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GENERATING PROPERTIES OF INVERTIBLE POLYNOMIAL MAPS IN THREE VARIABLES, WHICH HAVE A SMALL COMPOSITIONAL-TRIANGULAR LENGTH

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It is shown that each k-triangular invertible map (choosing in advance) for $k = 1, 2, 3$ with linear maps generate the group of tame polynomial automorphisms in three variables.

Key words: invertible polynomial map, affine space, affine group, infinitely dimensional algebraic group.

Invertible polynomial maps of an affine space A^n over a field K form a group GA_n (see [1],[2]), which sometimes is called the affine Cremona group. Elements of GA_n can be written down as tuples of polynomials

$$\langle f_1(x_1,\dots,x_n), f_2(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n)\rangle, \tag{1}$$

$f_i \in K[x]$, with composition of tuples as group operation and $X = \langle x_1,\dots,x_n\rangle$ as the unit of GA_n. It is useful to introduce vectors of the standard basis $\vec{e_i} = (0,\dots,0,1,0,\dots,0), i = 1,2,\dots,n$ and represent the elements (1) in the form

$$g = \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\vec{e}_i \tag{2}$$

For such a polynomial map $g \in GA_n$ let $\deg g = \max_i \deg f_i$. It is evident that maps with $\deg g \leqslant 1$ form an isomorphic copy of the affine group in GA_n. In particular, the elements $c_i = X + \vec{e}_i$ form a basis of A_n^+ as a vector space over K . Everywhere below we will identify AGL_n with the image of this standard enclosure. In this sense we shall also understand the matrix and permutational notations. For instance, the cycle $(1,2,3)$ in the dimension 3 means the transformation $\langle x_2, x_3, x_1\rangle$ in the form (1) and $A_{ij} = X + x_j\vec{e}_i \in GL_n$ in the form (2). It is easy to check that tuples (1) of the kind

$$\langle x_1 + h_1, x_2 + h_2(x_1),\dots,x_i + h_i(x_1,\dots,x_{i-1}),\dots x_n + h_n(x_1,\dots x_{n-1})\rangle \tag{3}$$

or

$$X + \sum_{i=1}^{n} h_i(x_1,\dots,x_{i-1})\vec{e}_i,$$

in the form (2), are invertible polynomial maps for arbitrary polynomials h_i and make up the subgroup U_n of the unitriangular transformations. U_n can be considered as iterated algebraic wreath product of K^+. There is a semidirect decomposition $U_n = U_{n-1}F_n$, where U_{n-1} (F_n) consist of the elements of form (3) with $h_n \equiv 0$ $(h_i \equiv 0,\ i = 1, \ldots, n-1)$. For $n = 3$ we have $U_3 = U_2F_2$. There is a partial order $\prec$ on U_n, which is the extension of the inverse lexicographical order of monomials. We will say that an element u_1 has height less than u_2, for $u_1, u_2 \in U_n$ if $u_1 \prec u_2$.

The normalizer $B_n = N_{GA_n}(U_n) = T_n \cdot U_n$ is a subgroup of triangular automorphisms whose elements have the form

$$\langle \alpha_1 x_1 + h_1, \alpha_2 x_2 + h_2(x_1), \ldots, \alpha_i x_i + h_i(x_1, \ldots, x_{i-1}), \ldots \alpha_n x_n + h_n(x_1, \ldots x_{n-1}) \rangle, \quad (4)$$

where T_n is an algebraic torus, $\alpha_i \in K^*$.

The subgroup GA_n^0 is a stabilizer of zero $(f_i(0) = 0)$ and contains a descended chain of the normal subgroups $GA_n^m, m = 0, 1, 2, 3, \ldots$, whose elements have the form $\langle x_1 + H_1^{m+1} + \ldots, x_2 + H_2^{m+1} + \ldots, \ldots, x_n + H_n^{m+1} + \ldots \rangle$, where H_2^{m+1} are homogeneous forms of degree $m+1$ and the dots mean items of higher degrees. Here is a simplest example of the element from GA_n^m : $\sigma^{(m)} = \langle x_1, x_2, \ldots, x_n + x_1^{m+1} \rangle$. There is a series of natural epimorphisms $\phi_k : GA_n^0 \to GA_n^0/GA_n^k$, moreover the corresponding quotient-group is a finite-dimensional algebraic groups. In particular, we have the semidirect decomposition , $GA_n^0 = GL_n \cdot GA_n^1$. Let us recall next definitions from [3]

Definition 1. An *elementary polynomial map* is defined as a transformation of kind

$$\langle x_1, x_2, \ldots, x_{i-1}, x_i + a(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \ldots, x_n \rangle.$$

Definition 2. A polynomial map, which is a finite composition of elementary or linear maps is called a *tame* map.

Tame automorphisms form group which can be defined as $TGA_n = \langle AGL_n, B_n \rangle$. Indeed, each elementary polynomial map is conjugate by a transposition (i, n) with an unitriangular one. On the other hand, triangular elements of kind $\langle x_1, x_2, \ldots, x_{i-1}, x_i + a(x_1, \ldots, x_{i-1}), x_{i+1}, \ldots, x_n \rangle$ are elementary maps and generate the group U_n.

One of the most difficult questions about affine Cremona groups is (see [2]): is TGA_n a proper subgroup of GA_n (the answer is negative for $n = 2$)?

Theorem 1. ([4]). *$GA_2 = B_2 * AGL_2$, where $*$ stands for the amalgamated product with the intersection $B_2 \cap AGL_2$, consisting of linear triangular maps (4).*

Corollary 1. $TGA_2 = GA_2$.

Corollary 2. *Let $\sigma^{(m)} = \langle x_1, x_2 + x_1^{m+1} \rangle \in GA_2^{(m)} \cap U_2$. Then the groups $Q_m = < AGL_2, \sigma^{(m)} >$ form an ascending chain $AGL_2 = Q_0 < Q_1 < \ldots Q_m < Q_{m+1}, \ldots$ with $GA_2 = \cup_m Q_m$.*

Proof. It follows from a uniqueness of element's decomposition in amalgamated products. Hence, the element $\sigma^{(k)}, k > m$ can not belong to Q_m. On the other hand, given element $\sigma^{(k)}$, one can calculate commutators with translations from A_n^+ and gets all elements $\sigma^{(m)}, m < k$. Thus $Q_m < Q_k$.

In particular, this means that in the dimension 2 the affine group is not maximal.

In the dimension more then 2 we have a more complicated situation. The question about the coincidence TGA_n with GA_n is open even in the dimension 3.

Conjecture (Nagata [2]) *The automorphism*

$$\langle x_1 - 2x_2(x_2^2 + x_3x_1) - x_3(x_2^2 + x_3x_1)^2, x_2 + x_3(x_2^2 + x_3x_1), x_3\rangle$$

is wild.

On the other hand, as follows from [5], the affine group is a maximal closed algebraic subgroup of GA_n as an ∞-dimensional group, correspondent definition was introduced in [1]. Comparing results with Nagata's conjecture, it is natural to pose a question: *is does AGL_n a maximal subgroup of TGA_n?*

As was proved in [6], one can replace the whole subgroup B_3 in the equality $TGA_3 = \langle AGL_3, B_3\rangle$ by any its nonlinear element. More precise,

Theorem 2. *([6]) Let $t \in B_3$ be an arbitrary nonlinear triangular map. Then*

$$TGA_3 = \langle t, AGL_3\rangle.$$

The aim of this paper is to show that bitriangular and three-triangular elements $g \in TGA_3$ have similar property $TGA_3 =< AGL_3, g >$.

Definition 3. A map $q \in GA_n$ is called k-triangular if it can be presented in the form

$$q = A_1 \cdot t_1 \cdot A_2 \cdot t_2 \cdot, \cdots, \cdot A_k \cdot t_k \cdot A_{k+1}, \qquad (5)$$

where $A_i \in AGL_n, t_i \in B_n$. The smallest number k satisfying (5) is called *the triangular-compositional length of q.*

The term bitriangular map means that triangular-compositional length of the map equals 2.

Theorem 3. *Let q be an arbitrary bitriangular transformation. Then*

$$TGA_3 = \langle q, AGL_3\rangle.$$

Let $G = \langle q, AGL_3\rangle$. Without lost of generality, we may assume that $q = t^A \cdot t'$, $t, t' \in U_3 \cap GA_3^0, A \in GL_3$. Let $A = B_1WB_2$ be a Brua decomposition, where $B_1, B_2 \in GL_3 \cap U_3$ and W is a permutational matrix. Then we have the bitriangular element $q^{B_2^{-1}} = t_1^W \cdot t_2 \in G$, where $t_1 = t^{B_1}, t_2 = (t')^{B_2^{-1}}$. Let us preserve the notation q for this new element of G

$$q = t_1^W \cdot t_2, \qquad (6)$$

and represent the triangular elements t_1, t_2 in the the form

$$t_1 = X + a_1(x_1)\vec{e}_2 + a_2(x_1, x_2)\vec{e}_3, \quad t_2 = X + b_1(x_1)\vec{e}_2 + b_2(x_1, x_2)\vec{e}_3 \in GA_3^1 \cap U_1.$$

Without lost of generality, one can suppose that $t_1, t_2 \in GA_n^0$ and polynomials a_1, a_2, b_1, b_2 have no linear parts. The idea of the proof is simple - to find an affine map a which permutable with t_2 (or t_1^W), calculate a commutator $[a, q^{-1}]$ (or $[a, q]$) and get a triangular element from G. In particular, one can use the element $a = c_3$ from the center of U_3 to get an 1-triangular element $t_1^{-Wc_3}t_1^W$ and apply Theorem 2.

Unfortunately, for some kind of triangular elements t_1, t_2 the correspondent commutators will be linear triangular elements and direct application of Theorem 2 is impossible. The situations, when it can be happened, is described in the next proposition.

Proposition 1. *If* $L = \{l(x_1, x_2) = \alpha x_1 + \beta x_2 + \gamma\}$ *is a set of linear polynomials, then*

1^0. $\Delta h = h(x_1, x_2 + a(x_1) + 1) - h(x_1, x_2 + a(x_1)) \in L$ *for some polynomial* $a(x_1)$, $\deg a > 1$ *iff polynomial* h *has the form* $h = h_{02}x_2^2 - 2h_{02}a(x_1)x_2 + h_{11}x_1x_2 + r(x_1) + l$, *where* $l \in L$.

2^0. $\Delta h = h(x_1 + 1, x_2 - \alpha x_1^2) - h(x_1, x_2 - \alpha x_1^2) \in L$ *iff* $h = r(x_2) + h_{11}x_1x_2 + \frac{1}{3}h_{11}\alpha x_1^3 + h_{20}x_1^2 + l$.

3^0. $\Delta h = h(x_1 + 1, x_2 - \beta(2x_1 + 1)) - h(x_1, x_2) \in L$ *iff* $h = f(x_2 - \beta x_1^2) + h_{11}x_1(x_2 - \beta x_1^2) - \frac{1}{3}h_{11}\alpha x_1^3 + h_{20}x_1^2 + l$, *where* $f = f(x_2)$ *is an arbitrary polynomial.*

Proof. 1^0. For $h = \sum_{i=0}^{m} r_i(x_1)x_2^i$ we have $\Delta h = mr_l(x_1)(x_2^{m-1}) + \ldots$, where dots mean items of lower degree than x_2. From this, it follows, that if $\Delta h \in L$ then $\deg r_l + m - 1 \leqslant 1$. Thus, we have the form of $h = r_2x_2^2 + r_l(x_1)x_2 + r_0(x_1)$, where $r_2 \in K, \deg r_1 \leqslant 1$. For such a form of h we have $\Delta h = 2r_2(x_2 + a(x_1)) + r_1(x_1) + r_2$. Since $\deg a > 1$, this polynomial can be linear when $2r_2a(x_1) + r_1(x_1) \in L$. Hence we get the form of h, pointed in 1^0.

In the case 2^0 select a highest monomial (in the sense $\prec$) containing x_1 : $h = f(x_2) + x_1^kx_2^m + \ldots (k > 0)$. Then we get $\Delta h = x_1^{k-1}x_2^l + \cdots \in L \to k + l \leqslant 2$. If $m = 0$ then $h = f(x_2) + g(x_1)$, $\deg g \leqslant 2$. If $f \neq 0$, then $h = h_{11}x_1x_2 + g_1(x_1) + f(x_2)$ and $\Delta h = h_{11}x_2 - \alpha x_1^2 + g_1(x_1 + t) - g_1(x_1)$. This polynomial can be linear if $\deg g_1 = 3$ and its highest item have the form $\frac{1}{3}h_{11}\alpha x_1^3$. In that way this case is exhausted. The case 3^0 can be proved by analogy.

Now we are ready to prove Theorem 3.

Proof. Let us analyze five cases corresponding to the forms of permutation matrix W in the formula (6).

Case 1. $W = (1, 2)$ is a transposition. In this case we get
$t_1^W = X + a_1(x_1)\vec{e_1} + a_2(x_1, x_2)\vec{e}_3$ and

$$q = X + a_1(x_2 + b_1(x_1))\vec{e_1} + b_1(x_1)\vec{e}_2 + (b_2(x_1, x_2) + a_2(x_2 + b(x_1), x_1)\vec{e}_3. \quad (7)$$

Consider the linear transformation $A_{32} = X + x_2\vec{e}_3 \in GL_n$ and obtain a map $q^{A_{32}} \cdot q^{-1} = X - b_1(x_1 - a_1(x_2))\vec{e}_5$. This element can be linear only if $b_1(x_1) \equiv 0$. If this happens then (6) implies that the element q is not bitriangular.

Case 2. $W = (2, 3)$.

Remark. A polynomial a_2 (b_2) can not be independent of x_2, because, in this case t_1^W (t_2^W) and then q^W is triangular.

Let us calculate the commutator with the translation c_2, which is a triangular map

$$q^{-1}q^{c_2} = X + (b_2(x_1, x_2 + t) - b_2(x_1, x_2))\vec{e}_3.$$

It will be linear iff $b_2 = b_{02}x_2^2 + b(x_1)$. In this situation let us consider the element

$$q^{c_1} \cdot q^{-1} = X + (a_2(x_1, x_3 - a_1(x_1 + 1)) - a_2(x_1, x_3 - a_1(x_1)))\vec{e}_5.$$

Accordance to Proposition 1 it will be linear iff $a_2 = a_{02}x_2^2 + 2a_{02}a_{11}(x_1)x_2 + a_{11}x_1x_2 + a(x_1)$. Let us use $A_{31} = X + x_1\vec{e}_3 \in U_3 \cap GL_3$ and calculate the double commutator $r = [q, c_3], r_1 = [A_{31}, r^{-1}]$. Next element will be obtained

$$r_1 = X + 2a_{02}x_1\vec{e}_2 - 2(2b_{02}a_{02}x_1x_2 + (b_{02}a_{02} - b_{11})a_{02}x_1^2)\vec{e}_3.$$

This element can be linear in the cases
[2.1] $b_{02} = b_{11} = 0$;
[2.2] $a_{02} = 0$,

In both cases the element r will be triangular, with $x_3 + (b_{11}a_{11} + b_{02}a_{11}^2)x_1^2 - 2b_{02}a_{11}x_1x_2$ as a third coordinate and a linear second coordinate. In the case 2.1 the polynomial $b_2 = b_2(x_1)$ does not depend on the variable x_2. As was pointed in the remark at the beginning of this case, it contradicts to the assumption that q is bitriangular element. In the case 2.2, if 2.1 is not hold, then the element r is linear when $a_{11} = 0$. The equalities $a_{02} = a_{11} = 0$ imply that a_2 does not depend on x_2, which (by the remark) leads to a contradiction also.

Case 3. $W = (1,3)$. This is the hardest case in which we have

$$q = X + a_2(x_1 + b_2(x_1,x_2), x_2 + b_2(x_1))\vec{e}_1 + (b_1(x_1) + a_1(x_3 + b_2(x_1,x_2)))\vec{e}_2 + b_2(x_1,x_2)\vec{e}_3.$$

Let us calculate the commutator

$$q^{c_3}q^{-1} = X + (a_1(x_1+1) - a_1(x_1))\vec{e}_2 + (a_2(x_1+1, x_2 - a(x_1)) - a_2(x_1, x_2 - a(x_1)))\vec{e}_3$$

Clearly that this triangular element can be linear only if $a_1(x_1) = \alpha x_1^2$ and (by Proposition 1) $a_2(x_1,x_2) = a(x_2) + a_{11}x_1x_2 + \frac{1}{3}a_{11}\alpha x_1^3 + a_{20}x_1^2$.

One the other hand, we have

$$q^{-c_1}q^{-1} = X + (b_1(x_1) - b_1(x_1+1))\vec{e}_2 + (b_2(x_1,x_2) - b_2(x_1+1, x_2 + b(x_1) - b(x_1+1)))\vec{e}_3,$$

which implies $b_1 = \beta x_1^2$ and (by p. 3^0 of Proposition 1) $b_2(x_1,x_2) = b(x_2 + \beta x_1^2) + b_{11}x_1(x_2 + \beta x_1^2) - \frac{1}{3}b_{11}\beta x_1^3 + b_{20}x_1^2$

Let us consider the

Case 3.1 when $\alpha = \beta = 0$. In this situation we can use the linear element $A_{32} = X + x_2\vec{e}_3$ again and get the element

$$r = q^{A_{32}} \cdot q^{-1} = X + [(a_{11} + a_{20})x_2^2 + 2a_{20}x_2x_3]\vec{e}_1.$$

Clearly, that $r^{(1,3)}$ is a triangular element and it will be linear if $a_{11} = a_{20} = 0$ only. If we replace the element q with q^{-W} then the similar procedure leads to the conclusion $b_{11} = b_{20} = 0$. So, the situation when $t_1 = X + a(x_1)\vec{e}_3, t_2 = X + b(x_1)\vec{e}_3$, $\deg a, \deg b > 1$ and $q = X + a(x_2)\vec{e_1} + b(x_2)\vec{e}_3$ should be considered. In this way we get the nonlinear triangular element $q^{(1,2)} = X + a(x_1)\vec{e_2} + b(x_1)\vec{e}_3$.

Case 3.2, in which $\alpha \neq 0, \beta = 0$. Let us put $Q_1 = X - b_{11}x_1\vec{e}_3 \in GL_3$ and calculate the element $r = q^{c_2} \cdot Q_1 \cdot q^{-1} \in G$. The element $A_{13} = X + x_3\vec{e}_1$ belongs to the center of $U_3^W \cap GA_3^0$ and so, one can get the element

$$r_1 = r^{A_2} \cdot r^{-1} = X + (b(x_2 - \alpha x_3^2) - b(x_2 + 1 - \alpha x_3^2))\vec{e_1}.$$

Thus we can get the triangular element r_1^W moreover, since $\alpha \neq 0$, and $\deg a \neq 1$ it can be linear only if $b \equiv 0$, i.e. $r_1 = X$. In this situation the element r has form

$$r = t_1^{Wc_2} \cdot t_1^{-W} = X + (a(x_2 + 1 - \alpha x_3^2) - a(x_2 - \alpha x_3^2) + a_{11}x_2)\vec{e}_1.$$

Similarly we can get $a \equiv 0$. Let us use the element $Q_2 = X - 2b_{20}x_1\vec{e}_3 \in AGL_3$ permutable with t_1, t_2 and get the element

$$r_3 = q^{c_1}Q_2q^{-1} \in G.$$

The calculation of the commutator with a linear element A_{13}, which was used above leads to the equality $r_4 = r_3^{A_{13}} r^{-1} = X + (b_{11}\alpha x_3^2 - b_{11}x_2 - b_{20})\vec{e_1}$. The triangular element r_4^W can be linear if $b_{11} = 0$. If this holds, then the element of torus $\tau = \langle x_1, sx_2, x_3\rangle$ is permutable with t_2 for any $s \in K^*$. Therefore one can get the triangular element $(q^\tau q^{-1})^W = t_1^\tau t_1^{-1}$, whose second coordinate equals $x_2 + \alpha\frac{1-s}{s}x_1^2$. Since $\alpha \neq 0$ this element is nonlinear for $s \neq 1$. This completes the analysis of case 3.2 .

Case 3.3. $\alpha = 0, \beta \neq 0$. Replacing q with q^{-W} it can be reduced to the previous case.

Case 3.4. $\alpha, \beta \neq 0$. This item is central in the case 3. Put

$$Q_3 = X - \frac{sb_{20} + a_{20}}{\alpha} x_2 \vec{e}_1 \in GL_3$$

where s is a parameter. It is convenient to introduce the element $q_1 = Q_3 t_1^W t_2$. The reason will be explained bellow. It is easy to see that monomial structures of the elements t_2^{-1} and t_1 are similar. Taking this in account, we could choose an element of the torus $T = \langle\lambda x_1, \mu x_2, \nu x_3\rangle$ in such a manner that the element $r_5 = t_2^W(Q_3 t_1^W)^T$ has the form

$$X + f(x_2)\vec{e}_1. \tag{8}$$

If we succeed in this then, without special difficulties we could derive a triangular nonlinear element from G. For first coordinates of the elements $(Q_3 t_1^W)^T$ and t_2^{-W} let us equate coefficients of their monomials $x_3^2, x_3^3, x_2 x_3$. In this way we get three equations with unknown λ, μ, ν

$$\frac{a_{11}\mu\nu}{\lambda} = -b_{11};\ \frac{a_{11}\alpha\nu^3}{\lambda} = b_{11}\beta;\ \frac{\nu^2 s b_{20}}{\lambda} = b_{20}. \tag{9}$$

It is evident that the solution $\lambda, \mu, \nu \in K^*$ exists if either both a_{11}, b_{11} equal to zero or both not. Now let us remark that the element Q_5 was introduced to avoid the same problem with the coefficients a_{02}, b_{02}.

In the Case 3.4.1 $a_{11}, b_{11} \neq 0$ we have the solution (9)

$$\lambda = \frac{b_{11}^2\beta^2 s^3}{a_{11}^2\alpha^2};\quad \mu = -\frac{b_{11}^2\beta s^2}{a_{11}^2\alpha};\quad \nu = \frac{b_{11}\beta s}{a_{11}\alpha}.$$

After the substitution of these values in $t_2^W(Q_3 t_1^W)^T$ we get an element of the necessary form (8), where

$$f(x_2) = s^{-3}\left(\frac{a_{11}\alpha}{b_{11}\beta}\right)^2 a\left(-\frac{b_{11}^2\beta s^2 x_2}{a_{11}^2\alpha}\right) + b(x_2) + (b_{20} + a_{20}s^{-1})\frac{x_2}{\beta}.$$

Since A_{31} is permutable with the unitriangular elements t_1, t_2, for the element $g = q_1^W(q_1)^T = t_1 r t_2^T$, we get a triangular element $g^{A_{31}}g^{-1} = \tilde{t}_1 r_5^{A_{31}}\tilde{t}^{-1}$, where $\tilde{t}_1 = (A_{31})^W t_1 \in U_3$. Direct calculations lead to the following formula

$$g^A g^{-1} = X+$$

$$(s^{-3}\left(\frac{a_{11}\alpha}{b_{11}\beta}\right)^2 a\left(-\frac{b_{11}^2\beta s^2(x_2 - \alpha x_1^2)}{a_{11}^2\alpha}\right) +$$

$$b(x_2-\alpha x_1^2)+(b_{20}+a_{20}s^{-1})\frac{x_2-\alpha x_1^2}{\beta})\vec{e}_3 \qquad (10)$$

Let us analyze conditions under which it will be linear. If a_k, b_k are coefficients at k-degrees of polynomials a, b then the coefficient of a monomial $x_2^k, k \geqslant 2$ is a polynomial from s

$$\left(\frac{a_{11}\alpha}{b_{11}\beta}\right)^2\left(\frac{b_{11}^2\beta}{a_{11}^2\alpha}\right)^k a_k s^{2k-3}+b_k.$$

It can be equal to zero if $a_k=b_k=0$. Hence, the element $g^A g^{-1}$ is linear iff $a \equiv b \equiv 0, a_{20}=b_{20}=0$, i.e. $t_2^W(Q_3 t_1^W)^T = X$. In this situation we can put $s=1$ in the polynomial f_2 and suppose that ν is a parameter. Then we have a triangular element $t=t_1 t_2^W t_1^{WT} t_2^T = t_1 t_2^T$ whose second coordinate is equal to

$$x_2-(\nu^4\left(\frac{a_{11}}{b_{11}}\right)^2-1)\alpha x_1^2.$$

Since $\alpha, a_{11} \neq 0$, one can choose a value of ν in such a manner that t is a nonlinear element.

Case 3.4.2. $a_{11}=0, b_{11}\neq 0$.

Let us put $g=q^{-c_2}q$ and use A_{31} again. Since A_{31} is permutable with t_1 it is easy to calculate

$$g^{-A_{31}}g=t_2^{-1}\left(t_1^{-W}t_1^{Wc_2}\right)^{A_{31}}t_1^{-Wc_2}t_1^W t_2 = X+(a(x_2+\beta x_1^2)-a(x_2+\beta x_1^2+1))\vec{e}_3.$$

Since $\beta\neq 0$, it follows that this triangular element can be linear only if $a\equiv 0$. In this situation the element $t_1^W=\langle x_1+a_{20}x_3^2, x_2+\alpha x_3^2, x_3\rangle$ is permutable with c_2, hence,

$$q^{-c_2}q=t_2^{-c_2}t_2 = X+(b(x_2+\beta x_1^2)-b(x_2+\beta x_1^2+1)-b_{11}x_1)\vec{e}_3.$$

This element can be linear if $b\equiv 0$.

In this case it is easy to verify that the linear transformation $Q_4=\langle x_1,(1-\frac{ca_{20}}{\alpha})x_2+cx_1,x_3\rangle$ is permutable with t_1^W for each value $c\neq\frac{\alpha}{a_{20}}$, hence, one can get the element $q^{-1}q^{Q_4}$. It turned out, that its second coordinate $x_2+\frac{\beta a_{20}c}{\alpha-a_{20}c}x_1^2$ is a nonlinear polynomial if $a_{20}\neq 0$. In the opposite case $a_{20}=0$ is the third coordinate of this element $x_3+b_{11}x_1^2c$ is a nonlinear polynomial.

Case 3.4.3. $a_{11}\neq 0, b_{11}=0$ can be reduced to the previous one replacing q by q^{-W}.

Case 3.4.4. $a_{11}=b_{11}=0$. In this case we use $\lambda=\nu^2 s, \mu=\alpha\nu^2/\beta$ and parameter ν as a solution of equations (9) and get the element

$$r_5=t_2^W(Q_4t_1^W)^T = X+\left(\frac{a\left(-\frac{\alpha\nu^2(x_2-\alpha x_1^2)}{\beta}\right)}{\nu^2 s}+b(x_2-\alpha x_1^2)+\frac{sb_{20}+a_{20}}{s\beta}x_2\right)\vec{e}_1.$$

One can repeat the argument done after the formula (10) and conclude that r_5 can be linear only if $a\equiv b\equiv 0$ and $a_{20}=b_{20}=0$. But under these conditions the element q is triangular and we get a contradiction.

Case 4. $W=(1,2,3)$. Our standard procedure leads to equalities :

$$(q^{c_3}q^{-1})^{W^2} = X+(a_2(x_1,x_2+a_1(x_1)+1)-a_2(x_1,x_2+a_1(x_1),))\vec{e}_3,$$

$$q^{c_1}q^{-1} = X+(b_1(x_1+1)-b_1(x_1))\vec{e}_2+(b_2(x_1+1,x_2)-b_2(x_1,x_2+b_1(x_1+1)+b_1(x_1)))\vec{e}_3$$

and the next conditions on the polynomials a_1, a_2, b_1, b_2, under which the obtained triangular elements are linear. $b_2(x_1) = \beta x_1^2$, and (by Proposition 1)

$$a_2 = a_{02}x_2^2 + 2a_{02}a_1(x_1)x_2 + a_{11}x_1x_2 + r(x_1)$$

$$b_2 = f(x_2 + \beta x_1^2) + b_{11}x_1(x_2 + \beta x_1^2) + \frac{1}{3}b_{11}\beta x_1^3 + b_{20}x_1^2,$$

where $r(x_1), f(x_1)$ are polynomials.

At the same time, together with the element $q = t_1^W t_2$ the group G contains the element $q_1 = q^{-W^{-1}} = t_2^{-W^2}t_1^{-1}$, for which one can to calculate the commutator $q_2 = q_1^{-1}q_1^{c_1}$ and check that it is not triangular. But the element

$$q_3 = q_2^{A_{31}}q_2^{-1} = X - b_{11}x_1\vec{e}_2 - b_{11}(2a_{02}x_1x_2 + (a_{11} - a_{02})x_1^2)\vec{e}_3 \in G$$

is triangular. Thus we have the alternative cases 4.1 and 4.2

Case 4.1. $b_{11} \neq 0, a_{02} = a_{11} = 0$, where we have

$$q = X + r(x_2+\beta x_1^2)\vec{e}_1 + \beta x_1^2\vec{e}_2 + [f(x_2+\beta x_1^2) + b_{11}x_1x_2 + \frac{2}{3}b_{11}\beta x_1^3 + a_1(x_2+\beta x_1^2)]\vec{e}_3.$$

One can get a triangular element $q^{-1}q^{A_{13}} = X - r(x_2 + \beta x_1^2)\vec{e}_3$, which can be linear under condition $r \equiv 0$. This yields a contradiction that q is triangular.

Case 4.2. $b_{11} = 0$ Direct calculation of a commutator leads to the triangular element

$$q_1^{-1}q_1^{c_1} = X - 2b_{20}(2x_1 + 1)\vec{e}_2 - (4a_{02}b_{20}x_1x_2 + \dots)\vec{e}_3,$$

which could be linear if $a_{02}b_{20} = 0$. Here dots mean items of the lower height.

Case 4.2.1. $b_{20} = 0$. The element $q^{-1}q^{A_{31}} = X + \beta(2x_1x_3 + x_3^2)$ is 1-triangular and can be linear only if $\beta = 0$.

Case 4.2.2. $a_{02} = 0$. In this case one can use $A_{21} = X + x_1\vec{e}_2$ and calculate the double commutator $g = q_1^{A_{31}}q_1^{-1}, g_1 = g^{A_{21}}g^{-1}$, which conjugate by the transposition $(2,3)$ with a triangular element of the form $X - (\beta^3x_3t + \dots)\vec{e}_5$. Just as in the the previous case the case $\beta = 0$ should be treated. But in both cases the equality $\beta = 0$ gives contradiction: q isn't bitriangular.

Case 5. $W = (1,3,2)$. The group G contains the element q together with $q_1 = q^{-W^{-1}} = t_2^{-W^{-1}}t_1^{-1}$. Since $W^{-1} = (1,2,3)$ this case is reduced to the previous one and this completes the proof of the theorem.

The previous proof was based on calculations with commutators $[q,c], c \in A_3^+$, which have the compositional-triangular length less then the element q. For 3-triangular maps, as a rule, we will get 3-triangular elements also. But the height of the intermediate triangular element of the new elements will decrease. The proof of the next theorem is based on this simple remark.

Theorem 4. *Let q be a 3-triangular element of GA_3. Then*

$$TGA_3 = \langle AGL_3, q\rangle.$$

Proof. If $G = \langle AGL_3, q\rangle$, then without loss of generality one can suppose that q has a form $q = t_1^{A_1} t_2 t_3^{A_3}$. The Brua decomposition leads to the equality

$$q = B_1^{-1} \cdot {t'_1}^{W_1} \cdot B_1 \cdot t_2 \cdot B_3^{-1} \cdot {t'_3}^{W_3} \cdot B_3$$

and we get an element

$$B_1 \cdot q \cdot B_3 = {t'_1}^{W_1} \cdot t'_2 \cdot {t'_3}^{W_3} \in G,$$

where $t'_1 = B_1 \cdot t_1 \cdot B_1^{-1}, t'_3 = B_3 \cdot t_3 \cdot B_3^{-1}, t'_2 = B_1 \cdot t_2 \cdot B_3^{-1}$. Bellow we will preserve notations and suppose that the group G contains a map of the form

$$q = {t_1}^{W_1} \cdot t_2 \cdot {t_3}^{W_3}. \tag{11}$$

Case 1. $W_1 = W_3 = (1,2)$.
Put $A_{32} = X + x_2\vec{e}_3$. Since it is permutable with ${t_i}^{W_i}$, $i = 1,2$, we get a triangular element

$$q^{A_{32}} \cdot q^{-1} = X - b_1(x_1 - a_1(x_2))\vec{e}_3,$$

which could be linear only if $b_1 \equiv 0$. But in this situation q is a bitriangular element, because the element

$${t_1}^{(1,2)} \cdot t_2 = \left(t_1 \cdot t_2^{(1,2)}\right)^{(1,2)}$$

is 1-triangular. This contradiction completes analysis of this case.

Case 2. $W_1 = W_3 = (2,3)$.
Since $c_2 = X + \vec{e}_2$ is permutable with ${t_i}^{W_i}$, $i = 1,2$ we get the element

$$q_1 = q^{c_2} \cdot q^{-1} = {t_1}^{W_1} \cdot {t_2}^{c_2} \cdot {t_2}^{-1} \cdot {t_1}^{-W_1} \in G.$$

If a_2 is independent of x_2, then ${t_1}^{W_1}$ is triangular and q isn't 3-triangular, hence, a_2 depends on x_2.

Therefore one can proceed to calculate commutators $q_{i+1} = {q_i}^{c_2} q^{-1} = {t_1}^{W_1}\tau_i \cdot {t_1}^{-W_1}$ till τ_i will be of the form

Case 2.1. $\tau_i = X + \alpha x_1^k \vec{e}_3, \quad k > 0$;

Case 2.2. $\tau_i = X + (\beta x_2 + \gamma)\vec{e}_3$.

In the case 2.1 we get a 1-triangular element $q_i = \left(t_1 \tau_i^{W_1} t_1^{-1}\right)^{W_1}$, where $\tau_i^{W_1} = X + x_1^k \vec{e}_2$ is a triangular element and hence the element q_i is a nonlinear 1– triangular element.

In the case 2.2, τ_i is a linear element and hence the element q_i is bitriangular.

Case 3. $W = (1,3)$. Consider the case when t_2 doesn't depend on x_1, i.e. $t_2 = X + b_2(x_2)\vec{e}_3$. Then the commutator $q^{A_{13}} \cdot q^{-1} = X - b_2(x_2 - a_1(x_3))\vec{e}_1$ is a nonlinear 1-triangular element. In a similar way let us consider the case when t_1 doesn't depend on x_1, i.e. $t_2 = X + a_2(x_2)\vec{e}_3$. Remark that in this case $b_1 \not\equiv 0$. Indeed, if $b_1 \equiv 0$, then $t_1^{W_1} t_2$ is 1-triangular and q is not 3-triangular. Therefore the element $q^{A_{31}} q^{-1} = X + (a_2(x_2 - b_1(x_1 - a_2(x_2))) - a_2(x_2))\vec{e}_3$ is a nonlinear triangular one.

Let us consider the general case and calculate the commutator

$$q_1 = q^{c_1} \cdot q^{-1} = {t_1}^{W_1} \cdot {t_2}^{c_1} \cdot {t_2}^{-1} \cdot {t_1}^{-W_1} \in G,$$

where

$t_2^{c_1} \cdot t_2^{-1} = X + (b_1(x_1+1) - b_1(x_1))\vec{e}_2 + (b_2(x_1+1, x_2 - b_1(x_1)) - b_2(x_1, x_2 - b_1(x_1)))\vec{e}_3.$

Let us put $q_{i+1} = q_i^{c_1} \cdot q_i^{-1} = t_1^{W_1} \cdot \tau_i \cdot (t_1^{-W_1}) \in G$. One can proceed the process till the $\deg_{x_1} \tau_1 = 1$ (the case when $\deg_{x_1} t_2 = 0$, was considered above). Thus we will stop a process when the element $\tau_i = X + (\alpha x_1 + \beta)\vec{e}_2 + (x_1 r(x_2) + r_0(x_2))\vec{e}_3$ will be obtained. If $\deg r \leqslant 0$, then one can pick out the linear part

$$\tau_i = L \cdot \tau_i' = (X + (\alpha x_1 + \beta)\vec{e}_2 + (\alpha_1 x_1 + \beta_1)\vec{e}_3) \cdot (X + r_0(x_2)\vec{e}_3)$$

and to join it to $t_1^{W_1}$. It could be done by replacing q_{i+1} with $L^{-1} \cdot q_{i+1} \in G$. In this way we get the element $\bar{q}_{i+1} = t_1^{W \cdot L} \tau_i' t_1^{-W}$. It is easy to check that the map

$$\bar{q}_{i+1}^{c_1} q_{i+1}^{-1} = t_1^{W \cdot L} \cdot t_1^{-W}$$

is a nonlinear bitriangular one. If $\deg r > 0$ then the element

$$\tau_i = X + (\alpha x_1 + \beta)\vec{e}_2 + (x_1 r_1 + r_0)\vec{e}_3$$

is linear and q_{i+1} is a bitriangular or 1-triangular. It is easy to check that the last case can be realized only if $a_2 = a_2(x_1), r_1 = 0$. Then it can be linear if $a_2 \equiv 0$, but it yields the contradiction that q is 1-triangular map.

In the case when $\deg r > 0$ one can proceed the process of the calculations τ_i until an element of the kind $\tau_i = X + (\alpha x_1 + \beta)\vec{e}_3$ appears. Similarly to the case of $\deg r < 1$ one can get a nonlinear bitriangular element.

The case when $W_1 = W_2 = (1, 2, 3)$ can be investigated by the previous procedure of an iterated commutators with c_1.

In the case $W_1 = W_2 = (1, 3, 2)$ we can calculate commutators $q_1 = q^{c_2} \cdot q^{-1} = t_1^{W_1} \cdot \tau_1 \cdot (t_1^{-W_1}) \in G$, where τ_1 has the form $X + r(x_1, x_2)\vec{e}_3$. If $\deg r > 1$, then one can consider the element $q_1^{(1,2)} = t_1^{(1,3)} \tau_1 t_1^{-(1,3)} \in G$ and reduce this case to the previous one. Let us investigate the situations, when $\deg \tau_1 \leqslant 1$. If $\deg \tau_1 = 1$, then we have the map q_1 which is bitriangular unless the case when a_1, a_2 doesn't depend on x_1. But the last case is impossible because it contradicts to the suggestion that q is 3-triangular. We can obtain the element q_1 with $\deg \tau_1 = 0$, when b_2 doesn't depend on x_2. In this case the element $q^{(2,3)} = t_1^{(1,2)} t_2^{(1,2)} t_1^{-(1,2)} \in G$ have the form of the case 1.

If q has the form (11), where $W_1 \neq W_2$ one can choose the linear element A_{ij}, permutable with $t_1^{W_1}$ or $t_2^{W_2}$ but not permutable with t_2. Then the map $q^{A_{ij}} \cdot q^{-1}$ or $q^{-1} \cdot q^{A_{ij}}$ will be 3-triangular and has the form (11) with $W_1 = W_2$.

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ПОРОДЖУЮЧІ ВЛАСТИВОСТІ ОБОРОТНИХ ПОЛІНОМІАЛЬНИХ ВІДОБРАЖЕНЬ ВІД ТРЬОХ ЗМІННИХ, ЩО МАЮТЬ МАЛУ КОМПОЗИЦІЙНО-ТРИКУТНУ ДОВЖИНУ

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Показано, що кожне k-трикутне оборотне відображення (наперед вибране) для $k = 1, 2, 3$ разом з лінійними відображеннями породжує групу ручних поліноміальних автоморфізмів від трьох змінних.

Ключові слова: оборотне поліноміальне перетворення, афінний простір, афінна група, афінна група Кремони, нескінченновимірна алгебрична група.

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