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GENERALIZED NILPOTENT GROUPS WITH THE WEAK π-MINIMAL AND THE WEAK π-MAXIMAL CONDITIONS

Mykola CHERNIKOV, Mykola KHMELNITSKIY

Institute of Mathematics National Academy of Sciences of Ukraine,
3 Tereshchenkivska Str. 01601 Kyiv-4, Ukraine

Locally nilpotent and generalized radical groups with the weak π-minimal and the weak π-maximal conditions are investigated.

Key words: weak π-minimal condition, weak π-maximal condition, soluble minimax group, locally nilpotent group, locally finite group, Chernikov group.

Below π is some set of primes. Recall that a group G satisfies the *π-minimal* condition or, briefly, the condition π-min if G has no infinite chains $G_1 \supset G_2 \supset \ldots \supset G_i \supset G_{i+1} \supset \ldots$ of subgroups such that for each i the difference $G_i \setminus G_{i+1}$ contains a π-element (S.N. Chernikov, 1958). Recall that a group G satisfies the *weak π-minimal* (resp., the *weak π-maximal*) condition or, briefly, the π-min-∞ (resp., π-max-∞) condition if it has no infinite chains $G_1 \supset G_2 \supset \ldots \supset G_i \supset G_{i+1} \supset \ldots$ (resp., $G_1 \subset G_2 \subset \ldots \subset G_i \subset G_{i+1} \subset \ldots$) of subgroups such that for each i the index $|G_i : G_{i+1}|$ (resp., $|G_{i+1} : G_i|$) is infinite and the difference $G_i \setminus G_{i+1}$ (resp., $G_{i+1} \setminus G_i$) contains some π-element (N.S. Chernikov, see [1]).

The main results of the present paper are as follows.

Theorem 1 [1]. *For a locally nilpotent group G the following assertions are equivalent:*

1. G satisfies the π-min *condition.*
2. G satisfies the π-min-∞ *condition.*
3. The Sylow π-subgroup P of G is Chernikov.

Theorem 2 [2]. *Let G be a locally nilpotent group and P is the Sylow π-subgroups of G. The group G satisfies the* π-max-∞ *condition iff P is finite or G is a soluble minimax group.*

Theorem 3 [3]. *Let a group G have an infinite normal π-subgroup and possess an ascending series with locally nilpotent and locally finite factors. Then G satisfies the* π-max-∞ *condition iff it is almost soluble minimax.*

Lemma 1. *Let a group G satisfy the π-min-∞ or the π-max-∞ condition. Then in arbitrary direct decomposition of G the number of multipliers with nontrivial π-elements is finite.*

Proof is analogous to the proof of Lemma from [4].

Proposition 1. *Let a locally finite π-group G satisfy the π-min-∞ or the π-max-∞ condition. Then G is Chernikov.*

Proof. Let $G \neq 1$, A be a nontrivial abelian p-subgroup of G and B be the subgroup of all its elements of the orders $\leqslant p$. By the First Prufer's theorem A is a direct product of subgroups of order p. Therefore Lemma 1 implies $|B| < \infty$ and by Lemma 1.10 of [5], A is Chernikov. Further, any abelian subgroup $K \neq 1$ of G is a direct product of its nonidentity Sylow p-subgroups. By Lemma 1 the number of direct multipliers is finite. Consequently K is Chernikov. Then by results from [6, 7], G is Chernikov.

Note that according to Lemma 1.2 of [8], an abelian group G satisfies the min-∞ condition (i.e. the weak minimal condition for subgroups) or the max-∞ condition (i.e. the weak maximal condition for subgroups) iff G is minimax (i.e. G has a finite series such that each its factor satisfies the minimal or maximal condition for subgroups).

Proposition 2. *Let a group G satisfy the π-max-∞ condition and has some infinite normal locally finite π-subgroup H. Then G satisfies the* max$-\infty$ *condition for abelian subgroups (or, equivalently, all abelian subgroups of G are minimax).*

Proof. In view of Proposition 1, H is Chernikov. Let $K \leqslant H$, $|H : K| < \infty$ and K is a direct product of quasicyclic subgroups; K_i is the subgroups of K which consists of all its elements with orders $\leqslant i$, $i \in \mathbb{N}$. Then $K \trianglerighteq G$, and $|K_i| < \infty$, $K_{i+1} \supseteq K_i \triangleright G$ and $K = \bigcup_{i \in \mathbb{N}} K_i$. Let some abelian subgroup $A \subseteq G$ is not minimax. Since $A \cap K$ is Chernikov, it is easy to see that there exists some nonminimax subgroup $L \subseteq A$ such that $L \cap K = 1$. By Lemma 1.2 of [8] there is some ascending chain $L_1 \subset L_1 \subset \ldots \subset L_\omega$ of subgroups of L such that each index $|L_{i+1} : L_i|$ is infinite. Then for the chain $K_1L_1 \subset K_2L_2 \subset \ldots \subset KL_\omega$ of subgroups in G every index $|K_{i+1}L_{i+1} : K_iL_i|$ is infinite and, also, the set of all differences $K_{i+1}L_{i+1} \setminus K_iL_i$ possessing π-elements is infinite. Thus G does not satisfy the π-max condition, a contradiction.

Proposition 3. *Let a group G with minimax abelian subgroups has an ascending series with locally nilpotent factors and locally finite factors. Then G is minimax and almost soluble.*

Proof. Let H be a subgroup of G generated by all its normal radical (in the sense of B. I. Plotkin) subgroups. It is easy to see that H is radical. Obviously, G/H has the same series as G has, and, also, the locally nilpotent radical of G/H is identity. Let L/H be the locally finite radical of G/H, and A/H be arbitrary abelian subgroup of G/H. Then A is radical. Therefore in view of Theorem 4.2 of [8], A is minimax. Consequently, A/H is minimax too. Since A/H is periodic, it follows that A/H is Chernikov. Then by results from [6,7] L/H is Chernikov. Let $R \leqslant L/H$,

$|L/H : R| < \infty$ and R is a direct product of quasicyclic subgroups or $R = 1$. Then R is contained in the locally nilpotent radical of G/H. Consequently, $R = 1$ and $|L/H| < \infty$.

Thus, if $G/L = 1$, we have $|G : H| < \infty$.

Let $G/L \neq 1$. According to Theorem 1.2 from [9, Chapter V, §5] the locally finite radical of arbitrary group X contains all ascendant locally finite subgroups of X. Therefore G/L has no nonidentity ascendant locally finite subgroups. Then G/L has some nontrivial ascendant locally nilpotent subgroup. Therefore by the same theorem from [10] the locally nilpotent radical S/L of the group G/L is nontrivial. Further, obviously, $C_{S/H}(L/H) \trianglerighteq G/H$ and $C_{S/H}(L/H)$ is locally nilpotent. Therefore $C_{S/H}(L/H) = 1$. Since $|L/H| < \infty$ it follows that $|S/H : C_{S/H}(L/H)| < \infty$. So $|S/H| < \infty$. Then $S/H = L/H$, a contradiction.

Thus $G/L = 1$ and $|G : H| < \infty$. In view of Theorem 4.2 from [8], H is minimax and soluble. Consequently, G is minimax and almost soluble. Proposition is proven.

Proof of Theorem 1. Obviously, the assertion 1 is as a consequence of the assertion 2. Suppose the assertion 2 holds and $P \neq 1$. Since group G is locally nilpotent, P is a direct product of nontrivial Sylow p-subgroups by some primes $p \in \pi$. By Proposition 1 these Sylow p-subgroups are Chernikov, and by Lemma 1 their number is finite. Consequently P is Chernikov. Let the assertion 3 hold. Then by Lemma 1 of [10] G satisfies the π-min.

Proof of Theorem 2. Necessity. Let G satisfy the π-max-∞ condition and P is infinite. Then by Proposition 2 and Theorem 4.2 of [8], G is soluble minimax.

Sufficiency. Let G be soluble minimax. Then by Lemmas 1.1 and 1.2 from [8], G satisfies the max-∞ condition. Further, let $|P| < \infty$. Then the set of all π-elements of G is finite and, obviously, G satisfies the π-max-∞.

Proof of Theorem 3. Let G satisfy the π-max-∞. Then by Propositions 2 and 3, G is minimax and almost soluble.

Let G be minimax and almost soluble. Then by Lemmas 1.1, 1.2 of [8], G satisfies the max-∞.

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УЗАГАЛЬНЕНО НІЛЬПОТЕНТНІ ГРУПИ ЗІ СЛАБКИМИ УМОВАМИ π-МІНІМАЛЬНОСТІ ТА π-МАКСИМАЛЬНОСТІ

М. Хмельницький, М. Черніков

Інститут математики НАН України,
вул. Терещенківська, 3 01601 Київ-4, Україна

Досліджено локально нільпотентні та узагальнено радикальні групи зі слабкими умовами π-мінімальності та π-максимальності.

Ключові слова: слабка умова π-мінімальності, слабка умова π-максимальності, розв'язна мінімальна група, локально нільпотентна група, локально скінченна група, група Чернікова.

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