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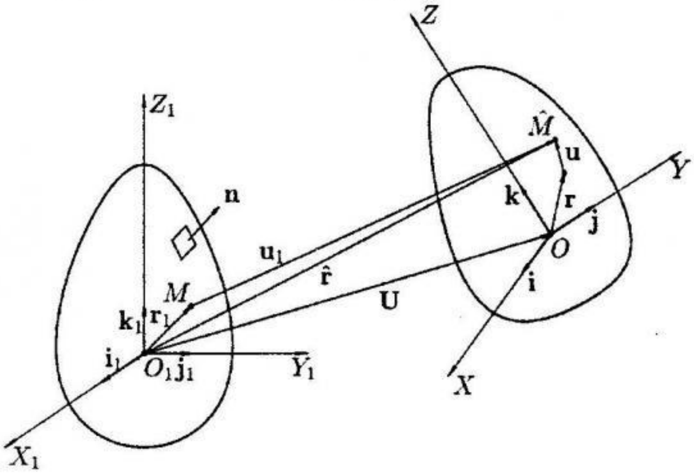
STEADY ROTATION OF ELASTIC SOLIDS: THE FIRST INTEGRAL OF EULER EQUATIONS AND ITS APPLICATION IN THE THEORY OF WAVE GYROSCOPES

Introduction. Dynamics of spatial motion of elastic solids. A thesis that elastic waves and vibrations in the solids suffer some qualitative and quantitative changes, if they move in the space, is a background of the theory of wave gyroscopes and its applications in technique. Generally, mathematical consideration in this field is restricted to the case of steady or uniformly accelerated rotation of the resonator about some fixed axis, and so, questions to be replied to are what effect this rotation brings in concrete eigenfrequency spectrum and how the picture of normal modes change. When these parameters of wave micro-motion are found for some operating regime it is suggested the problem is solved well. But, sometimes it may appear that the axis of rotation can also displace in the space with respect to initial unloaded state of a gyroscope. In absence of kinematical connections this displacement results in the precession of rotation axis, and if the resonator is clamped, some pendulum-like vibrations of a whole system may appear. Both effects are essentially the change of spatial macro-motion of a gyroscope due to wave micro-motion of a resonator. Such deduction follows from general theory of spatial motion of elastic solids [2]. Mathematical consideration of that theory is built on the principle of decomposition of complicated spatial motion of elastic solid in translational motion and relative motion (Galileo transform). Application of this transform to relations of time-depended elastic deformation results in mathematically closed boundary-valued problem, which realization requires consequent solution of coupled vector equations. These are generalized Lamé equation, which determine relative elastic displacements and Euler equation, which determine instant rotation of a relative coordinate system. Such equations have the same sense as their prototypes in classic elastodynamics and kinematics of rigid solids, but both are nonlinear due to the coupling of elastic displacements and components of a vector of angular velocity.

Lamé equation. Consider general statement. Suppose that in a moment $t > 0$ unbalanced external loads with the resultant of volume forces $\mathbf{P}$ and the resultant of surface forces $\mathbf{F}_n$ are applied to elastic solid. They will set a solid into complex spatial motion with the non-uniform, time-dependent bulk deformation. From the viewpoint of classic elasticity a boundary-valued problem, if formulated in absolute coordinate system, becomes non-linear and very complicated due to the large displacements of material points of a solid [3, 4].

The use of Galileo transform provides partial linearization of such problem. Let V_0 is a volume of elastic solid closed by the surface S_0, both are in



the initial unloaded state, and let ρ_0 denotes its volume density. Choose the origin O_1 of the absolute coordinate system $O_1X_1Y_1Z_1$ in the center of inertia in the unloaded state, and introduce the relative coordinate system $OXYZ$, which corresponds to spatial motion of some absolutely rigid solid, having the same volume and density and subjected to the action of the same loads $\mathbf{P}$ and $\mathbf{F}_n$ (See Figure). The radius-vector $\hat{\mathbf{r}}$ of some point $\hat{M}$ in deformable state can be represented in two ways

$$\hat{\mathbf{r}} = \mathbf{r}_1 + \mathbf{u}_1 \quad \text{or} \quad \hat{\mathbf{r}} = \mathbf{U} + \mathbf{r} + \mathbf{u}, \tag{1}$$

what gives

$$\mathbf{u}_1 = \mathbf{U} + \mathbf{r} + \mathbf{u} - \mathbf{r}_1. \tag{2}$$

In these formulae $\mathbf{r}_1$ and $\mathbf{u}_1$ determine initial position and absolute displacement of a point M in absolute coordinate system, $\mathbf{r}$ is the radius-vector of that point in relative system, and $\mathbf{u}$ are the relative displacements, describing small deviation of $\hat{M}$ from its initial position M. Displacement of the mass center $\mathbf{U}$ is determined from the equation

$$\rho_0 V_0 \frac{d^2\mathbf{U}}{dt^2} = \mathbf{R}, \quad \mathbf{R} = \oiint\limits_{S_0} \mathbf{F}_n ds + \iiint\limits_{V_0} \mathbf{P}\, dv, \tag{3}$$

where $\mathbf{R}$ is resultant of external volume and surface forces. To perform decomposition of relations of elastic deformation with finite displacements [3, 4] according to vector equalities (1)–(3), we take into account the following relations between the unit vectors of relative and absolute coordinate systems

$$\mathbf{i} = \mathbf{i}_1 \cos\alpha_1 + \mathbf{j}_1 \cos\beta_1 + \mathbf{k}_1 \cos\gamma_1,$$

$$\mathbf{j} = \mathbf{i}_1 \cos\alpha_2 + \mathbf{j}_2 \cos\beta_2 + \mathbf{k}_1 \cos\gamma_2,$$

$$\mathbf{k} = \mathbf{i}_1 \cos\alpha_3 + \mathbf{j}_2 \cos\beta_3 + \mathbf{k}_1 \cos\gamma_3. \tag{4}$$

Change of cosine functions of $\alpha_1(t)$, $\beta_1(t)$, $\gamma_1(t)$ ... determines the instant rotation of a relative coordinate system with the vector of angular velocity $\mathbf{\Omega} = \mathbf{i}\Omega_x + \mathbf{j}\Omega_y + \mathbf{k}\Omega_z$. Then, generalized Lamé equation formulated in the relative coordinates takes the form

$$\frac{2(1-\nu)}{1-2\nu} \operatorname{grad} \operatorname{div} \mathbf{u} - \operatorname{rot} \operatorname{rot} \mathbf{u} - \frac{1}{G}\left[\mathbf{P} - \frac{1}{V_0}\mathbf{R}\right] =$$

$$= \frac{\rho_0}{G}\left[\ddot{\mathbf{u}} + (\dot{\mathbf{\Omega}} \times \mathbf{r}) + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + (\dot{\mathbf{\Omega}} \times \mathbf{u}) + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}) + 2(\mathbf{\Omega} \times \dot{\mathbf{u}})\right], \qquad (5)$$

where ν is Poisson ratio and G is shear modulus. The use of this equation is to describe dynamic phenomena concerned with wave propagation in movable solids. It can be extended to steady and unsteady vibrations of thin-walled elements of wave gyroscopes.

Euler equations. At the first sight, Lamé equation (5) cannot be solved for relative displacements $\mathbf{u}$ until components of angular velocity $\mathbf{\Omega}$ are found. The most convenient method to establish an instant rotation of the relative coordinate system about a mass center O consist of application of the theorem of a change of angular momentum. So, taking into account (1) and assuming that $\mathbf{U}$ is already found from (3) we can write

$$\frac{d}{dt}\iiint\limits_{V_0}\rho_0\left[(\mathbf{r}+\mathbf{u})\times\left(\frac{d'\mathbf{r}}{dt}+\frac{d'\mathbf{u}}{dt}\right)\right]dv = \mathbf{M}, \qquad (6)$$

where

$$\mathbf{M} = \oiint\limits_{S_0}[(\mathbf{r}+\mathbf{u})\times\mathbf{F}_n]\,ds + \iiint\limits_{V_0}[(\mathbf{r}+\mathbf{u})\times\mathbf{P}]\,dv \qquad (7)$$

is a principal couple of external loads. After some transformations of (6) Euler equations can be represented in coordinate form

$$\frac{d}{dt}\left[\Omega_x A - \Omega_y F - \Omega_z E\right] - \Omega_y f + \Omega_z e - \dot{d} + \Omega_y\Omega_z(C-B) - \Omega_x\Omega_y E +$$
$$+\Omega_x\Omega_z F - (\Omega_y^2 - \Omega_z^2)D = M_x,$$

$$\frac{d}{dt}\left[\Omega_y B - \Omega_z D - \Omega_x F\right] + \Omega_x f - \Omega_z d - \dot{e} + \Omega_x\Omega_z(A-C) +$$
$$+\Omega_x\Omega_y D - \Omega_x\Omega_z F - (\Omega_z^2 - \Omega_x^2)E = M_y,$$

$$\frac{d}{dt}\left[\Omega_z C - \Omega_x E - \Omega_y D\right] - \Omega_x e + \Omega_y d - \dot{f} + \Omega_x\Omega_y(B-A) +$$
$$+\Omega_x\Omega_z F - \Omega_y\Omega_z D - (\Omega_x^2 - \Omega_y^2)F = M_z,$$

where

$$A(t) = \iiint\limits_{V_0}\rho_0\left[(y+u_y)^2 + (z+u_z)^2\right]dv, \quad B(t) = \ldots, \quad C(t) = \ldots, \qquad (8)$$

are the principal moments of inertia about coordinate axes, and

$$D(t) = \iiint\limits_{V_0}\rho_0(y+u_y)(z+u_z)\,dv, \quad E(t) = \ldots, \quad F(t) = \ldots, \qquad (9)$$

are the centrifugal moments of inertia. Quantities of the same dimension

$$d(t) = \iiint\limits_{V_0}\rho_0\left[(z+u_z)\dot{u}_y - (y+u_y)\dot{u}_z\right]dv, \quad e(t) = \ldots, \quad f(t) = \ldots, \qquad (10)$$

set into Euler equations due to elastic deformation of a solid. To the best of author's knowledge they have no conventional definition in the literature. Mathematical difficulties in the course of solution of classic Euler equations are well known. Moreover, in our case we collide with a necessity to determine components of relative displacements $\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z$ from the solution of Lamé equation (5). Therefore, both equations are coupled with quantities $\mathbf{u}$ and Ω.

First integral of Euler equations. Steady or uniformly accelerated rotation about fixed axis are two exceptions of such coupling. Usually, trivial solution of Euler equations are known beforehand and boundary-valued problems are formulated a priori for Lamé equation. Numerous works on the theory of wave gyroscopes (see for example [1]) are restricted essentially to this case. In other words, mutual influence and interaction between vibrations of a resonator (micro-motion) and rotational macro-motion of it are excluded in the statement of a problem. In particular, questions: how can wave processes or vibrations act towards a rotation of a solid and how significant this influence can be, remains unsolved in the literature. Further we consider one example of possible reply to these questions

Consider steady rotation of elastic solid about fixed axis. To realize such situation we must suggest that the solid once have been loaded by the self-balanced loads $\mathbf{P}$ and $\mathbf{F}_n$, in which case $\mathbf{M} = 0$. It clear up a way to simple integration of (6) in time

$$\iiint\limits_{V_0} \rho_0 \left[(\mathbf{r} + \mathbf{u}) \times \left(\frac{d'\mathbf{r}}{dt} + \frac{d'\mathbf{u}}{dt} \right) \right] dv = \mathbf{K}, \tag{11}$$

where the kinetic momentum $\mathbf{K}$ is a constant vector. Simple transformations of (11) yield the first integral of Euler equations

$$\begin{aligned} &\Omega_x A - \Omega_y F - \Omega_z E - d = K_x, \\ &-\Omega_x F + \Omega_y B - \Omega_z D - e = K_y, \\ &-\Omega_x E - \Omega_y D - \Omega_z C - f = K_z, \end{aligned} \tag{12}$$

from which we can deduce: (**i**) under the specified distribution of external forces $\mathbf{P}$ and $\mathbf{F}_n$, which do not change the kinetic momentum $\mathbf{K}$, and (**ii**) under the specified values of relative elastic displacements, which arise in a solid due to the action of these forces, components of the angular velocity are determined from the first integral of Euler equations by simple algebraic operations.

Consider, for example, changes of angular velocity of a solid of revolution arising due to such loading. Choose the axes of coordinate system in the unloaded state in directions of principal axes of inertia and nominate OZ to be the axis of rotation. In this case centrifugal moments of inertia equal to zero, what simplifies a problem. The components of angular velocity in the state of free rotation are $\Omega_x = 0$, $\Omega_y = 0$, $\Omega_z = \Omega_0$, $\Omega_0 = \text{const}$. The components of displacements due to centrifugal force (static deformation) can be easily defined as this force is known. Let $u_x^{(0)}, u_y^{(0)}, u_z^{(0)}$ are such displacements. Obviously $\dot{u}_x^{(0)} = \dot{u}_y^{(0)} = \dot{u}_z^{(0)} = 0$, as they are independent of time. Under such conditions $A^{(0)} = B^{(0)}$ and it follows from (8)–(10) that $K_x = 0$, $K_y = 0$, and $K_z = C^{(0)}\Omega_0$, where $C^{(0)}$ is the momentum of inertia about rotational axis. If we apply instantaneously some forces $\mathbf{F}_n$ this will give rise to disturbance of angular velocity $\mathbf{\Omega}$

$$\mathbf{\Omega} = \mathbf{k}\,\Omega_0 + \tilde{\mathbf{\Omega}} = \mathbf{i}\,\tilde{p} + \mathbf{j}\,\tilde{q} + \mathbf{k}(\Omega_0 + \tilde{r}) \tag{13}$$

and change relative elastic displacements, so that

$$u_x = u_x^{(0)} + \tilde{u}_x,\; u_y = u_y^{(0)} + \tilde{u}_y,\; u_z = u_z^{(0)} + \tilde{u}_z. \tag{14}$$

Now $\dot{u}_x = \dot{\tilde{u}}_x$, $\dot{u}_y = \dot{\tilde{u}}_y$, $\dot{u}_z = \dot{\tilde{u}}_z$. Substituting (15) in (9), and then in (12) we can write

$$\tilde{p}(t) = \frac{\Omega_0}{A^{(0)}} \iiint\limits_{V0} \rho_0 (x\,\tilde{u}_z + z\,\tilde{u}_x)\,dv + \frac{1}{A^{(0)}} \iiint\limits_{V0} \rho_0 (z\,\dot{\tilde{u}}_y - y\,\dot{\tilde{u}}_z)\,dv\,,$$

$$\tilde{q}(t) = \frac{\Omega_0}{B^{(0)}} \iiint\limits_{V0} \rho_0 (z\,\tilde{u}_y + y\,\tilde{u}_z)\,dv + \frac{1}{B^{(0)}} \iiint\limits_{V0} \rho_0 (x\,\dot{\tilde{u}}_z - z\,\dot{\tilde{u}}_x)\,dv\,,$$

$$\tilde{r}(t) = -\frac{2\Omega_0}{C^{(0)}} \iiint\limits_{V0} \rho_0 (x\,\tilde{u}_x + y\,\tilde{u}_y)\,dv + \frac{1}{C^{(0)}} \iiint\limits_{V0} \rho_0 (y\,\dot{\tilde{u}}_x - x\,\dot{\tilde{u}}_y)\,dv\,, \tag{15}$$

These formulae describe approximate magnitudes of small disturbances of angular velocity (13) and can be used in the design of solid-state sensors of angular velocity of different types.

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РІВНОМІРНИЙ ОБЕРТОВИЙ РУХ ПРУЖНОГО ТІЛА: ПЕРШИЙ ІНТЕГРАЛ РІВНЯНЬ РУХУ ЕЙЛЕРА ТА ЙОГО ЗАСТОСУВАННЯ В ТЕОРІЇ ХВИЛЬОВИХ ГІРОСКОПІВ

Для випадку рівномірного обертового руху одержано прості формули, за допомогою яких оцінюють малі відхилення кутової швидкості обертання пружного твердого тіла, які виникають завдяки коливанням або поширенню пружних хвиль. Базовою моделлю для вивчення цього ефекту служить загальна теорія просторового руху пружних тіл.

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