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# ON CLOSED EMBEDDINGS OF FREE TOPOLOGICAL ALGEBRAS

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Let $\mathcal{K}$ be a complete quasivariety of completely regular universal topological algebras of continuous signature $\mathcal{E}$ (which means that $\mathcal{K}$ is closed under taking subalgebras, Cartesian products, and includes all completely regular topological $\mathcal{E}$-algebras algebraically isomorphic to members of $\mathcal{K}$). For a topological space $X$ by $F(X)$ we denote the free universal $\mathcal{E}$-algebra over $X$ in the class $\mathcal{K}$. Using some extension properties of the Hartman-Mycielski construction we prove that for a closed subspace $X$ of a metrizable (more generally, stratifiable) space $Y$ the induced homomorphism $F(X) \to F(Y)$ between the respective free universal algebras is a closed topological embedding. This generalizes one result of V.Uspenskiĭ [11] concerning embeddings of free topological groups.

*Key words:* universal topological algebra, free universal algebra, closed embedding, stratifiable space, metrizable space, Hartman-Mycielski construction

One of important recent achievements in the theory of free topological groups is a charming theorem by O. Sipacheva [9] asserting that the free topological group $F(X)$ of a subspace $X$ of a Tychonov space $Y$ is a topological subgroup of $F(Y)$ if and only if any continuous pseudometric on $X$ can be extended to a continuous pseudometric on $Y$, see [9]. The "only if" part of this theorem was proved earlier by V. Pestov [8] while the "if" part was proved by V. V. Uspenskiĭ [11] for the partial case of metrizable (or more generally, stratifiable) $Y$. To prove their theorems both Uspenskiĭ and Sipacheva used a rather cumbersome technique of pseudonorms on free topological groups which makes their method inapplicable for studying some other free objects.

In this paper, using a categorial technique based on extension properties of the Hartman-Mycielski construction we shall generalize the Uspenskiĭ theorem and prove some general results concerning embeddings of free universal algebras. It should be mentioned that the Hartman-Mycielski construction has been exploited in [2] for proving certain results concerning embeddings of free topological inverse semigroups.

Now we remind some notions of the topological theory of universal algebras developed by M. M. Choban and his collaborators, see [5]. Under a *continuous signature* we shall understand a sequence $\mathcal{E} = (E_n)_{n\in\omega}$ of topological spaces. A *universal topological algebra of continuous signature* $\mathcal{E}$ or briefly a *topological* $\mathcal{E}$*-algebra* is a topological space $X$ endowed with a sequence of continuous maps $(e_n : E_n \times X^n \to X)_{n\in\omega}$ called *algebraic operations* of $X$. A subset $A \subset X$ is called a *subalgebra* of $X$ if

$e_n(E_n \times A^n) \subset A$ for all $n \in \omega$. Under *a homomorphism* between topological $\mathcal{E}$-algebras $(X, (e_n^X)_{n\in\omega})$ and $(Y, (e_n^Y)_{n\in\omega})$ we understand a map $h : X \to Y$ such that

$$e_n^Y(c, h(x_1), \dots, h(x_n)) = h(e_n^X(c, x_1, \dots, x_n))$$

for any $n \in \omega$, $c \in E_n$, and points $x_1, \dots, x_n \in X$. Two topological $\mathcal{E}$-algebras $X$, $Y$ are *(algebraically) isomorphic* if there is a bijective homomorphism $h : X \to Y$. If, in addition, $h$ is a homeomorphism, then $X$ and $Y$ are *topologically isomorphic*.

Under a *free universal algebra* of a topological space $X$ in a class $\mathcal{K}$ of topological $\mathcal{E}$-algebras we understand a pair $(F(X), i_X)$ consisting of a topological $\mathcal{E}$-algebra $F(X) \in \mathcal{K}$ and a continuous map $i_X : X \to F(X)$ such that for any continuous map $f : X \to K$ into a topological $\mathcal{E}$-algebra $K \in \mathcal{K}$ there is a unique continuous homomorphism $h : F(X) \to K$ such that $f = h \circ i_X$. It follows that for any continuous map $f : X \to Y$ between topological spaces there is a unique continuous homomorphism $F(f) : F(X) \to F(Y)$ such that $F(f) \circ i_X = i_Y \circ f$. Our aim in the paper is to find conditions on $f$ guaranteeing that the homomorphism $F(f)$ is a topological embedding.

According to [5], a free universal algebra $(F(X), i_X)$ of a topological space $X$ exists (and is unique up to a topological isomorphism) provided $\mathcal{K}$ is *a quasivariety*, which means that the class $\mathcal{K}$ is closed under taking subalgebras and arbitrary Cartesian products. A quasivariety $\mathcal{K}$ of topological $\mathcal{E}$-algebras is called *a complete quasivariety* if any completely regular topological $\mathcal{E}$-algebra $X$, algebraically isomorphic to a topological algebra $Y \in \mathcal{K}$, belongs to the class $\mathcal{K}$.

Finally we remind that a regular topological space $X$ is called *stratifiable* if there exists a function $G$ which assigns to each $n \in \omega$ and a closed subset $H \subset X$, an open set $G(n, H)$ containing $H$ so that $H = \bigcap_{n\in\omega} \overline{G(n, H)}$ and $G(n, K) \supset G(n, H)$ for every closed subset $K \supset H$ and $n \in \omega$. It is known that the class of stratifiable spaces includes all metrizable spaces and is closed with respect to many countable operations over topological spaces, see [3], [6].

Now we are able to state one of our main results.

**Theorem 1.** *Let $\mathcal{K}$ be a complete quasivariety of completely regular topological $\mathcal{E}$-algebras of continuous signature $\mathcal{E}$. For any closed topological embedding $e : X \to Y$ between stratifiable spaces the induced homomorphism $F(e) : F(X) \to F(Y)$ between the corresponding free algebras is a closed topological embedding.*

In fact, Theorem 1 follows from a more general result involving the construction of Hartman and Mycielski. This construction appeared in [7] and was often exploited in topological algebra, see [4]. For a topological space $X$ let $HM(X)$ be the set of all functions $f : [0; 1) \to X$ for which there exists a sequence $0 = a_0 < a_1 < \dots < a_n = 1$ such that $f$ is constant on each interval $[a_{i-1}, a_i)$, $1 \leqslant i \leqslant n$. A neighborhood subbase of the Hartman-Mycielski topology of $HM(X)$ at an $f \in HM(X)$ consists of sets $N(a, b, V, \varepsilon)$, where

1) $0 \leqslant a < b \leqslant 1$, $f$ is constant on $[a; b)$, $V$ is a neighbourhood of $f(a)$ in $X$ and $\varepsilon > 0$;
2) $g \in N(a, b, V, \varepsilon)$ means that $|\{t \in [a; b) : g(t) \notin V\}| < \varepsilon$, where $|\cdot|$ denotes the Lebesgue measure on $[0, 1)$.

If $X$ is a Hausdorff (Tychonov) space, then so is the space $HM(X)$, see [7], [4]. The construction $HM$ is functorial in the sense that for any continuous map $p : X \to Y$ between topological spaces the map $HM(p) : HM(X) \to HM(Y)$, $HM(p) : f \mapsto p \circ f$, is continuous, see [7], [4], [10].

The space $X$ can be identified with a subspace of $HM(X)$ via the embedding $hm_X : X \to HM(X)$ assigning to each point $x$ the constant function $hm_X(x) : t \mapsto x$. This embedding $hm_X : X \to HM(X)$ is closed if $X$ is Hausdorff. It is easy to see that for any continuous map $f : X \to Y$ we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle hm_X}\downarrow & & \downarrow{\scriptstyle hm_Y} \\ HM(X) & \xrightarrow{HM(f)} & HM(Y). \end{array}$$

Our interest in the Hartman-Mycielski construction is stipulated by the following important extension result proven in [1].

**Proposition 1.** *For a closed subspace $X$ of a stratifiable space $Y$ there is a continuous map $r : Y \to HM(X)$ extending the embedding $hm_X : X \subset HM(X)$.*

It will be convenient to call a subspace $X$ of a space $Y$ *an $HM$-valued retract of* $Y$ if there is a continuous map $r : Y \to HM(X)$ extending the canonical embedding $hm_X : X \subset HM(X)$. In these terms, Proposition 1 asserts that each closed subspace of a stratifiable space $Y$ is an HM-valued retract of $Y$.

As a set, the space $HM(X)$ can be thought of as a subset of the Cartesian power $X^{[0,1)}$. Moreover, if $(X, (e_n)_{n\in\omega})$ is a topological $\mathcal{E}$-algebra then $HM(X)$ is a subalgebra of $X^{[0,1)}$. Let $\{e_n^{HM} : E_n \times HM(X)^n \to HM(X)\}_{n\in\omega}$ denote the induced algebraic operations on $HM(X)$. That is, $e_n^{HM}(c, f_1, \dots, f_n)(t) = e_n(c, f_1(t), \dots, f_n(t))$ for $n \in \omega$, $(c, f_1, \dots, f_n) \in E_n \times HM(X)^n$, and $t \in [0,1)$. It is easy to verify that the continuity of the operation $e_n$ implies the continuity of the operation $e_n^{HM}$ with respect to the Hartman-Mycielski topology on $HM(X)$). Thus we get

**Proposition 2.** *If $(X, (e_n)_{n\in\omega})$ is a topological $\mathcal{E}$-algebra, then $(HM(X), (e_n^{HM})_{n\in\omega})$ is a topological $\mathcal{E}$-algebra too. Moreover the embedding $hm_X : X \to HM(X)$ is a homomorphism of topological $\mathcal{E}$-algebras.*

Since $HM(K)$ is algebraically isomorphic to a subalgebra of $X^{[0,1)}$, we conclude that for each completely regular topological $\mathcal{E}$-algebra $X$ belonging to a complete quasivariety $\mathcal{K}$ of topological $\mathcal{E}$-algebras the $\mathcal{E}$-algebra $HM(X)$ also belongs to the quasivariety $\mathcal{K}$. Now we see that Theorem 1 follows from Proposition 1 and

**Theorem 2.** *Let $\mathcal{K}$ be a quasivariety of (Hausdorff) topological $\mathcal{E}$-algebras of continuous signature $\mathcal{E}$. Then for a subspace $X$ of a topological space $Y$ the homomorphism $F(e) : F(X) \to F(Y)$ induced by the natural inclusion $e : X \to Y$ is a (closed) topological embedding provided $X$ is an $HM$-valued retract of $Y$ and $HM(F(X)) \in \mathcal{K}$.*

*Proof.* Suppose that $HM(F(X)) \in \mathcal{K}$ and $X$ is a $HM$-valued retract of $Y$. The latter means that there is a continuous map $r : Y \to HM(X)$ such that $hm_X = r \circ e$ where $hm_X : X \to HM(X)$ and $e : X \to Y$ are natural embeddings. Applying to the maps $hm_X$, $e$ and $r$ the functor $F$ of the free universal $\mathcal{E}$-algebra in the quasivariety $\mathcal{K}$, we get the equality $F(hm_X) = F(r) \circ F(e)$.

Applying the functor $HM$ to the canonical map $i_X : X \to F(X)$ of $X$ into its free universal algebra, we get a continuous map $HM(i_X) : HM(X) \to HM(F(X))$. Taking into account that the $\mathcal{E}$-algebra $HM(F(X))$ belongs to the quasivariety $\mathcal{K}$, by the definition of the free algebra $(F(HM(X)), i_{HM(X)})$, we can find a unique continuous homomorphism $h : F(HM(X)) \to HM(F(X))$ such that $h \circ i_{HM(X)} = HM(i_X)$. Let us show that $h \circ F(hm_X) = hm_{F(X)}$. Since the maps $h \circ F(hm_X)$ and $hm_{F(X)}$ are homomorphisms from the free algebra $F(X)$ of $X$, to prove the equality $h \circ F(hm_X) = hm_{F(X)}$ it suffices to verify that $h \circ F(hm_X) \circ i_X = hm_{F(X)} \circ i_X$.

By the definition of the homomorphism $F(hm_X)$, we get the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & F(X) \\ {\scriptstyle hm_X}\downarrow & & \downarrow{\scriptstyle F(hm_X)} \\ HM(X) & \xrightarrow{i_{HM(X)}} & F(HM(X)) \end{array}$$

which implies that $h \circ F(hm_X) \circ i_X = h \circ i_{HM(X)} \circ hm_X = HM(i_X) \circ hm_X$ by the choice of the homomorphism $h$.

On the other hand, by the naturality of the transformations $\{hm_Z\}$, we get the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & F(X) \\ {\scriptstyle hm_X}\downarrow & & \downarrow{\scriptstyle hm_{F(X)}} \\ HM(X) & \xrightarrow{HM(i_X)} & HM(F(X)) \end{array}$$

which implies that $HM(i_X) \circ hm_X = hm_{F(X)} \circ i_X$. Thus $h \circ F(hm_X) \circ i_X = hm_{F(X)} \circ i_X$ which just yields $hm_{F(X)} = h \circ F(hm_X) = h \circ F(r) \circ F(e)$. Observe that the map $hm_{F(X)}$ is an embedding. Moreover, it is closed if $F(X)$ is Hausdorff (which happens if the quasivariety $\mathcal{K}$ consists of Hausdorff $\mathcal{E}$-algebras). Now the following elementary lemma implies that $F(e)$ is a (closed) embedding. □

**Lemma.** *Let $f : X \to Y$, $g : Y \to Z$ be continuous maps. If $g \circ f : X \to Z$ is a (closed) topological embedding, then so is the map $f$.*

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## ПРО ЗАМКНЕНІ ВКЛАДЕННЯ ВІЛЬНИХ ТОПОЛОГІЧНИХ АЛГЕБР

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Нехай $\mathcal{K}$ – повний квазімноговид цілком регулярних універсальних топологічних алгебр неперервної сигнатури $\mathcal{E}$. Для топологічного простору $X$ через $F(X)$ позначимо вільну універсальну $\mathcal{E}$-алгебру над $X$ в класі $\mathcal{K}$. Використовуючи конструкцію Гартмана-Мицельського, доводимо, що для замкненого підпростору $X$ метризовного простору $Y$ індукований гомоморфізм $F(X) \to F(Y)$ вільних універсальних алгебр – замкнене вкладення.

*Ключові слова:* універсальна топологічна алгебра, вільна топологічна алгебра, замкнене вкладення, стратифікований простір, метризовний простір, конструкція Гартмана-Мицельського.

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