$2(1-\nu)u_{,xx} + (1-2\nu)u_{,yy} + v_{,xy} = 0\,,\ (1-2\nu)v_{,xx} + 2(1-\nu)v_{,yy} + u_{,xy} = 0 \quad (1)$$

with the following boundary conditions

$$\tau_{xy}(x,H) = 0, \qquad |x| < \infty\,, \quad (2)$$

$$\sigma_{yy}(x,H) = 0, \qquad |x| > l\,, \qquad v(x,H) = \delta - g(x), \qquad |x| \le l\,, \quad (3)$$

$$\tau_{xy}(x,0) = 0, \qquad |x| < \infty\,, \quad (4)$$

$$v(x,0) = 0, \qquad |x| > a\,, \qquad \sigma_{yy}(x,0) = k\,v(x,0), \qquad |x| \le a\,, \quad (5)$$

where Oxy is the system of Cartesian co-ordinates; u, v are components of the displacement vector; σ_{yy}, τ_{xy} are components of the stress tensor; ν is Poisson's ratio; k is the stiffness of Winkler material; δ is the punch rigid displacement and $g(x)$ is the function describing the punch geometry.

The solution of equations (1) obtained by the Fourier transforms [3] has the form

$$u(x,y) = -\sqrt{\frac{2}{\pi}}\frac{1}{2\mu}\int_0^\infty \{A(\alpha)\sinh(\alpha y) + B(\alpha)[2(1-\nu)\cosh(\alpha y) + \alpha y\sinh(\alpha y)] +$$

$$+ C(\alpha)\cosh(\alpha y) + D(\alpha)[2(1-\nu)\sinh(\alpha y) + \alpha y\cosh(\alpha y)]\}\sin(\alpha x)\,d\alpha\,, \quad (6)$$

$$v(x,y) = \sqrt{\frac{2}{\pi}}\frac{1}{2\mu}\int_0^\infty \{-A(\alpha)\cosh(\alpha y) + B(\alpha)[(1-2\nu)\sinh(\alpha y) - \alpha y\cosh(\alpha y)] -$$

$$- C(\alpha)\sinh(\alpha y) + D(\alpha)[(1-2\nu)\cosh(\alpha y) - \alpha y\sinh(\alpha y)]\}\cos(\alpha x)\,d\alpha\,, \quad (7)$$

$$\sigma_{yy}(x,y) = -\sqrt{\frac{2}{\pi}}\int_0^\infty \alpha\{A(\alpha)\sinh(\alpha y) + B(\alpha)\,\alpha y\sinh(\alpha y)] +$$

$$+ C(\alpha)\cosh(\alpha y) + D(\alpha)\,\alpha y\cosh(\alpha y)]\}\cos(\alpha x)\,d\alpha\,, \quad (8)$$

$$\tau_{xy}(x,y) = -\sqrt{\frac{2}{\pi}}\int_0^\infty \alpha\{A(\alpha)\cosh(\alpha y) + B(\alpha)[\sinh(\alpha y) + \alpha y\cosh(\alpha y)] -$$

$$- C(\alpha)\sinh(\alpha y) + D(\alpha)[\cosh(\alpha y) + \alpha y\sinh(\alpha y)]\}\sin(\alpha x)\,d\alpha\,, \quad (9)$$

where μ is the shear module of the layer and functions $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $D(\alpha)$ are unknown. For the determination of these unknowns the boundary conditions (2)–(5) must be used. Note here that the boundary conditions (3), (5) are of mixed type at both upper and lower surfaces of the layer.

Applying the method of dual equations we obtain the final form of system of two integral equations of the contact problem under consideration for the contact pressure $p(x)$ and the function $h(t)$

$$h(t) - t\int_0^a h(t')\,R_1(t,t')\,dt' - \frac{t}{2(\lambda+\mu)}\int_0^l p(x')\,R_2(x',t)\,dx' = 0, \qquad 0 \le t \le a \quad (10)$$

$$\frac{2(1-\nu)}{1-2\nu}\frac{2}{\pi}\int_0^a h(t')\,R_2(x,t')\,dt' + \frac{1-\nu}{\mu}\frac{2}{\pi}\int_0^l p(x')\,R_4(x',x)\,dx' = \delta - g(x), \qquad 0 \le x \le l \quad (11)$$

where λ is Lame constant and

$$R_1(t',t) = \int_0^\infty \alpha\left[1 - F_1(\alpha) - \frac{\theta}{\alpha H}\right]J_0(\alpha t')\,J_0(\alpha t)\,d\alpha\,,$$

$$R_2(x',t) = \int_0^\infty F_2(\alpha)\cos(\alpha x')\,J_0(\alpha t)\,d\alpha\,, \qquad R_4(x',x) = \int_0^\infty F_4(\alpha)\cos(\alpha x')\cos(\alpha x)\frac{d\alpha}{\alpha}\,,$$

$$F_1(\alpha) = \frac{\sinh^2(\alpha H) - \alpha^2 H^2}{\sinh(\alpha H)\cosh(\alpha H) + \alpha H}\,, \qquad F_2(\alpha) = \frac{\alpha H\cosh(\alpha H) + \sinh(\alpha H)}{\sinh(\alpha H)\cosh(\alpha H) + \alpha H}\,,$$

$$F_4(\alpha) = \frac{\sinh^2(\alpha H)}{\sinh(\alpha H)\cosh(\alpha H) + \alpha H}\,.$$

Here $J_0(\cdot)$ is the Bessel function of the first kind and $\theta = (1-\nu)\,k\,H/\mu$.

The system (10), (11) must be completed by the equilibrium equation

$$2\int_0^l p(x)\,dx = P \tag{12}$$

where P is the load.

Introducing dimensionless variables and functions

$$t = a\tau\,,\quad t' = a\,\tau'\,,\quad x = l\xi\,,\quad x' = l\,\xi'\,,\quad \beta = \alpha\,H\,,\quad \kappa_1 = a/H\,,\quad \kappa_2 = l/H\,,$$

$$p(x) = \frac{P}{l}\,p^*(\xi)\,,\quad h(t) = \frac{1-2\nu}{2\mu}\,P\,h^*(\tau)\,,\quad g(x) = \frac{1-\nu}{\mu}\,P\,g^*(\xi)$$

and displaying the singular part of the kernel $R_4(\cdot)$, the system (10)–(12) can be transformed to the form

$$h^*(\tau) - \kappa_2^2\,\tau\int_0^1 h^*(\tau')\,R_1^*(\tau',\tau)\,d\tau' - \frac{\kappa_2\tau}{2}\int_{-1}^1 p^*(\xi')\,R_2^*(\xi',\tau)\,d\xi' = 0, \qquad 0 \le \tau \le 1\,, \tag{13}$$

$$\frac{2\kappa_1\kappa_2}{\pi}\int_0^1 h^*(\tau')\,R_2^*(\xi,\tau')\,d\tau' + \frac{1}{\pi}\int_{-1}^1 p^*(\xi')\left\{\frac{1}{\xi-\xi'} - R_{40}^*(\xi',\xi)\right\}d\xi' = -g^{*\prime}(\xi), \quad -1 \le \xi \le 1\,, \tag{14}$$

$$\int_{-1}^1 p^*(\xi)\,d\xi = 1\,, \tag{15}$$

where the dimensionless kernels $R_1^*(\cdot), R_2^*(\cdot), R_{40}^*(\cdot)$ are some known functions. The singular integral equation (14) is the Cauchy-type for the function of contact pressure. Knowing the function $h^*(\tau)$, the deflection in the excavation zone can be calculated as

$$w_0(\xi) = \frac{2}{\pi}\int_\xi^1 \frac{h^*(\tau)}{\sqrt{\tau^2-\xi^2}}\,d\tau, \qquad 0 \le \xi \le 1\,. \tag{16}$$

3. Contact of the parabolic punch with the layer. In this case, presented in fig. 1a, the function $g^{*\prime}(\xi)$ has the form

$$g^{*\prime}(\xi) = -\frac{2}{\pi}\frac{l}{l_H^2}\frac{P_H}{P}\,\xi\,, \tag{17}$$

where l_H, P_H are the contact size and the load in the Hertz problem. In the future analysis we will assume that the contact size l is equal to l_H, but the ratio P_H/P is unknown.

The distribution of the contact pressure in this case can be expressed as

$$p^*(\xi) = \varphi(\xi)\,\sqrt{1-\xi^2}\,, \tag{18}$$

where $\varphi(\xi)$ is a new unknown function.

Using the Gauss-Chebyshev quadrature formulae of the first kind [1], the system (13)–(15) can be transformed to the equivalent system of linear algebraic equations which was solved numerically.

The numerical analysis was performed to display the effect of stiffness of Winkler material θ on the distribution of the contact pressure $p^*(\xi)$ and the deflection in excavation zone $w_0(\xi)$ for different values of parameters κ_1, κ_2. The obtained results showed that the stiffness of material in the excavation has small influence on the contact pressure but the deflection $w_0(\xi)$ depends strongly with the stiffness θ.

4. Contact of the flat punch with the layer. In this case, presented in fig. 1,b, the function $g^{*\prime}(\xi)$ has the form $g^{*\prime}(\xi) = 0$.

The distribution of the contact pressure for the index equal to 1 can be written as $p^*(\xi) = \varphi(\xi)/\sqrt{1-\xi^2}$, where $\varphi(\xi)$ is a new unknown function.

Using the Gauss-Chebyshev quadrature formulae of the second kind [1], the system (13)–(15) can be transformed to the equivalent system of linear algebraic equations which was solved numerically.

The obtained results showed that the stiffness θ has as before small influence on the contact pressure. Moreover, the geometry of the punch has no influence on the distribution of the deflection $w_0(\xi)$.

5. Conclusions. The elastic layer supported by the combined foundation can be considered as a model of soil structure with the excavation. It is known that old non-used mines are filled by the sand which is modeled here as the Winkler medium. These problems are very important in underground building and mining engineering.

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ПЛОСКІ КОНТАКТНІ ЗАДАЧІ ДЛЯ ШАРУ НА КОМБІНОВАНІЙ ОСНОВІ

Розглянуто плоскі контактні задачі для смуги, яка лежить на жорсткому півпросторі з приповерхневою заглибиною, що заповнена матеріалом Вінклера. З використанням перетворення Фур'є задачі зведені до систем інтегральних рівнянь, які розв'язані чисельно. Досліджено характер контактного тиску і прогинів над заглибиною.

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