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INVERSE PROBLEM OF DETERMINATION OF TIME DEPENDENT SOURCE FOR PSEUDOPARABOLIC EQUATION

Introduction. We consider an inverse problem of determination of unknown time dependent source for pseudoparabolic equation with nonlocal boundary conditions and nonlocal overdetermination condition. The conditions for existence and uniqueness of solution of the inverse problem are obtained.

The method of [1] is used. Note that in this case unlike to many other parers for similar problems (see [2-4]), this problem includes nonlocal boundary conditions and a nonlocal overdetermination condition.

Formulation of problem and denotation. Let $Q_T = (0,H)\times(0,T)$, $T>0$, $H>0$. We introduce the following notation:

$$C^{1,0}(\overline{Q}_T) = \{u \in C(\overline{Q}_T) : u_x \in C(\overline{Q}_T)\},$$
$$C^{(2,1)}(Q_T) = \{u \in C(Q_T) : u_x, u_t, u_{xx}, u_{xt}, u_{xxt} \in C(Q_T)\}.$$

In the domain Q_T we consider the problem to find a pair of functions $(u(x,t), f(t)) \in C^{(2,1)}(Q_T)\cap C^{1,0}(\overline{Q}_T)\times C[0,T]$ which satisfy the conditions:

$$Lu \equiv u_{xxt} - k(x,t)u_t + \eta(x,t)u_{xx} + c(x,t)u_{xt} + a(x,t)u_x + b(x,t)u = f(t)h(x,t) + g(x,t),\ (x,t)\in Q_T, \tag{1}$$

$$u(x,0) = u_0(x),\ \ x\in[0,H], \tag{2}$$

$$\begin{aligned}(P_1u)(t) \equiv{}& \alpha_1(t)u(0,t) + \alpha_2(t)u(H,t) + \alpha_3(t)u_x(0,t) + \alpha_4(t)\int_0^H u(x,t)dx = \varphi_1(t),\\ (P_2u)(t) \equiv{}& \beta_1(t)u(H,t) + \beta_2(t)u_x(0,t) + \beta_3(t)u_x(H,t) + \beta_4(t)\int_0^H u(x,t)dx = \varphi_2(t), \quad t\in[0,T],\end{aligned} \tag{3}$$

$$(\Phi u)(t) \equiv \gamma_1(t)u(x_0,t) + \gamma_2(t)\int_0^H \sigma(x,t)u(x,t)dx = \delta(t),\ \ t\in[0,T], \tag{4}$$

where x_0 is a fixed point, $x_0\in(0,H)$.

Let

$$a_{11}(t) = \alpha_1(t) + \alpha_2(t)\chi(H,t) + \alpha_4(t)\int_0^H \chi(x,t)dx,$$

$$a_{12}(t) = \alpha_2(t)\lambda(H,t) + \alpha_3(t) + \alpha_4(t)\int_0^H \lambda(x,t)dx,$$

$$a_{21}(t) = \beta_1(t)\chi(H,t) + \beta_3(t)\chi_x(H,t) + \beta_4(t)\int_0^H \chi(x,t)dx,$$

$$a_{22}(t) = \beta_1(t)\lambda(H,t) + \beta_2(t) + \beta_3(t)\lambda_x(H,t) + \beta_4(t)\int_0^H \lambda(x,t)dx,$$

where functions χ and λ are solutions of the corresponding Cauchy problems:

$$\chi_{xx}(x,t) + c(x,t)\chi_x(x,t) - k(x,t)\chi(x,t) = 0,\ \chi(0,t) = 1,\ \chi_x(0,t) = 0;$$
$$\lambda_{xx}(x,t) + c(x,t)\lambda_x(x,t) - k(x,t)\lambda(x,t) = 0,\ \lambda(0,t) = 0,\ \lambda_x(0,t) = 1.$$

Theorem. *Let the following conditions be satisfied:*
1) $k, c, \sigma, k_t, c_t, \eta, a, b, h, g, \sigma_t \in C(\overline{Q}_T)$; $\alpha_i, \beta_i \in C^1[0,T], i = \overline{1,4}$; $\varphi_i \in C^1[0,H]$, $i = 1,2$; $u_0 \in C^2[0,H]$; $\delta \in C^1[0,T]$; $\gamma_i \in C^1[0,T]$, $i = 1,2$, $\gamma_1^2(t) + \gamma_2^2(t) \neq 0, t \in [0,T]$; $k(x,t) \geqslant k_0 > 0$, $(x,t) \in \overline{Q}_T$;

2) $\operatorname{rank}\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 & \alpha_4 \\ 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} = 2;$

$$\alpha_1(0)u_0(0) + \alpha_2(0)u_0(H) + \alpha_3(0)u_0'(0) + \alpha_4(0)\int_0^H u_0(x)dx = \varphi_1(0),$$

$$\beta_1(0)u_0(H) + \beta_2(0)u_0'(0) + \beta_3(0)u_0'(H) + \beta_4(0)\int_0^H u_0(x)dx = \varphi_2(0),$$

$$\gamma_1(0)u_0(x_0) + \gamma_2(0)\int_0^H \sigma(x,0)u_0(x)dx = \delta(0);$$

3) $\Delta(t) = a_{11}(t)a_{22}(t) - a_{12}(t)a_{21}(t) \neq 0, \quad t \in [0,T]$;

4) $(\Phi w)(t) \neq 0, t \in [0,T]$, *where the function* w *is a solution of the problem*

$$w_{xx}(x,t) + c(x,t)w_x(x,t) - k(x,t)w(x,t) = h(x,t), \quad x \in (0,H),$$

$$\alpha_1(t)w(0,t) + \alpha_2(t)w(H,t) + \alpha_3(t)w_x(0,t) + \alpha_4(t)\int_0^H w(x,t)dx = 0,$$

$$\beta_1(t)w(H,t) + \beta_2(t)w_x(0,t) + \beta_3(t)w_x(H,t) + \beta_4(t)\int_0^H w(x,t)dx = 0.$$

Then there exists an unique solution of problem (1)-(4).

Remark 1.
It was proved in [1] that $\lambda(x,t) > 0$, $\chi(x,t) > 1$, $\lambda_x(x,t) > 0$, $\chi_x(x,t) > 0$, $\gamma(x,t) = \chi(x,t)\lambda_x(x,t) - \chi_x(x,t)\lambda(x,t) > 0$,
$q(x,t) = \lambda(x,t)\int\limits_0^x \chi(s,t)ds - \chi(x,t)\int\limits_0^x \lambda(s,t)ds > 0$,
$\forall x \in (0,H], \forall t \in [0,T]$.

Since the function $y(x,t) = \lambda_x(x,t)\int\limits_0^x \chi(s,t)ds - \chi_x(x,t)\int\limits_0^x \lambda(s,t)ds$ is a solution of the Cauchy problem

$$y_x(x,t) + c(x,t)y(x,t) = \gamma(x,t) + k(x,t)q(x,t), \quad y(0,t) = 0,$$

we see that $y(x,t) > 0, \forall x \in (0,H], \forall t \in [0,T]$.

Let δ_{ij} be the determinants composed of the i,j-th columns of the matrix from condition 2) of the Theorem.

Suppose $\delta_{25} \neq 0$, then

$$\Delta(t) = \delta_{13}(t) + \delta_{23}(t)\chi(H,t) + \delta_{12}(t)\lambda(H,t) + \delta_{14}\lambda_x(H,t) -$$
$$-\delta_{34}(t)\chi_x(H,t) + \delta_{15}(t)\int\limits_0^H \lambda(x,t)dx - \delta_{35}(t)\int\limits_0^H \chi(x,t)dx +$$
$$+\delta_{24}(t)\gamma(H,t) - \delta_{25}(t)q(H,t) - \delta_{45}(t)y(H,t) \neq 0, \forall t \in [0,T],$$

if

$$\delta_{13}/\delta_{25} \leqslant 0, \delta_{23}/\delta_{25} \leqslant 0, \delta_{12}/\delta_{25} \leqslant 0, \delta_{14}/\delta_{25} \leqslant 0, \delta_{34}/\delta_{25} \geqslant 0,$$
$$\delta_{15}/\delta_{25} \leqslant 0, \delta_{35}/\delta_{25} \leqslant 0, \delta_{24}/\delta_{25} \leqslant 0, \delta_{45}/\delta_{25} \geqslant 0.$$

We obtain the similar conditions for $\Delta \neq 0$, considering other 9 cases $\delta_{13} \neq 0, \delta_{23} \neq 0, \delta_{12} \neq 0, \delta_{14} \neq 0, \delta_{34} \neq 0, \delta_{15} \neq 0, \delta_{35} \neq 0, \delta_{24} \neq 0, \delta_{45} \neq 0$. There exists a unique solution of direct problem (1)-(3) [1] under conditions 1)-3) of the Theorem.

Remark 2. As a general solution of homogeneous equation of Theorem condition 4) can be written in the form $w(x,t) = C_1\chi(x,t) + C_2\lambda(x,t)$, where $\forall t \in [0,T]$, C_1, C_2 are arbitrary constants, a homogeneous boundary value problem has only trivial solution due to conditions of the Theorem.

Proof. As our problem (1)-(4) is linear, its solution can be found in the form:

$$(u,f) = (u^1,0) + (u^2,f), \text{ where } Lu^1 = g(x,t),\ u^1(x,0) = u_0(x),$$
$$(P_1u^1)(t) = \varphi_1(t),\ (P_2u^1)(t) = \varphi_2(t);$$
$$Lu^2 = f(t)h(x,t),\ u^2(x,0) = 0,\ (P_1u^2)(t) = 0,\ (P_2u^2)(t) = 0,$$
$$(\Phi u^2)(t) = \delta(t) - (\Phi u^1)(t).$$

It is easy to show that we can investigate only the following problem:

$$Lu = f(t)h(x,t), \quad (x,t) \in Q_T, \tag{5}$$
$$u(x,0) = 0, \quad x \in [0,H], \tag{6}$$
$$(P_1u)(t) = (P_2u)(t) = 0, \quad t \in [0,T], \tag{7}$$
$$(\Phi u)(t) = \delta(t), \quad t \in [0,T], \quad \delta(0) = 0. \tag{8}$$

Let the function f be known. Then we shall search the solution of problem (5)-(7) in the form [1]:

$$u(x,t) = \chi(x,t)\mu(t) + \lambda(x,t)\nu(t) + \int_0^t k_1(x,t,\tau)\mu(\tau)d\tau +$$

$$+ \int_0^t k_2(x,t,\tau)\nu(\tau)d\tau + \int_0^x \int_0^t G(x,t,s,\tau)h(s,\tau)f(\tau)d\tau ds, \tag{9}$$

where

$$\mu(t) = u(0,t)\,,\ \nu(t) = u_x(0,t)\,,\ t \in [0,T];$$

$$G(x,t,s,\tau) = p(x,t,s,\tau) + \int_s^x \int_\tau^t p(x,t,\xi,t_1)R(\xi,t_1,s,\tau)dt_1 d\xi;$$

$$p(x,t,s,\tau) = (x-s)\omega_1(t,s,\tau) + \omega_2(x,s,\tau) - (x-s);$$

$R(x,t,s,\tau)$ is the resolvent of the kernel

$$\begin{aligned} A(x,t,s,\tau) =& k(x,t)(x-s)\omega_{1t}(t,s,\tau) - \eta(x,t)\omega_{2xx}(x,s,\tau) - \\ & - c(x,t)\omega_{1t}(t,s,\tau) - a(x,t)\Big(\omega_1(t,s,\tau) + \omega_{2x}(x,s,\tau) - 1\Big) - \\ & - b(x,t)\Big((x-s)\omega_1(t,s,\tau) + \omega_2(x,s,\tau) - (x-s)\Big); \end{aligned}$$

$$\begin{aligned} k_i(x,t,\tau) =& \int_0^x \Bigg[p(x,t,s,\tau)P_i(s,\tau) - \frac{\partial[p(x,t,s,\tau)q_i(s,\tau)]}{\partial\tau} - \\ & - \int_0^s \int_\tau^t p(x,t,s,t_1)R_i(s,t_1,\xi,\tau)dt_1 d\xi\Bigg] ds\,,\ i = 1,2; \\ & q_1 = k(s,\tau)\,,\ q_2 = sk(s,\tau) - c(s,\tau); \end{aligned}$$

$$\begin{aligned} R_i(s,t_1,\xi,\tau) =& R(s,t_1,\xi,\tau)z_i(\xi,\tau) + \frac{\partial[R_i(s,t_1,\xi,\tau)q_i(\xi,\tau)]}{\partial\tau}\,,\ i = 1,2; \\ & z_1 = b(\xi,\tau)\,,\ z_2 = a(\xi,\tau) + \xi b(\xi,\tau); \end{aligned}$$

$$P_i(s,\tau) = - z_i(s,\tau) + \int_0^s A(s,\tau,\xi,\tau)q_i(\xi,\tau)d\xi\,,\ i = 1,2;$$

$\omega_1(t,s,\tau)$ and $\omega_2(x,s,t)$ are the solutions of the following Cauchy problems:

$$\omega_{1t}(t,s,\tau) + \eta(s,t)\omega_1(t,s,\tau) = 0\,,\ t \in (0,T)\,,\ \omega_1(\tau,s,\tau) = 1;$$

$$\omega_{2xx}(x,s,t) + c(x,t)\omega_{2x}(x,s,t) - k(x,t)\omega_2(x,s,t) = 0, \quad x \in (0,H),$$

$$\omega_2(s,s,t) = 0, \quad \omega_{2x}(s,s,t) = 1.$$

We obtain the following system of Volterra equations of the second kind with continuously differentiable kernels and right parts by substituting (9) in (7) and taking into account the conditions of the Theorem:

$$\begin{cases} \mu(t) &= \int\limits_0^t \Big(K_{11}(t,\tau)\mu(\tau) + K_{12}(t,\tau)\nu(\tau) \Big) d\tau + \psi_1(t), \\ \nu(t) &= \int\limits_0^t \Big(K_{21}(t,\tau)\mu(\tau) + K_{22}(t,\tau)\nu(\tau) \Big) d\tau + \psi_2(t), \end{cases} \tag{10}$$

where

$$\begin{aligned} K_{i,j}(t,\tau) = &(-1)^i \Delta^{-1}(t) \Big[(\alpha_2(t) a_{2\ 3-i}(t) - \beta_j(t) a_{1\ 3-i}(t)) k_j(H,t,\tau) - \\ &- \beta_3(t) a_{1\ 3-i}(t) k_{j_x}(H,t,\tau) + (\alpha_4(t) a_{2\ 3-i}(t) - \\ &- \beta_4(t) a_{1\ 3-i}(t)) \int\limits_0^H k_j(x,t,\tau) dx \Big], \quad i,j = 1,2; \end{aligned}$$

$$\begin{aligned} G_i(t,\tau) = &(-1)^i \Delta^{-1}(t) \int\limits_0^H \Big[(\alpha_2(t) a_{2\ 3-i}(t) - \beta_1(t) a_{1\ 3-i}(t)) \times \\ &\times G(H,t,\xi,\tau) - \beta_3(t) a_{1\ 3-i}(t) G_x(H,t,\xi,\tau) + (\alpha_4(t) a_{2\ 3-i}(t) - \\ &- \beta_4(t) a_{1\ 3-i}(t)) \int\limits_\xi^H G(x,t,\xi,\tau) dx \Big] h(\xi,\tau) d\xi, \quad i = 1,2, \end{aligned}$$

$$\psi_i(t) = \int\limits_0^t G_i(t,\tau) f(\tau) d\tau.$$

From the first equation of system (10) we obtain the following representation:

$$\mu(t) = \int\limits_0^t A_1(t,\tau)\nu(t,\tau) d\tau + \Psi_1(t), \tag{11}$$

where

$$A_1(t,\tau) = K_{12}(t,\tau) + \int\limits_\tau^t R_1(t,s) K_{12}(s,\tau) ds, \ \Psi_1(t) = \int\limits_0^t R_1(t,\tau)\psi_1(\tau) d\tau + \psi_1(t),$$

$R_1(t,\tau)$- resolvent of kernel K_{11}.

Substituting (11) to the second equation of (10) we obtain:

$$\nu(t) = \int\limits_0^t A_2(t,\tau)\nu(\tau) d\tau + \Psi_2(t), \tag{12}$$

where

$$A_2(t,\tau) = K_{22}(t,\tau) + \int\limits_\tau^t K_{21}(t,s) A_1(s,\tau) ds,$$

$$\Psi_2(t) = \int_0^t K_{21}(t,\tau)\Psi_1(\tau)d\tau + \psi_2(t).$$

Consequently, there exists the following representation of the solution of system (10):

$$\mu(t) = \int_0^t A_5(t,\tau)f(\tau)d\tau, \quad \nu(t) = \int_0^t B_3(t,\tau)f(\tau)d\tau, \tag{13}$$

where

$$A_5(t,\tau) = \int_\tau^t \Big(A_4(t,s)G_1(s,\tau) + A_3(t,s)G_2(s,\tau)\Big)ds + G_1(t,\tau),$$

$$A_4(t,\tau) = \int_\tau^t \Big(R_1(t,\tau)\int_s^t A_3(t,s_1)K_{21}(s_1,s)ds_1 + A_3(t,s)K_{12}(s,\tau)\Big)ds + R_1(t,\tau),$$

$$B_3(t,\tau) = \int_\tau^t \Big(B_2(t,s)G_1(s,\tau) + R_2(t,s)G_2(s,\tau)\Big)ds + G_2(t,\tau),$$

$$A_3(t,\tau) = \int_\tau^t A_1(t,s)R_2(s,\tau)ds, \quad B_2(t,\tau) = B_1(t,\tau) + \int_\tau^t B_1(t,s)R_1(s,\tau)ds,$$

$$B_1(t,\tau) = K_{21}(t,\tau) + \int_\tau^t R_2(t,s)K_{21}(s,\tau)ds,$$

$R_2(t,\tau)$ is the resolvent of the kernel $A_2(t,\tau)$.

The solution of problem (5)-(7) can be written by substituting (13) to (9) in the form :

$$u(x,t) = \int_0^t G_3(x,t,\tau)f(\tau)d\tau. \tag{14}$$

where

$$G_3(x,t,\tau) = \chi(x,t)A_5(t,\tau) + \lambda(x,t)B_3(t,\tau) + \int_\tau^t k_1(x,s,\tau)A_5(s,\tau)ds +$$

$$+ \int_\tau^t k_2(x,t,s)B_3(s,\tau)ds + \int_0^x G(x,t,s,\tau)h(s,\tau)ds.$$

Differentiating (8) on t and accounting (14) we obtain

$$(\Phi w)(t)f(t) + \int_0^t G_4(t,\tau)f(\tau)d\tau = \delta'(t), \tag{15}$$

where

$$w(x,t) = G_1(t,t)\chi(x,t) + G_2(t,t)\lambda(x,t) + \int_0^x \omega_2(x,s,t)h(s,t)ds, \quad (16)$$

$$G_4(t,\tau) = \gamma_1(t)G_{3t}(x_0,t,\tau) + \gamma_1'(t)G_3(x_0,t,\tau) + \gamma_2'(t)\int_0^H \sigma(x,t)G_3(x,t,\tau)dx +$$

$$+ \gamma_2(t)\int_0^H \sigma_t(x,t)G_3(x,t,\tau)dx + \gamma_2(t)\int_0^H \sigma(x,t)G_{3t}(x,t,\tau)dx.$$

It is easy to show that the following statement is valid due to conditions of the Theorem:
If (u,f) is the solution of problem (5)-(8), then f is the solution of equation (15). Vice versa, if f is the solution of equation (15), then (u,f) is the solution of problem (5)-(8), where u is reperesenting by (14).

Hence, it is enough to define solvability conditions of equation (15).

So, if $(\Phi w)(t) \neq 0, t \in [0,T]$, we obtain a second-order Volterra equation with continuous functions $\frac{1}{(\Phi w)(t)}G_4(t,\tau)$, $\frac{1}{(\Phi w)(t)}\delta'(t)$ for determination of f.

Hence, the theorem is proved.

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ОБЕРНЕНА ЗАДАЧА ЗНАХОДЖЕННЯ ДЖЕРЕЛА ЗАЛЕЖНОГО ВІД ЧАСУ ДЛЯ ПСЕВДОПАРАБОЛІЧНОГО РІВНЯННЯ

У праці досліджено обернену задачу знаходження залежного від часу джерела для псевдопараболічного рівняння. Отримано умови існування та єдиності розв'язку зазначеної задачі.

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