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SOME OPEN PROBLEMS IN TOPOLOGICAL ALGEBRA

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In the paper we consider and comment some open problems in topological algebra posed by participants of the conference dedicated to the 20th anniversary of the Chair of Algebra and Topology of Lviv National University, that was held in September, 2001.
Key words: topological group, paratopological group, topological semigroup.

This is the list of open problems in topological algebra posed on the conference dedicated to the 20th anniversary of the Chair of Algebra and Topology of Lviv National University, that was held in September, 2001.

Problem 1 (Čhoban). *Is every topological group a quotient group of a zero-dimensional topological group of the same weight? When an almost metrizable topological group is a quotient of a zero-dimensional group of the same weight?*

A space X is zero-dimensional if $\dim X = 0$. A topological group is *almost metrizable* if contains a compact subset of countable character. Let us make some comments to this problem. If a topological group G is a quotient of an almost metrizable group H with $\operatorname{ind} H = 0$, then G contains a zero-dimensional compact subgroup of countable character in G.

If H is a zero-dimensional subgroup of countable character in a group G and $xh = hx$ for every $x \in G$ and $h \in H$, then G is a quotient group of some almost metrizable zero-dimensional group of the same weight. In particular the answer to the first part of Problem 1 is positive for metrizable groups (see [10], [8]) and for almost metrizable abelian groups. A. V. Arhangel'skii proved that every topological group is a quotient of a zero-dimensional σ-discrete group. For universal algebras this fact was proved in [9].

Problem 2 (Čhoban). *Under which conditions is the free universal algebra of an uncountable signature over a metrizable space X paracompact? In particular, is the free topological linear space $L(X)$ of X over the discrete field of real numbers paracompact?*

A semigroup S with the identity e endowed with a topology is called a *left* (resp. *right*) *bounded* if for every neighborhood U of e there is a finite subset F of S such that $S = FU$ (resp. $S = UF$); S is called *bounded* if S is both left and right bounded. Replacing the "finite subset F" by "countable subset F" we get the definition of a (*left, right*) *ω-bounded* semigroup.

For a topological space X let $S(X) \subset X^X$ be the semigroup of all continuous selfmappings of X endowed with the topology of pointwise convergence (i.e., the topology inherited from the Tychonov product X^X). A topological space X is called *homogeneous* if for any points $x_1, x_2 \in X$ there is a homemorphism h of X with $h(x_1) = x_2$.

Problem 3 (Protasov). *Is the semigroup $S(X)$ left bounded for every zero-dimensional compact homogeneous space?*

The answer is affirmative provided X has a base of the topology consisting of pairwise homeomorphic clopen subsets, see [22].

Problem 4 (Protasov). *Is the semigroup $S(X)$ right bounded for any zero-dimensional homogeneous space X?*

The answer is affirmative provided X has a base of the topology consisting of clopen subsets homeomorphic to X, see [22].

Let X be a topological space. A subgroup H of $S(X)$ is called *distal* if for any distinct points $x_1, x_2 \in X$ and any point $x \in X$ there is a neighborhood U of x such that $\{h(x_1), h(x_2)\} \not\subset U$ for all $h \in H$. It is proven in [22, Theorem 4] that a left bounded subgroup $H \subset S(X)$ is distal provided H acts transitively on X.

Problem 5 (Protasov). *Let X be a compact space and H be a distal subgroup of $S(X)$ acting transitively on X. Is H left bounded?*

Under a *left-topological group* we understand a pair (G, τ) consisting of a group G and a topology τ invariant with respect to the left shifts $l_g : x \mapsto gx$. If, in addition τ is invariant with respect to the right shifts, then (G, τ) is called a *semitopological group*. A semitopological group G with continuous inverse mapping $x \mapsto x^{-1}$ is called a *quasitopological group*. If the group operation of G is continuous with respect to the topology τ, then (G, τ) is a *paratopological group*. If, additionally, the operation of taking the inverse is continuous, then (G, τ) is a topological group.

It is well known that σ-compact topological groups have countable cellularity [29] while compact topological groups support a strictly positive probability measure (i.e., a Borel probability measure μ such that $\mu(U) > 0$ for any nonempty open subset U of the group). Recently T. Banakh and O. Ravsky [4] (see also [6]) proved that any bounded paratopological group G has countable cellularity (moreover, each cardinal of uncountable cofinality is a precaliber of G). On the other hand, according to [23] for every infinite cardinal τ there is a left bounded left topological group of cellularity τ.

Problem 6 (Protasov). *Let G be a bounded semitopological (quasitopological) group. Has G countable cellularity?*

The answer to this problem is positive for bounded first countable left topological groups. This follows from an easy observation that a first countable left topological group G has countable cellularity provided G is *left ω-narrow* in the sense that for each neighborhood $U \subset G$ of the unit and each uncountable subset $F \subset G$ there are two distinct elements $x, y \in F$ with $xU \cap yU \neq \emptyset$. It is easy to see that a left topological group is left ω-narrow provided it is right ω-bounded. Therefore each right ω-bounded first countable left topological group has countable cellularity.

Problem 6 is related to another

Problem 7 (Protasov). *Does every zero-dimensional compact homogeneous space admit a structure of a left topological group?*

According to [25], each homogeneous topological (T_1-)space X is the quotient space G/H of the homeomorphism group G of X endowed with a suitable left invariant topology by a (closed discrete) subgroup H of G. For countable spaces there is a much stronger result proved recently by E. Zelenyuk: for each countable group G and each regular homogeneous countable topological space X there is a left topological group H homeomorphic to X and algebraically isomorphic to G. Thus each countable regular space is homeomorphic to a semitopological group. This Zelenyuk's result is specific for countable spaces and cannot be generalized onto zero-dimensional homogeneous spaces: there are examples of zero-dimensional homogeneous spaces homeomorphic to no semitopological group. Many such examples can be constructed with help of a recent result [18] implying that *each Tychonov almost Čech-complete semitopological group is a Čech-complete topological group.* A Tychonov space X is defined to be *almost Čech-complete* if X contains a dense Čech-complete subspace. Examples of compact left topological groups which are not topological groups (see [19]) show that the mentioned result of [18] cannot be generalized onto left topological groups. Nonetheless we do not know the answer to

Problem 8 (Banakh). *Is every almost Čech complete left topological group Čech complete?*

Let us note that each almost Čech complete left topological group G is *non-expandable* in the sense that G coincides with any Tychonov left topological group $\tilde{G}$ containing G as a dense subgroup. The non-expandable left topological groups can be thought as "complete" in some sense.

Problem 9 (Banakh). *Investigate the class of non-expandable left topological groups. In particular, is every metrizable non-expandable left topological group Čech-complete?*

According to a remarkable theorem of [14], the countable power X^ω of a zero-dimensional metrizable space X is homogeneous provided X is meager or almost Čech-complete. This theorem in combination with [18] allows us to construct simple examples of homogeneous almost Čech complete metrizable spaces which are not Čech-complete and thus fail to support the structure of a semitopological group. On the other hand, by the technique of [13] and [15] it can be shown that the countable power X^ω of any meager separable metrizable space X is homeomorphic to a Boolean topological group.

There are also simple examples of homogeneous compact first countable spaces homeomorphic to no semitopological groups. Such spaces can be simply constructed with help of the Motorov Theorem asserting that the countable power K^ω of each first-countable zero-dimensional compact space K is homogeneous, see [2] and [12]. Observe that such a power K^ω admits the structure of a semitopological group if and only if the compactum K is metrizable (in this case K^ω is homeomorphic to the Cantor set).

In constrast, the structure of a left-topological group does not impose so strict restrictions on the topological structure of homogeneous compacta. For example, the Aleksandrov "two-arrows" space T is a non-metrizable zero-dimensional first-countable compactum carrying the structure of a left-topological group (algebraically isomorphic to the semi-direct product $S^1 \rtimes \mathbb{Z}_2$ of the circle and $\mathbb{Z}_2 = \{0, 1\}$). It follows that T^ω carries a structure of a left-topological group.

Problem 10 (Banakh). *Does the countable power K^ω of a compact first-countable zero-dimensional space K carry a structure of a left topological group?*

Note that the affirmative answer to Problem 6 would follow from the affirmative answer of

Problem 11 (Protasov). *Does every compact left topological group G support a strictly positive probability Borel measure?*

There exists a compact left-topological group admitting no *invariant* probability Borel measure, see [19].

Problem 12 (Protasov). *Is every topological group algebraically generated by a nowhere dense subset?*

Let us mention that each countable topological group is algebraically generated by some closed discrete subset (see reference in [24]) while every left topological group is algebraically generated by some subset with empty interior [24].

It is known that each regular countably compact paratopological group is a bounded topological group, see [21]. I. Guran [17] asked if any Hausdorff countably compact paratopological group is a topological group. O. Ravsky and E. Reznichenko [26] (see also [27]) observed that this question is equivalent to the problem of the ω-boundedness of any Hausdorff countably compact paratopological group. In the same paper [26] an example of a Hausdorff countably compact paratopological group which is not a topological group was constructed under Martin Axiom.

Problem 13 (Guran). *Is there a ZFC-example of a Hausdorff countably compact paratopological group which is not a topological group?*

Problem 14 (Guran). *Is a paratopological group G right ω-bounded if it is left ω-bounded?*

The answer to the last question is positive provided G is *saturated* (in the sense that the inverse U^{-1} of any neighborhood $U \subset G$ of the unit has non-empty interior in G) or *quasi-balanced* (in the sense that for each neighborhood $U \subset G$ of the unit there is a countable family $\mathcal{U}$ of neighborhoods of the unit such that for any $a \in G$ there are a neighborhood $U \in \mathcal{U}$ and an element $b \in G$ with $Wb \subset aU$), see [4] and [5].

A subset A of a topological group G is called *o-bounded* if for any sequence $(U_n)_{n\in\omega}$ of neighborhoods of the origin of G there is a sequence $(F_n)_{n\in\omega}$ of finite subsets of G such that $A \subset \bigcup_{n\in\omega} F_n U_n$. It is clear that each σ-compact topological group is o-bounded while each o-bounded group is ω-bounded. Next, given a subset A of a topological group G, consider the following game OF(A) (abbreviated from Open-Finite). Two players, I and II, choose at every step $k \in \omega$ a neighborhood $U_n \subset G$ of the origin, and a finite subset F_n of G, respectively. At the end of the game, player II is declared the winner if $A \subset \bigcup_{n\in\omega} F_n U_n$. It is easy to see that for a σ-compact group G player II has a winning strategy in the game $\mathrm{OF}(G)$. On the other hand, if a topological group G is not o-bounded, then player I has a winning strategy in $\mathrm{OF}(G)$.

Problem 15 (Banakh). *Is there a (metrizable) o-bounded topological group G such that player I has a winning strategy in the game OF(G)?*

Such a group, if exists, cannot be analytic and abelian (more generally, cannot be an analytic SIN-group). We remind that a topological space X is analytic if it is a metrizable continuous image of a separable complete metric space. On the other hand, the group $\mathbb{Z}^\omega$ contains a dense o-bounded G_δ-subset A such that the first player has a winning strategy in the game $\mathrm{OF}(A)$, see [7].

Problem 16 (Banakh). *Let n be a positive integer. Is there a compact subset K of the real line $\mathbb{R}$ such that the difference $K - K = \{x - y : x, y \in K\}$ is a neighborhood of zero in $\mathbb{R}$ but the sum $\underbrace{K + \cdots + K}_{n}$ is nowhere dense in $\mathbb{R}$?*

For $n = 2$ the answer is affirmative: By computer calculations, S. Ravsky has found that the compact subset

$$K = \Big\{ \sum_{n=1}^{\infty} \frac{x_n}{19^n} : x_n \in \{0, 1, 2, 10, 13, 16, 17, 18\} \Big\}$$

of the closed interval $[0, 1]$ has the following properties $K - K \supset (-1, 1)$ but $K + K \subset \big\{ \sum_{n=1}^{\infty} \frac{x_n}{19^n} : x_n \in \{0, \dots, 18\} \setminus \{6\} \big\}$ and thus is nowhere dense in $\mathbb{R}$.

Added in proofs. Recently T. Banakh and O. Hryniv [3] have answered Problems 3, 4 and 7 in negative. A suitable counterexample is supplied by the van Douwen homogeneous compactum Ξ constructed in [11] (see also [20]). The space has a very simple description: $\Xi = (A \times \{-1, 1\}) \cup ([0, 1] \setminus A) \times \{0\}$ for a suitable subset $A \subset (0, 1)$ and Ξ is endowed with the interval topology generated by the natural lexicographic order. The space Ξ has many surprising properties, in particular: 1) Ξ is a first-countable linearly ordered zero-dimensional homogeneous compactum admitting a continuous map $\pi : \Xi \to [0, 1]$ such that $|\pi^{-1}(x)| \leqslant 2$ for each $x \in [0, 1]$; 2) Ξ possesses a unique Borel probability measure μ that projects onto the Lebesgue measure by the map π; 3) two closed-and-open subsets of Ξ are homeomorphic if and only if their μ-measures coincide; 4) each homeomorphism of Ξ is measure preserving and each continuous self-mapping of Ξ does not increase the measure; 5) Ξ contains a countable dense subset $\mathbb{Q}$ such that $\mathbb{Q} \cap h(\mathbb{Q}) \neq \emptyset$ for any homeomorphism of Ξ 6) Ξ is homeomorphic to no left-topological group; 7) the group of homeomorphisms $H(\Xi)$ of Ξ is neither left nor right-bounded in the semigroup $C(\Xi)$ of continuous self-mappings of Ξ.

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ДЕЯКІ ВІДКРИТІ ПРОБЛЕМИ З ТОПОЛОГІЧНОЇ АЛГЕБРИ

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