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ON THE BRAUER GROUP AND THE HASSE PRINCIPLE FOR PSEUDOGLOBAL FIELDS

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We prove the Tate critérion for the Hasse principle in finite extensions of an algebraic function field K with pseudofinite constant field. Also any central simple algebra of finite dimension over such a field is cyclic and its index and exponent coincide.

Key words: class field theory, algebraic function field, Brauer group, Hasse principle, finite dimensional central simple algebra.

The aim of this paper is to show that the basic properties of the Brauer group of a global field hold as well for the Brauer group of an algebraic function field with pseudofinite [1] constant field. We call such a field pseudoglobal field. We prove also that any central simple algebra of finite degree over a pseudoglobal field K is cyclic and its index and exponent coincide. Besides, we discuss the Hasse principle in finite extensions of a pseudoglobal field.

The basic properties of the Brauer group of a pseudoglobal field will follow as the simple corollaries from the fundamental sequence

$$0 \to \mathrm{Br}K \xrightarrow{\ i\ } \bigoplus_{v\in V_K} \mathrm{Br}K_v \xrightarrow{\ j\ } \mathbb{Q}/\mathbb{Z} \to 0, \qquad (1)$$

which is exact both for global and pseudoglobal fields [1,2]. Here V_K denotes the set of all the valuations of pseudoglobal field K (which are trivial on the constant field), $\mathrm{Br}K$ is the Brauer group of K, and $\mathrm{Br}K_v$ is the Brauer group of the corresponding completion of field K at the valuation $v \in V_K$.

The elements of $\mathrm{Br}K$ are the equivalence classes of central simple K-algebras A of finite dimension with respect to the following equivalence relation: two algebras A and B are equivalent if there exist two natural numbers $m, n \geqslant 1$ such that the algebras $A\otimes_K M_n(K)$ and $B\otimes_K M_m(K)$ are isomorphic. All matrix algebras over K are equivalent and form the zero element of Brauer group. The class of the opposite algebra A° (that is A° is the additive group A equipped with the new multiplication $*$ such that $a*b = ba$) is the inverse for the class of A. We shall denote the class of A in the Brauer group by $[A]$.

The field extension L of K is said to be a splitting field of algebra A, if the algebras $A\otimes_K L$ and $M_m(L)$ are isomorphic. Two equivalent algebras have the same splitting fields. The subset $\mathrm{Br}(L/K)$ of $\mathrm{Br}K$, consisting of all the elements of $\mathrm{Br}(K)$ which split in L, is a subgroup of $\mathrm{Br}K$.

In the exact sequence (1) the map i sends $[A] \in \mathrm{Br}K$ to $(..., [A \otimes_K K_v], ...) \in \oplus \mathrm{Br}K_v$ (notice that there are only finitely many of valuations v of K such that $[A \otimes_K K_v]$ is a nontrivial element of $\mathrm{Br}K_v$ (see e.g. [3], p. 441, or [4])). Any local algebra $A_v = A \otimes K_v$ is a simple central algebra over a general local field K_v, so it determines an element of the Brauer group $\mathrm{Br}\, K_v$. It is known [5] that for a general local field K_v there exists an isomorphism $\mathrm{inv}_v : \mathrm{Br}K_v \longrightarrow \mathbb{Q}/\mathbb{Z}$.

The image of the element $[A_v]$ under this isomorphism is said to be the invariant of A_v (or the local invariant of the algebra A at the valuation v). It is denoted by $\mathrm{inv}_v(A)$. The homomorphism j maps an element of the group $\oplus_{v \in V_K} \mathrm{Br}K_v$ into sum of all corresponding local invariants.

The following proposition is an analogue for pseudoglobal fields of the classical Albert-Brauer-Hasse-Noether theorem on central simple algebras over global fields.

Proposition 1. *A central simple K-algebra A splits over pseudoglobal field K if and only if it splits locally everywhere, that is all its local invariants vanish.*

Proof. If the algebra A splits locally everywhere, then all its local invariants vanish. The injectivity of the homomorphism i in the exact sequence (1) show that the only trivial element of the group $\mathrm{Br}K$ (that is the matrix algebras over K) may have all trivial local invariants.

Proposition 2. *Suppose that a central simple K-algebra A over a pseudoglobal field K splits locally at all the valuations of K except possibly the valuation v_0. Then it splits over K.*

Proof. The exact sequence (1) yields that the sum of all local invariants of A is zero, so the local invariant of A_{v_0} must be zero as well. Thus the algebra A splits locally everywhere, and by Proposition 1 it splits over K.

The following proposition is an analogue for pseudoglobal fields of the Hasse norm theorem for cyclic extensions of global fields.

Proposition 3. *Let L/K be a cyclic extension of a pseudoglobal field K. An element $a \in K$ is a norm from L if and only if a is a norm locally everywhere, that is $a \in N_{L^v/K_v} L^v$ for all $v \in V_K$, where L^v denotes the completion of L at an extension of valuation $v \in V_K$ to L.*

Proof. Consider the exact sequence $0 \to L^* \to J_L \to C_L \to 0$, where J_L and C_L are the idèle group and the idèle class groups of the field L respectively. It was proved in [2] that $H^1(Gal(L/K), C_L) = 0$, thus we have the short exact cohomological sequence

$$0 \to H^2(G, L^*) \to \Pi_{v \in V_K} H^2(G_v, L^{v*}), \tag{2}$$

where L^v is the completion of L at some extension of the valuation v, and G_v is the decomposition group of the valuation v.

Since the extension L/K is cyclic, we have

$$H^2(G, L^*) \simeq K^*/N_{L/K}L^*,\ H^2(G_v, L^{v*}) \simeq K_v^*/N_{L^v/K_v}L^{v*},$$

so the exact sequence (2) may be written as follows

$$0 \to K^*/N_{L/K}L^* \to \Pi_{v \in V_K} K_v^*/N_{L^v/K_v}L^{v*}.$$

This exact sequence yields the desired statement.

Remark. Using the arguments given in the hints to the exercise 4 in [4, pp. 465–469], one can prove that Proposition 3 implies the Minkowski-Hasse theorem on quadratic forms: a nonsingular quadratic form over a pseudoglobal field K is isotropic if and only if it is isotropic over all the completions K_v of K.

To formulate the next Proposition, let us denote by $\hat{i}_K$ the composition of homomorphism i from sequence (1) and the isomorphism $\mathrm{Br}K_v \simeq \mathbb{Q}/\mathbb{Z}$. Then $\hat{i}_K$ maps the class of algebra A over K into the collection of its local invariants, $\hat{i}_K(A) = (\dots, \hat{i}_{K_v}[A \otimes_K K_v], \dots)$.

Proposition 4. *Let K be a pseudoglobal field.*

a) $\hat{i}_K$ defines an injective homomorphism $BrK \to \oplus_{v \in V_K} \mathbb{Q}/\mathbb{Z}$.

b) Two K-algebras A and B are equivalent if and only if $\hat{i}_K(A) = \hat{i}_K(B)$.

c) Two K-algebras A and B are isomorphic if and only if $\hat{i}_K(A) = \hat{i}_K(B)$ and $\deg A = \deg B$.

Proof. a) $\hat{i}_K(A)$ depends only on the class $[A]$ of A. The injectivity of $\hat{i}_K : \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$ follows from the injectivity of the homomorphism i in the exact sequence (1).

b) This follows immediately from a).

c) If the algebras A and B are isomorphic, it is obvious that $\hat{i}_K(A) = \hat{i}_K(B)$ and $\deg A = \deg B$.

Conversely, if $\hat{i}_K(A) = \hat{i}_K(B)$, then the algebras A and B are equivalent, so there exists a skew field D of finite dimension over K such that $A = M_m(D), B = M_n(D)$ for some natural numbers $m, n \geqslant 1$. Since $\deg A = \deg B$, we have $m = n$ and $A \simeq B$.

Proposition 5. *a) Let L/K be a finite Galois extension of a pseudoglobal field K, A be a central simple algebra of finite dimension over K, $v \in V_K$, $w \in V_L$, w is an extension of the valuation v to the field L, K_v and L_w be the corresponding completions of the fields K and L. The algebra A splits over L if and only if $[L_w; K_v] \cdot \mathrm{inv}_v(A) = 0$.*

b) The field L is isomorphic to a strongly maximal subfield of the algebra A if and only if $\deg A = [L : K]$, and $[L_w : K_v] \cdot \mathrm{inv}_v(A) = 0$ for all valuations v of K and their extensions w to L.

Proof. a) By Proposition 1 the field L splits A if and only if $\mathrm{inv}_w(A \otimes_K L_w) = 0$ for all valuations w of L. It is easy to check that the following diagram

$$\begin{array}{ccc} \mathrm{Br}K_v & \xrightarrow[\mathrm{inv}_v]{\simeq} & \mathbb{Q}/\mathbb{Z} \\ \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle n} \\ \mathrm{Br}L_w & \xrightarrow[\mathrm{inv}_w]{\simeq} & \mathbb{Q}/\mathbb{Z} \end{array}$$

commutes, and we have $\mathrm{inv}_w(A \otimes_K L) = [L_w : K_v] \, \mathrm{inv}_v(A)$.

b) Let L be a maximal subfield of A, then it is known (see e.g. [6] or [3]) that A splits over L, and $\deg A = [L : K]$.

Thus, by the above arguments $[L_w : K_v] \, \mathrm{inv}_v A = 0$. Conversely, if for a subfield L of A the condinions $\deg A = [L : K]$ and $[L_w : K_v] \, \mathrm{inv}_v A = 0$ hold, then the first of them implies that L is the maximal subfield of A, and the second one implies that the algebra A splits over K by statement a).

Proposition 6. *Let $\alpha_1, ..., \alpha_s$ be a finite set of elements of the group $\mathbb{Q}/\mathbb{Z}$. Then there is a simple central algebra A over a pseudoglobal field K with the invariants $\alpha_1, ..., \alpha_s$ if and only if $\sum_{i=1}^{s} \alpha_i = 0$.*

Proof. This is an immediate corollary from the exactness of sequence (1) in the middle term $\oplus_{v \in V_K} \mathrm{Br} K_v$.

Proposition 7. *Let L/K be a finite abelian extension of a pseudoglobal field K. If an element $a \in K$ is a local norm at all the completions K_v, where $v \in V_K \setminus \{v_0\}$, then a is a local norm in the field K_{v_0}.*

Proof. Consider the local and the global norm residue symbols θ_v i $\tilde{\theta}_K$ determined in [2] . They are related by the equality $\tilde{\theta}_K = \Pi_{v \in V_K} \theta_v$. Using the product formula for pseudoglobal field, we get $\tilde{\theta}_K(a) = 1 = \Pi_{v \in V_K} \theta_v(a) = \theta_{v_0}(a)$. The last equality and Proposition 7 follow from the fact that $\theta_v(a) = 1$, for $a \in K_v$, if and only if $a \in N_{L_w/K_v} L_w^*$.

The following Proposition 8 is a counterpart for pseudoglobal fields of Theorem 10 from [7, Chapter 6].

Proposition 8. *Let L/K be a finite abelian extension of a pseudoglobal field K, and let $\sigma_v \in G_v$ be the set of elements of the decomposition groups G_v such that almost all of them are trivial. Suppose that $\Pi_{v \in V_K} \sigma_v = 1$. Then there is an element $a \in K$ such that $\theta_v(a) = \theta_{L_w/K_v}(a) = \sigma_v$.*

Proof. Let $\theta_{L_w/K_v} : K_v/NL_w \to Gal(L_w/K_v)$ be the local norm residue symbol [2]. By the local class field theory generalized to general local fields (see [5]), one can find an idèle $(a_v) \in J_K$ such that $\theta_{L_w/K_v}(a_v) = \sigma_v$. Since $\Pi_{v \in V_K} \sigma_v = 1$, we have $\Pi_{v \in V_K} \theta_v(a_v) = \tilde{\theta}_K((a_v)) = 1$. Then $(a_v) \in K^* N_{L/K} J_L, (a_v) = a(b_v)$, where $(b_v) \in N_{L/K} J_L$. Thus $\tilde{\theta}_K((b_v)) = 1$, and we have $\theta_v(a) = \sigma_v$.

The following Proposition asserts that the conditions for a valuation v of K to be unramified or to split completely in a given Galois extension L/K can be formulated in terms of subgroups of idèle class group of the field K exactly in the same manner as for the global fields (see [7, Chapter 8, Theorem 3]).

Proposition 9. *Let L/K be a finite abelian extension of a pseudoglobal field K. The valuation v of the field K is unramified in the field L if and only if $U_v \subset N_{L/K} C_L$. The valuation v of the field K splits completely in the field L if and only if $K_v \subset N_{L/K} C_L$.*

Proof. In Proposition 9 it is assumed that the completion K_v is embedded into the group C_K by using the composition $K_v \hookrightarrow J_K \longrightarrow C_K$. To prove Proposition 9 we follow the arguments which were used in the case of global field (see [7, Ch.8]). First, we show that $N_{L/K} C_L \bigcap K_v = N_{L_w/K_v} L_w$, where w is an extension of the valuation v to L. It is enough to prove the inclusion $KN_{L/K} J_L \bigcap KK_v \subset KN_{L_w/K_v} L_w$. Let $a \in K, a_v \in K_v$, and $aa_v = N_{L/K}((a_w))$, where $(a_w) \in J_L$. Thus a is a local norm at all the valuation, except possibly at v, but then it follows from Proposition 7 that a is a local norm everywhere, so a_v is a local norm in K_v, and the desired inclusion is proved.

Let U_v be the unit group of K_v. If the valuation v is unramified, then all elements of U_v are norms by [5], thus $U_v \subset N_{L/K} C_L$. If $U_v \subset N_{L/K} C_L$, we have

$U_v \subset (N_{L/K}C_L \bigcap K_v) = N_{L_w/K_v}L_w$. Using the local class field theory for general local fields [5], we get that the valuation v is unramified in L.

To finish the proof of Proposition 9, it is enough to consider in the above argument the group K_v instead of U_v and the property "to split completely" instead of "to be unramified".

Let X be an algebraic curve defined over the field k. The Brauer group $\mathrm{Br}(X)$ of the curve X is the kernel of the homomorphism $\mathrm{Br}K \longrightarrow \oplus_{v \in V_K} \mathrm{Br}K_v$, where K is the function field on X.

Proposition 10. *For a pseudoglobal field K with constant field k the following equivalent properties hold:*

a) the reciprocity law holds for K/k;

b) for any finite cyclic extension L/K the sequence

$$Br(L/K) \longrightarrow \bigoplus_{v \in V_K} Br(L_w/K_v) \to [L:K]^{-1}\mathbb{Z}/\mathbb{Z} \to 0$$

us exact;

c) for any finite cyclic extension L/K,

$$H^1(Gal(L/K), Br(Y)) = 0,$$

where $Br(Y)$ is the Brauer group of the nonsingular projective algebraic curve Y with function field L;

d) for any finite cyclic extension L/K the map

$$K^*/N_{L/K}L^* \longrightarrow \bigoplus_{v \in V_K} K_v^*/N_{L_w/K_v}L_w^*$$

is injective;

e) $H^1(G(k), Jac_X(k_s)) = 0$, where $G(k)$ is the absolute Galois group of k and $Jac_X(k_s)$ is the jacobian of the curve X regarded over a separable closure k_s of the field k;

f) $Br(X) = 0$.

Proof. For a pseudoglobal field K/k assertion a) was proved in [2]. Assertion d) follows from Proposition 3. Conversely, as was proved in [2] d) implies a). The equivalence of a), b) and c) follows from Proposition A.12 [8, p. 167], and the equivalence of d),e),f) follows from Proposition A.13 [8, p. 168].

Now we shall show that the existence of class formation for pseudoglobal field K yields the same corollaries about the 3-dimensional Galois cohomology groups of the field K (respectively of idèle group and idèle class group of K) as in the case of a global field. Besides, it turns out that for abelian extensions of a pseudoglobal field one can get the Tate criterion for the Hasse principle.

Let L/K be a finite Galois extension of a pseudoglobal field K and let $G = Gal(L/K)$ be its Galois. Let H be a subgroup of G. Since the idèle classes of K form the class formation, it follows from Tate's theorem [4, p. 181] that the multiplication by the fundamental class $u_{L/K} \in H^2(G, C_L)$ defines the isomorphisms

$$H^n(H, \mathbb{Z}) \to H^{n+2}(H, C_L) \tag{3}$$

for all $n \in \mathbb{Z}$.

Let K' be the subfield of L corresponding to the subgroup H by Galois theory, $H = \mathrm{Gal}(L/K')$.

Proposition 11. *The diagrams*

$$\begin{array}{ccc} H^n(G,\mathbb{Z}) & \to & H^{n+2}(G,C_L) \\ \downarrow \mathrm{res} & & \downarrow \mathrm{res} \\ H^n(H,\mathbb{Z}) & \to & H^{n+2}(H,C_L) \end{array} \tag{4}$$

and

$$\begin{array}{ccc} H^n(G,\mathbb{Z}) & \to & H^{n+2}(G,C_L) \\ \uparrow \mathrm{cor} & & \uparrow \mathrm{cor} \\ H^n(H,\mathbb{Z}) & \to & H^{n+2}(H,C_L) \end{array} \tag{5}$$

commute.

Proof. The commutativity of (4) follows from the equalities

$$\mathrm{res}(a \cup u_{L/K}) = (\mathrm{res}\, a) \cup u_{L/K'} = \mathrm{res}\, a \cup \mathrm{res}\, u_{L/K},$$

and diagram (5) commutes according to

$$(\mathrm{cor}\, a) \cup u_{L/K} = \mathrm{cor}(a \cup u_{L/K'}) = \mathrm{cor}(a \cup \mathrm{res}\, u_{L/K}).$$

By using isomorhisms (3) one can prove, exactly in the same manner as in the case of global fields [4, c. 301], the Tate criterion for the Hasse principle for Galois extensions of pseudoglobal fields.

The kernel of the homomorphism

$$f_{0,L/K} : \hat{H}^0(G, L^*) \longrightarrow \hat{H}^0(G, J_L)$$

is called the obsruction for the Hasse principle for the Galois extension L/K with Galois group G. One says that the Hasse principle holds for L/K if $\mathrm{Ker}\, f_{0,L/K} = 0$.

Proposition 12. *Let L/K be a finite Galois extension of a pseudoglobal field K, $G = Gal(L/K)$. Then*

$$\mathrm{Ker}\, f_{0,L/K} \simeq \mathrm{Ker}(H^3(G,\mathbb{Z}) \to \Pi_{v \in V_K} H^3(G^v, \mathbb{Z})),$$

where G^v is a decomposition group G_w of an extension to L of the valuation v of K.

Proof. For the sake of completeness, we present the proof, despite it essentially coincides with that for the global field (see [4, p. 301]). Consider the exact sequence of G-modules

$$0 \to L^* \to J_L \to C_L \to 0, \tag{6}$$

and the corresponding sequence of Tate's Galois cohomology

$$\ldots \longrightarrow H^{n-1}(G, J_L) \xrightarrow{g_{n-1}} \hat{H}^{n-1}(G, C_L) \longrightarrow$$

$$\longrightarrow \hat{H}^n(G, L^*) \xrightarrow{f_n} H^n(G, J_L) \longrightarrow \ldots.$$

From the exactness of this last sequence we get

$$\mathrm{Ker}\, f_n \simeq \mathrm{Coker}\, g_{n-1}.$$

Now, by the local class field theory for general local fields [5],

$$H^{n-1}(G, J_L) \simeq \Pi_{v \in V_K} \hat{H}^{n-1}(G^v, L_v^*) \simeq \Pi_{v \in V_K} H^{n-3}(G^v, \mathbb{Z}),$$

and by (3),

$$\hat{H}^{n-1}(G, C_L) \simeq \hat{H}^{n-3}(G, \mathbb{Z}).$$

Using the fact that the groups $H^n(G, \mathbb{Z})$ and $H^{-n}(G, \mathbb{Z})$ are dual, one can write

$$\operatorname{Ker} f_n \simeq \operatorname{Coker}(\amalg_{v \in V_K} \hat{H}^{n-3}(G^v, \mathbb{Z}) \xrightarrow{g'_{n-1}} \hat{H}^{n-3}(G, \mathbb{Z}))$$

$$\simeq \operatorname{Ker}(H^{3-n}(G, \mathbb{Z}) \xrightarrow{h_{3-n}} \prod_{v \in V_K} H^{3-n}(G^v, \mathbb{Z})),$$

where $g'_{n-1}(\sum_v z_v) = \sum_v \operatorname{cor} z_v$, $h_{3-n}(z) = \prod_{v \in V_K} \operatorname{res} z$. Setting $n = 0$, we get

$$\operatorname{Ker} f_0 \simeq \operatorname{Ker}(H^3(G, \mathbb{Z}) \to \Pi_{v \in V_K} H^3(G^v, \mathbb{Z})),$$

as was to be proved.

Proposition 13. *Let L/K be a finite Galois extension of a pseudoglobal field K, $n = [L : K]$, g_v be the number of all distinct valuations w of L which are the extensions of a valuation v of K, d be the greatest common divisor of all g_v. Then, by identifiyng the group $H^2(G, L^*)$ with a subgroup of $H^2(G, J_L)$, the quotient group $H^2(G, J_L)/H^2(G, L^*)$ is a cyclic group of order $\frac{n}{d}$, and the image of the group $H^2(G, C_L)$ in $H^3(G, L^*)$ is a cyclic group of order d.*

Proof. We have $H^2(G, J_L) \simeq \bigoplus_{v \in V_K}(\frac{1}{n_v}\mathbb{Z}/\mathbb{Z})$, where $n_v = [L^v : K_v]$. On the other hand, $H^1(G, C_L) = 0$, and $H^2(G, C_L) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$, thus the exact Galois cohomology sequence corresponding to (6) can be written as follows

$$0 \to H^2(G, L^*) \to \bigoplus_{v \in V_K} (\frac{1}{n}\mathbb{Z}/\mathbb{Z}) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to H^3(G, L^*). \tag{7}$$

Consequently, the quotient group $H^2(G, J_L)/H^2(G, L^*)$ is isomorphic to a subgroup of $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, so it is cyclic. Let us find in this quotient group an element of maximal order.

Let $\{i_v\}_{v \in V_K}$ be the set of integers such that almost all of them are zero, and $\sum_v i_v g_v = d$. Since $n = n_v g_v$, we have $\sum_v \frac{i_v}{n_v} = \frac{g}{n}$. Hence, it follows that the element of the quotient group $H^2(G, J_L)/H^2(G, L^*)$ with representative $\left(\left(\frac{i_v}{n_v}\right) (\mathrm{mod} 1)\right)$ has the order $\frac{n}{d}$.

Futher, if $\left(\left(\frac{j_v}{n_v}\right) (\mathrm{mod} 1)\right)$ is the representative of another element $\bar{\alpha}$ of this quotient group, then one can find an integer m, such that $\sum j_v n_v = md$. Consequently, $\sum_v \frac{j_v}{n_v} = m\frac{d}{n}$, thus the order of $\bar{\alpha}$ divides $\frac{n}{d}$. Hence the order of considered quotient group is $\frac{n}{d}$.

Now, the exact sequence (7) shows that the image of $H^2(G, C_L)$ in the group $H^3(G, L^*)$ is a cyclic subgroup of order d generated by the image of fundamental class $u_{L/K} \in H^2(G, C_L)$.

Finally, we consider the central simple algebras of finite dimension over a pseudoglobal field. We shall show that any such algebra A is cyclic, its index and exponent coincide, and the reduced Whitehad group $SK_1(A)$ is trivial.

We shall use one result of Saltman [9] on existence of abelian extensions of valued fields, namely, a part of Theorem 5.10 from [9].

Theorem (Saltman [9]). *Let G be an abelian group, K be a field with real valued valuations $v_1, \dots, v_m$. Let r be the highest power of 2 dividing the exponent G. Let K_i be the completions of K with respect to $v_i, 1 \leqslant i \leqslant m$. Denote by $\rho(r)$ the primitive 2^r-th root of 1.*

a) Suppose that K has nonzero characteristic or $K_i(\rho(r))/K_i$ is cyclic for all i. Then if L_i/K_i are G Galois extensions, there is a G Galois extension L/K such that $L \otimes_K K_i = L_i$.

This result will play the same role in the proof of the theorem below that the Grunwald-Wang theorem [7, Chap.10] plays in the proof of classical result which asserts that any finite-dimensional central simple algebra over a global field is cyclic.

Theorem 1. *Any central simple algebra A of finite dimension over a pseudoglobal field K is cyclic and* $\operatorname{ind} A = \exp A$.

Proof. The proof we give is a slight modification of the proof for the classical case of algebras over global fields [3, p. 441-443]. Let $v_1, \dots, v_r$ be all the valuations of the pseudoglobal field K at which the algebra A has nontrivial local invariants. Set $n_i = \operatorname{ind} A_{v_i}$, where $A_{v_i} = A \otimes_K K_{v_i}$. Let m be the smallest common multiple of $n_1, \dots, n_r$. By [3, Proposition 13.4] $n_i | n$, where $n = \deg A = [A : K]^{\frac{1}{2}}$ for all $i, 1 \leqslant i \leqslant r$. Thus $m|n$. Now we use the Saltman theorem instead of the Grunwald-Wang theorem. By Saltman's theorem there are the cyclic extensions L and M of field K, of degrees m and n respectively, such that L/K and L_i/K_{v_i} are cyclic extensions of degree m, and M/K, M_i/K_{v_i} are cyclic extensions of degree n. Notice that one can take L_i and M_i to be the unramified extensions of K_{v_i} of degrees m and n respectively. All the number n_i divide m and n. It is easy to show that for the algebras A_{v_i} over general local fields K_{v_i}, we have, just as in the case of algebras over local fields, that n_i are the orders of local invariants of algebras A_{v_i}. Therefore it follows from Proposition 5 a) that A splits over the fields L and M. Then it follows from Proposition 5 b) that the field M is isomorphic to a strongly maximal subfield of A, thus the algebra A is cyclic.

It remains to prove that $\operatorname{ind} A = \exp A$. Since $\exp A | \operatorname{ind} A$ for an algebra over any field [3], it is enough to prove that $\operatorname{ind} A \leqslant \exp A$. Since the field L splits A, by [3, Proposition 13, p. 301] $\operatorname{ind} A \leqslant m$. But for the exponent e of A we have $e \cdot i[A] = i([A]^e) = 0$, where i is the homomorphism from the exact sequence (1). It follows that $e \cdot \hat{i}_{K_v} = 0$, where $\hat{i}_{K_v}$ is the local invariant of A for $v \in V_K$. Therefore $n_i | e, 1 \leqslant i \leqslant r$, hence $m|e$. Finally, $\operatorname{ind} A \leqslant m \leqslant e = \exp A$, and this completes the proof.

Let A be a central simple algebra of finite dimension over a field K. Let L be a maximal subfield of D. It is known [6] that there is an isomorphism $\varphi : A \otimes_K L \simeq M_n(L)$, where $n = [L : K]$. The composition map $N_{\text{red}} : A \to K$

$$x \mapsto x \otimes 1 \longrightarrow \phi(x \otimes 1) \longrightarrow det(\phi(x \otimes 1)).$$

is called the reduced norm. It turns out that the reduced norm does not depend either on a choice of maximal subfield L, or on a choice of a homomorphism φ, and its image is contained in K.

We shall prove that the reduced norm homomorphism is surjective for the algebras over pseudoglobal fields. For this purpose we will need the following simple lemma.

Lemma 1. *Any pseudoglobal field is a C_2-field.*

Proof. Recall that a field K is called C_i-field, if any homogeneous polynomial of degree d on $n > d^i$ variables has a nontrivial zero in K^n. Every condition C_i, in particular C_1, can be formulated as a sentence of the first order logic. Therefore, as the pseudofinite fields are elementarily equivalent to ultraproducts of finite fields, the pseudofinite constant field of K is a C_1-field.

S. Lang [10] and M. Nagata [11] proved that the property of a field to be a C_i - field is preserved under algebraic extensions. Besides, if k is a C_i–field, and K is an extension of k of transcendence degree n, then K is a C_{i+n} - field. Hence, a pseudoglobal field is a C_2-field.

Corollary. *Any quadratic form on $\geqslant 5$ variables defined over a pseudoglobal field K, has a nontrivial zero over K.*

Proposition 14. *Let A be a central simple algebra of finite dimension over a pseudoglobal field K. Then*

1) The reduced norm homomorphism $N_{\text{red}} : A \longrightarrow K$ is surjective.

2) The reduced Whitehad group $SK_1A = SL_1(A)/[A^*, A^*]$ of A is trivial. Here $SL_1(A) = \{a \in A | N_{\text{red}}(a) = 1\}$, $[A^*, A^*]$ is the commutant of multiplicative group A^* of algebra A.

Proof. 1) The algebra A is a matrix algebra over a skew field D. Clearly, it suffices to prove that $N_{\text{red}} : D \longrightarrow K$ is surjective. Let $[D : K] = n^2$. The map N_{red} is given by a homogeneous polynomial $\nu(\bar{x})$ of degree n on n^2 variables, and besides $\nu(\bar{x}) = 0$ if and only if $\bar{x} = \bar{0}$. We need to prove that the equation $\nu(\bar{x}) = a$ has a nontrivial solution over K for any $a \in K^*$. But, using that K is a C_2 - field by Lemma 1, this follows from the fact that the form $\nu(\bar{x}) = ax^n_{n^2+1}$ is of degree n and has $n^2 + 1$ variables.

2) The above arguments show that a pseudoglobal field is a C'_2 - field (a field K is called C'_2 - field if for any algebraic extension K'/K, and for any finite dimensional skew field D with center K', the reduced norm homomorphism $N_{\text{red}} : D \to K'$ is surjective). V. I. Yanchevskii [12] proved that, if K is a C'_2 – field, then $SK_1(A) = 0$ for any finite dimensional central simple K-algebra A. This completes the proof of Proposition 14.

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ПРО ГРУПУ БРАУЕРА ТА ПРИНЦИП ГАССЕ ДЛЯ ПСЕВДОГЛОБАЛЬНИХ ПОЛІВ

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Доведено критерій Тейта виконання принципу Гассе в скінченних розширеннях поля алгебричних функцій K із псевдоскінченним полем констант. Крім того, кожна скінченновимірна алгебра над таким полем є циклічною, а її індекс дорівнює експоненті.

Ключові слова: теорія полів класів, поле алгебричних функцій, група Брауера, принцип Гассе, скінченновимірні центральні прості алгебри.

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