ВІСНИК ЛЬВІВ. УН-ТУ
Серія мех.-мат.2003.Вип.61.С.37–40

VISNYK LVIV UNIV.
Ser.Mech-Math.2003.Vol.61.P.37–40

УДК 515.12+517.51

PSEUDOCOMPACTENESS OF THE SPACES OF ALMOST CONTINUOUS MAPPINGS

Bogdan BOKALO
Ivan Franko National University of Lviv, 1 Universitetska Str. 79000 Lviv, Ukraine

Given two Hausdorff spaces we study properties of the function space $AC_p(X,Y) \subset Y^X$ consisting of all almost continuous mappings $f : X \to Y$ (the almost continuity of f means that any nonempty subspace $A \subset X$ contains a point of continuity of the mapping $f|A : A \to Y$). We prove that for infinite Hausdorff spaces X, Y the space $AC_p(X,Y)$ is pseudocompact iff $AC_p(X,Y)$ is σ-pseudocompact iff Y^ω is pseudocompact and X each countable subspace of X is scattered.

Key words: almost continuous mappings, topology of pointwise convergence, pseudocompact space, scattered space.

In the paper we detect pseudocompact spaces $AC_p(X,Y) \subset Y^X$ consisting of all almost continuous mappings $f : X \to Y$. We remind that a mapping $f : X \to Y$ between topological spaces is called *almost continuous* if every non-empty subspace $A \subset X$ of X contains a continuity point of the map $f|A : A \to Y$. By $AC_p(X,Y) \subset Y^X$ we denote the space of all almost continuous functions $f : X \to Y$, endowed with the topology of point-wise convergence, see [1].

Observe that $AC_p(X,Y) = Y^X$ for any scattered space X. We recall that a topological space X is *scattered* if each subspace of X has an isolated point (equivalently, the identity map of X into X endowed with the discrete topology is almost continuous). We define a space X to be *ω-scattered* if each countable subspace of X is scattered. We shall prove that for an ω-scattered space X the subset $AC_p(X,Y)$ of Y^X still is very massive.

Given a function $f \in Y^X$ let $\Sigma(f) = \{g \in Y^X : |\{x \in X : f(x) \neq g(x)\}| \leqslant \aleph_0\}$. We call a subset $F \subset Y^X$ an *ω-tail set* in Y^X if $\Sigma(f) \subset F$ for any $f \in F$. Observe that each non-empty ω-tail subset $F \subset Y^X$ is *G_δ-dense* in the sense that $G \cap F \neq \emptyset$ for each non-empty G_δ-subset G of Y^X.

We shall say that a subspace Y of a space X is *C-embedded* into X if each continuous function $f : Y \to \mathbb{R}$ can be continuously extended over all X.

Proposition. *For a Hausdorff topological space X and an infinite Hausdorff space Y the following conditions are equivalent:*

1) *X is ω-scattered;*
2) *$AC_p(X,Y)$ is an ω-tail subset of Y^X;*
3) *$AC_p(X,Y)$ is G_δ-dense in Y^X.*

Moreover, if any finite power of Y is regular and Lindelöf, then the conditions (1)–(3) are equivalent to:

4) $AC_p(X,Y)$ is C-embedded into Y^X.

Proof. (1) $\Rightarrow$ (2) Assume that X is ω-scattered, $f : X \to Y$ is an almost continuous function and $g : X \to Y$ is a function such that the set $Z = \{x \in X : f(x) \neq g(x)\}$ is at most countable. We have to prove that g is almost continuous. Take any subset $A \subset X$. We consider two cases:

a) $A \cap Z$ is not dense in A. Then we can find a continuity point $a \in A \setminus \bar{Z}$ of the function $f|A \setminus \bar{Z}$ which also is a continuity point of the function $g|A$.

b) $A \cap Z$ is dense in A. The space $A \cap Z$, being a countable subspace of the ω-scattered space X, is scattered. Consequently, $A \cap Z$ contains an isolated point z which by the density of $A \cap Z$ in A is also isolated in A. Then z is a continuity point of the function $g|A$.

The implication (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Assume that $AC_p(X,Y)$ is G_δ-dense in Y^X but X contains a countable non-scattered subspace $A = \{a_n\}_{n\in\omega}$. Without loss of generality we can assume that A has no isolated points. The space Y, being infinite and Hausdorff, contains a countable collection $\{U_n\}_{n\in\omega}$ of non-empty pair-wise disjoint open subsets. Observe that the G_δ-subset $G = \{f \in Y^X : f(a_n) \in U_n \text{ for all } n \in \omega\}$ of Y^X misses the set $AC_p(X,Y)$ since G consists of functions everywhere discontinuous on A.

(2) $\Rightarrow$ (4) This implication follows from [2, 3.12.23(a)] asserting that for any Hausdorff space Y with Lindelöf finite powers Y^n and any $f \in Y^X$ the Σ-product $\Sigma(f)$ is C-embedded into Y^X.

The implication (4) $\Rightarrow$ (3) follows from the well-known fact asserting that each C-embedded subspace of a Tychonov space is G_δ-dense. □

Next we find conditions on infinite Hausdorff spaces X, Y under which the space $AC_p(X,Y)$ is (σ-)pseudocompact. We remind that a Hausdorff space X is *pseudocompact* if each locally finite collection of open subsets of X is finite. For Tychonov spaces this is equivalent to saying that each continuous real-valued function on X is bounded. A Hausdorff space X is defined to be *σ-pseudocompact* if X is the countable union of pseudocompact subspaces. It is easy to see that each dense pseudocompact subspace of a Hausdorff space X is G_δ-dense in X.

Theorem. *For infinite Hausdorff spaces X and Y the following conditions are equivalent:*

1) $AC_p(X,Y)$ is pseudocompact;

2) $AC_p(X,Y)$ is σ-pseudocompact;

3) X is ω-scattered and Y^ω is pseudocompact.

Proof. The implication (1) $\Rightarrow$ (2) is trivial.

(3) $\Rightarrow$ (1) Suppose X is ω-scattered and Y^ω is pseudocompact. To show that the space $AC_p(X,Y)$ is pseudocompact, assume that $\{U_n\}_{n\in\omega}$ is a locally finite collection of non-empty open subsets of $AC_p(X,Y)$. Without loss of generality, we can assume that for each set U_n there are a finite subset $C_n \subset X$ and an open set $W_n \subset Y^{C_n}$ such that $U_n = \pi_{C_n}^{-1}(W_n)$. The countable subspace $C = \bigcup_{n\in\omega} C_n$ of the ω-scattered space X is scattered. Consequently, the restriction operator $\pi_C : AC_p(X,Y) \to Y^C$, $\pi_C : f \mapsto f|C$, is surjective. This implies that $\{\pi_C(U_n)\}_{n\in\omega}$ is a locally finite collection of open sets in $\pi_C(AC_p(X,Y)) = Y^C$ which is not possible

because of the pseudocompactness of the space Y^ω. This contradiction shows that the space $AC_p(X,Y)$ is pseudocompact.

(2) $\Rightarrow$ (3) Suppose that the space $AC_p(X,Y)$ is σ-pseudocompact. First we show that the space Y^ω is pseudocompact.

The space X, being infinite and Hausdorff, contains a countable discrete subspace Z. Observe that the restriction operator $\pi_Z : AC_p(X,Y) \to Y^Z$, $\pi_Z : f \mapsto f|Z$, is surjective which implies that the space Y^ω is σ-pseudocompact. Using the fact that the spaces Y^ω and $(Y^\omega)^\omega$ are homeomorphic by the standard diagonal procedure it can be shown that the space Y^ω is pseudocompact.

Next we show that the space X is ω-scattered. Assume conversely that the space X contains a countable non-scattered subspace Z. Write $AC_p(X,Y) = \bigcup_{n\in\omega} B_n$, where B_n is pseudocompact for every $n \in \omega$.

Put $F_0 = \emptyset$ and $C_0 = Z$. By induction we shall construct countable sequences of function $(f_n)_{n\in\mathbb{N}} \in Y^X$, finite subsets $(F_n)_{n\in\omega}$ of Y and closed non-scattered subspaces $(C_n)_{n\in\omega}$ of Y such that

(a) $F_{n+1} \subset C_n$, $C_{n+1} \subset C_n \setminus F_{n+1}$;

(b) $g \notin \pi_{C_n}(B_n)$ for each function $g \in Y^{C_n}$ with $g|F_{n+1} = f_{n+1}|F_{n+1}$.

Assume that a non-scattered closed subspace C_n has been constructed. First we show that the projection $\pi_{C_n}(B_n)$ is not dense in Y^{C_n}. Assuming the converse we will get that the space $AC_p(C_n, Y)$ is pseudocompact since it contains a dense pseudocompact space $\pi_{C_n}(B_n)$. The pseudocompactness of $AC_p(C_n, Y)$ implies that it is G_δ-dense in Y^{C_n}. Applying the implication (3) $\Rightarrow$ (1) we conclude that the space C_n is ω-scattered which contradicts to the choice of C_n.

Hence $\pi_{C_n}(B_n)$ is not dense in Y^{C_n} and there are a function $f_{n+1} \in Y^X$ and a finite subset $F_{n+i} \subset C_n$ such that $g \notin \pi_{C_n}(B_n)$ for each $g \in Y^{C_n}$ with $g|F_{n+1} = f_{n+1}|F_{n+1}$. Finally take any non-scattered closed subspace $C_{n+1} \subset C_n$, disjoint with F_{n+1}. This completes the inductive step.

It follows that the subspace $F = \bigcup_{n\in\omega} F_n$ of X is scattered. Fix any point $y_0 \in Y$ and observe that the function $f : X \to Y$ defined by $f|X\setminus F \equiv y_0$ and $f|F_n = f_n|F_n$ for all n is almost continuous. On the other hand, by (b) $f \notin \bigcup_{n\in\omega} B_n = AC_p(X,Y)$ which is a contradiction. $\square$

---

1. *Bokalo B. M., Malanyuk O. P.* Some properties of topological spaces of almost continuous mappings // Matem. Studii. – 2000. – Vol. 14. – № 2. – P. 197-201.

2. *Engelking R.* General Topology. – Warszawa, 1977.

ПСЕВДОКОМПАКТНІСТЬ ПРОСТОРІВ МАЙЖЕ НЕПЕРЕРВНИХ ВІДОБРАЖЕНЬ

Б. Бокало

Львівський національний університет імені Івана Франка, вул. Університетська, 1 79000 Львів, Україна

Для заданих двох топологічних просторів X і Y вивчають властивості простору $AC_p(X,Y)$ майже неперервних відображень з простору X у простір Y в топології поточкової збіжності (відображення $f : X \to Y$ називається майже неперервним, якщо в кожному непорожньому підпросторі $A \subset X$ існує точка неперервності відображення $f|_A : A \to Y$). Доведено, що для нескінченних гаусдорфових просторів X і Y такі умови еквівалентні: 1) $AC_p(X,Y)$ – псевдокомпактний; 2) $AC_p(X,Y)$ є σ-псевдокомпактний; 3) кожний зліченний підпростір простору X є розрідженим і Y^ω – псевдокомпактним.

Ключові слова: майже неперервне відображення, топологія поточкової збіжності, псевдокомпактний простір, розріджений простір.

Стаття надійшла до редколегії 15.03.2002

Прийнята до друку 14.03.2003