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IDEALS OF GORENSTEIN TILED ORDERS WHOSE FACTOR RINGS ARE QUASI-FROBENIUS

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The Gorenstein tiled orders whose the exponent matrices are the Cayley tables of finite groups are studied. The ideals of orders such that the factor rings modulo these ideals being quasi-Frobenius, are described.

Key words: Gorenstein tiled order, exponent matrix, adjacency matrix, quasi-Frobenius ring.

1. Preliminaries. The following result giving a constructive description of one class of semidistributive rings was proved in [5]:

1.1. Theorem. *A right Noetherian semiperfect semiprime and semidistributive ring is isomorphic to a finite direct product of the full matrix rings $M_{m_k}(D_k)$ over skew fields D_k, and rings of the form:*

$$\Lambda = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix}, \qquad (1)$$

where $n \geqslant 1$ and $\mathcal{O}$ is a discrete valuation ring with a prime element π, α_{ij} are integers, moreover, $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ for all i, j, k and $\alpha_{ii} = 0$ for all i.

Conversely, all such rings are Noetherian semiperfect semiprime and semidistributive ones.

Any ring of form (1) is called a semimaximal order (a tiled order). It is a prime two-sided Noetherian semiperfect ring.

We shall use the following notation: $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E}(\Lambda) = (\alpha_{ij})$ is the exponent matrix of a ring Λ. If a tiled order is reduced then $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \dots, n$.

Every tiled order Λ can be embedded in a simple Artinian ring $Q = \sum_{i,j=1}^{n} e_{ij} D$, where D is the division ring of fractions of a ring $\mathcal{O}$. Here Q is both the left and right classical quotient ring of the ring Λ. Let I be a two-sided ideal of a tiled order Λ. Obviously, $I = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\beta_{ij}}\mathcal{O}$, where e_{ij} are the matrix units. Denote by $\mathcal{E}(I) = (\beta_{ij})$

the exponent matrix of an ideal I. Let I and J be two-sided ideals of the ring Λ, $\mathcal{E}(I) = (\beta_{ij})$ and $\mathcal{E}(J) = (\gamma_{ij})$. We have $\mathcal{E}(IJ) = (\delta_{ij})$, where $\delta_{ij} = \min_k(\beta_{ik} + \gamma_{kj})$. If R is the Jacobson radical of a reduced tiled order Λ then $\mathcal{E}(R) = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for $i = 1, \dots, n$. Let $Q(\Lambda)$ be the quiver of a reduced tiled order Λ [6] and $[Q(\Lambda)]$ the adjacency matrix of the quiver $Q(\Lambda)$. Evidently, $[Q(\Lambda)]$ is a (0, 1)-matrix and $[Q(\Lambda)] = \mathcal{E}(R^2) - \mathcal{E}(R)$.

1.2 Definition. A tiled order Λ will be called a Gorenstein tiled order if Λ is a bijective Λ-lattice, i.e. Λ^* is a projective left Λ-lattice (see [7]).

Further the Gorenstein tiled order will be often called the Gorenstein order.

1.3. Lemma. *[1, Lemma 3.2] The following conditions for a tiled order $\Lambda = \{\mathcal{O}, E(\Lambda) = (\alpha_{pq})\}$ are equivalent:*

a) there exists a bijective Λ-lattice;

b) there exist indices i, j such that $\alpha_{ik} + \alpha_{kj} = \alpha_{ij}$ for $k = 1, \dots, n$.

1.4. Theorem. *[2] The following conditions for a reduced tiled order $\Lambda = \{\mathcal{O}, E(\Lambda) = (\alpha_{pq})\}$ are equivalent:*

a) Λ is a Gorenstein order;

b) there exists a permutation $\sigma = \{i \to \sigma(i)\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i = 1, \dots, n$; $k = 1, \dots, n$.

Proof follows immediately from Lemma 1.3.

Below the permutation σ will be called a Kirichenko permutation.

1.5. Theorem. *[4] Let A_A be Noetherian.*

a) The following conditions are equivalent:

1) A_A is injective;

2) A_A is cogenerating;

3) ${}_AA$ is injective;

4) ${}_AA$ is cogenerating;

5) $\forall M \subset A_A\ [rl(M) = M] \wedge \forall N \subset {}_AA\ [lr(N) = N]$.

b) If the conditions from (p) hold then A is a two-sided Artinian ring.

(By $l(M)$ and $r(N)$ the annihilators of modules M and N are denoted, i.e. $l(M) = \{a \in A \mid aM = 0\}$, $r(N) = \{b \in A \mid Nb = 0\}$).

1.6 Definition. [4] A ring is called quasi-Frobenius (a QF-ring) if the conditions from the previous theorem are satisfied.

1.7 Theorem. *[2] Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a prime reduced Gorenstein tiled order with the Jacobson radical R and J be a two-sided ideal Λ such that $\Lambda \supset R^2 \supset J \supset R^n$ ($n \geqslant 2$). The factor ring Λ/J is quasi-Frobenius (QF) if and only if there exists $p \in R^2$ such that $J = p\Lambda = \Lambda p$.*

2. Finite groups and Gorenstein orders. Put $G_0 = \{0\}$. Denote by Γ_0 a Gorenstein tiled order with the exponent matrix $\mathcal{E}(\Gamma_0) = (0)$.

The matrix $\mathcal{E}(\Gamma_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Cayley table of a cyclic group (2) and also the exponent matrix of the Gorenstein tiled order Γ_1 with the Kirichenko permutation $\sigma = (12)$.

Denote by $U_n = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$ the square matrix of order n.

Clearly, the Cayley table of the Klein Viergruppe (2) × (2) can be written in the form

$$\mathcal{E}(\Gamma_2) = \begin{pmatrix} \mathcal{E}(\Gamma_1) & \mathcal{E}(\Gamma_1) + 2U_2 \\ \mathcal{E}(\Gamma_1) + 2U_2 & \mathcal{E}(\Gamma_1) \end{pmatrix}.$$

Let us consider the matrix

$$\mathcal{E}(\Gamma_k) = \begin{pmatrix} \mathcal{E}(\Gamma_{k-1}) & \mathcal{E}(\Gamma_{k-1}) + 2^{k-1}U_{2^{k-1}} \\ \mathcal{E}(\Gamma_{k-1}) + 2^{k-1}U_{2^{k-1}} & \mathcal{E}(\Gamma_{k-1}) \end{pmatrix}. \tag{2}$$

2.1. Proposition. *$\mathcal{E}(\Gamma_k)$ is the exponent matrix of a tiled order.*

Evidently,

$$\Gamma_k = \begin{pmatrix} \Gamma_{k-1} & \pi^{2^{k-1}}\Gamma_{k-1} \\ \pi^{2^{k-1}}\Gamma_{k-1} & \Gamma_{k-1} \end{pmatrix}. \tag{3}$$

Induction on k easily yields that Γ_k is a tiled order.

Let $G = H \times \langle g \rangle$ be a finite Abelian group, $H = \{h_1, \dots, h_n\}$, $g^2 = e$. We shall consider the Cayley table of the group H as the matrix $K(H) = (h_{ij})$ with the entries in H, where $h_{ij} = h_i h_j$. The following proposition is obvious.

2.2. Proposition. *The Cayley table of the group G is of the form*

$$K(G) = \begin{pmatrix} K(H) & gK(H) \\ gK(H) & K(H) \end{pmatrix}.$$

2.3. Proposition. *$\mathcal{E}(\Gamma_k)$ is the Cayley table of a group G_k of order 2^k.*

The proof is based on induction on k. The basis of induction have been already considered. If $\mathcal{E}(\Gamma_{k-1})$ is the Cayley table of a group of order 2^{k-1} then by 2.2. Proposition $\mathcal{E}(\Gamma_k)$ is the Cayley table of a group G_k of order 2^k.

2.4. Proposition. *A tiled order Γ_k is Gorenstein with the Kirichenko permutation* $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & 2^k-1 & 2^k \\ 2^k & 2^k-1 & 2^k-2 & \dots & 2 & 1 \end{pmatrix}$.

Let us prove by induction on k that the tiled order Γ_k is Gorenstein. For $k = 1$ this is obvious. Let the tiled order Γ_k be Gorenstein with the exponent matrix $\mathcal{E}(\Gamma_k) = (\alpha^k_{ij})$ ($i,j = 1,2,\dots,2^k$) and the Kirichenko permutation $\sigma = \sigma_k$, where $\sigma_k(i) = 2^k + 1 - i$. Then $\alpha^k_{ij} + \alpha^k_{j\sigma_k(i)} = \alpha^k_{i\sigma_k(i)}$ for all $i,j = 1,2,\dots,2^k$. Since

$$\alpha^{k+1}_{2^k+i,j} = \alpha^{k+1}_{i,2^k+j} = \alpha^k_{ij} + 2^k, \quad \alpha^{k+1}_{2^k+i,2^k+j} = \alpha^{k+1}_{ij} = \alpha^k_{ij} \text{ for all } i,j = 1,2,\dots,2^k, \tag{4}$$

we have $(\alpha^k_{ij} + 2^k) + \alpha^k_{j\sigma_k(i)} = \alpha^k_{ij} + (2^k + \alpha^k_{j\sigma_k(i)}) = \alpha^k_{i\sigma_k(i)} + 2^k$. Thus, taking into account (4) we obtain that

$$\alpha^{k+1}_{ij} + \alpha^{k+1}_{j,2^k+\sigma_k(i)} = \alpha^{k+1}_{i,2^k+\sigma_k(i)}, \quad \alpha^{k+1}_{i,2^k+j} + \alpha^{k+1}_{2^k+j,2^k+\sigma_k(i)} = \alpha^{k+1}_{i,2^k+\sigma_k(i)},$$

$$\alpha^{k+1}_{2^k+i,2^k+j} + \alpha^{k+1}_{2^k+j,\sigma_k(i)} = \alpha^{k+1}_{2^k+i,\sigma_k(i)}, \quad \alpha^{k+1}_{2^k+i,j} + \alpha^{k+1}_{j\sigma_k(i)} = \alpha^{k+1}_{2^k+i,\sigma_k(i)},$$

$i,j = 1,2,\dots,2^k$. Putting $\sigma_{k+1}(i) = 2^k + \sigma_k(i)$, $\sigma_{k+1}(2^k+i) = \sigma_k(i)$, we have $\alpha_{pq}^{k+1} + \alpha_{q\sigma_{k+1}(p)}^{k+1} = \alpha_{p\sigma_{k+1}(p)}^{k+1}$ for all $p,q = 1,2,\dots,2^{k+1}$, i.e. the tiled order Γ_{k+1} is Gorenstein with the Kirichenko permutation $\sigma = \sigma_{k+1}$, where $\sigma_{k+1}(i) = 2^{k+1}+1-i$.

2.5. Theorem. *The Cayley table of a finite group G is the exponent matrix of a reduced Gorenstein tiled order if and only if $G = G_k = (2) \times \ldots \times (2)$.*

Proof. The Cayley table of an Abelian 2-group G_k has form (2) and is the exponent matrix of a tiled order Γ_k.

Conversely, let G be a finite group and its Cayley table be the exponent matrix of a reduced Gorenstein tiled order. Then for any $g \in G$ we obtain $g^2 = e$ and G is an elementary Abelian 2-group. The theorem is proved.

Let us calculate the adjacency matrix of the quiver $Q(\Gamma_k)$. For this aim we present the tiled order Γ_k in form (3).

Let $R_k = rad\Gamma_k$ be the Jacobson radical of the ring Γ_k and $\mathcal{E}(\Gamma_k) = (\alpha_{ij}^k)$, $\mathcal{E}(R_k) = (r_{ij}^k)$, $\mathcal{E}(R_k^2) = (\beta_{ij}^k)$. Then

$$R_k = \begin{pmatrix} R_{k-1} & \pi^{2^{k-1}}\Gamma_{k-1} \\ \pi^{2^{k-1}}\Gamma_{k-1} & R_{k-1} \end{pmatrix}, R_k^2 = \begin{pmatrix} R_{k-1}^2 + \pi^{2^k}\Gamma_{k-1} & \pi^{2^{k-1}}R_{k-1}\Gamma_{k-1} \\ \pi^{2^{k-1}}R_{k-1}\Gamma_{k-1} & R_{k-1}^2 + \pi^{2^k}\Gamma_{k-1} \end{pmatrix}.$$

As $r_{ij}^{k-1} \leqslant 2^{k-1}$ so $\beta_{ij}^{k-1} \leqslant 2^k \leqslant 2^k + \alpha_{ij}^{k-1}$. Therefore $R_{k-1}^2 + \pi^{2^k}\Gamma_{k-1} = R_{k-1}^2$.

The equality $(radA)A = A(radA) = radA$ holds for any prime tiled order A. Hence $\pi^{2^{k-1}}R_{k-1}\Gamma_{k-1} = \pi^{2^{k-1}}R_{k-1}$. Since $\mathcal{E}(\pi^{2^{k-1}}R_{k-1}) - \mathcal{E}(\pi^{2^{k-1}}\Gamma_{k-1}) = (2^{k-1} + \mathcal{E}(R_{k-1}) - (2^{k-1} + \mathcal{E}(\Gamma_{k-1})) = E$, we obtain

$$\mathcal{E}(R_k^2) - \mathcal{E}(R_k) = \begin{pmatrix} \mathcal{E}(R_{k-1}^2) - \mathcal{E}(R_{k-1}) & E \\ E & \mathcal{E}(R_{k-1}^2) - \mathcal{E}(R_{k-1}) \end{pmatrix}.$$

From this follows that

$$[Q(\Gamma_k)] = \begin{bmatrix} [Q(\Gamma_{k-1})] & E \\ E & [Q(\Gamma_{k-1})] \end{bmatrix}.$$

Let us compute the characteristic polynomial $\chi_k(x) = \chi_{[Q(\Gamma_k)]}(x)$.

$$\chi_{k+1}(x) = |xE - [Q(\Gamma_{k+1})]| = \begin{vmatrix} xE - [Q(\Gamma_k)] & -E \\ -E & xE - [Q(\Gamma_k)] \end{vmatrix} =$$

$$= \begin{vmatrix} xE - [Q(\Gamma_k)] - E & 0 \\ -E & xE - [Q(\Gamma_k)] + E \end{vmatrix} =$$

$$= |(x-1)E - [Q(\Gamma_k)]| \cdot |(x+1)E - [Q(\Gamma_k)]|$$

Therefore

$$\chi_{k+1}(x) = \chi_k(x-1) \cdot \chi_k(x+1). \tag{5}$$

Since $\chi_1(x) = \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix} = x(x-2)$, we have $\chi_2(x) = (x-3)(x-1)(x-1)(x+1) = (x-3)(x-1)^2(x+1)$, $\chi_3(x) = (x-4)(x-2)^2x(x-2)x^2(x+2) = (x-4)(x-2)^3x^3(x+2)$.

2.6. Proposition. $\chi_m(x) = \prod_{i=0}^{m}(x-m-1+2i)^{C_m^i}$.

We shall prove this proposition by induction. The basis of induction is clear. Suppose that the formula is true for $m = k$. Then by formula (5) we have

$$\chi_{k+1}(x) = \prod_{i=0}^{k}(x-k-2+2i)^{C_k^i} \cdot \prod_{j=0}^{k}(x-k+2j)^{C_k^j} =$$

$$= (x-k-2)\prod_{i=1}^{k}(x-k-2+2i)^{C_k^i} \cdot \prod_{j=0}^{k-1}(x-k+2j)^{C_k^j}(x+k) =$$

$$= (x-k-2)\prod_{i=0}^{k-1}(x-k+2i)^{C_k^{i+1}} \cdot \prod_{j=0}^{k-1}(x-k+2j)^{C_k^j}(x+k) =$$

$$= (x-k-2)\prod_{i=0}^{k-1}(x-k+2i)^{C_k^i+C_k^{i+1}}(x+k).$$

As $C_k^i + C_k^{i+1} = C_{k+1}^{i+1}$, then $\chi_{k+1}(x) = (x-k-2)\prod_{i=0}^{k-1}(x-k+2i)^{C_{k+1}^{i+1}}(x+k) =$
$= (x-k-2)\prod_{j=1}^{k}(x-k+2(j-1))^{C_{k+1}^j}(x+k) = \prod_{j=0}^{k+1}(x-(k+1)-1+2j)^{C_{k+1}^j}$.

By induction it is easily to prove that $\sum_{i=1}^{2^k} q_{ij}(\Gamma_k) = k+1$, $\sum_{j=1}^{2^k} q_{ij}(\Gamma_k) = k+1$. So $[Q(\Gamma_k)] = (k+1)P_k$, where P_k is a twice stochastic matrix.

3. On quasi-Frobenius factor rings of Gorenstein tiled orders. In this section we describe all two-sided ideals I of a Gorenstein prime tiled order $A = \Gamma_k$ such that I alies in square of the Jacobson radical of the ring A and the factor ring A/I is quasi-Frobenius.

Recall that there is a one-to-one correspondence between a two-sided ideal I and the exponent matrix $\mathcal{E}(I)$ also, and , besides, the inequalities $i_{pq} + \alpha_{qt} \geqslant i_{pt}$ and $\alpha_{pq} + i_{qt} \geqslant i_{pt}$ hold.

From the results of paper [2] it follows that the factor ring A/I is quasi-Frobenius if and only if there are isomorphisms $I \simeq A_A$, $I \simeq_A A$.

3.1. Proposition. *Let* $I_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix}$ *be an ideal of the ring* $\Gamma_{k+1} = \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix}$. *Then* L, M, N, T *are ideals of the ring* Γ_k, $\hat{I}_{k+1} = \begin{pmatrix} N & T \\ L & M \end{pmatrix}$ *and* $\check{I}_{k+1} = \begin{pmatrix} M & L \\ T & N \end{pmatrix}$ *are ideals of the ring* Γ_{k+1}, *too.*

Proof. Since I_{k+1} is an ideal of the ring Γ_{k+1}, we have

$$I_{k+1}\Gamma_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix}\begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} =$$

$$= \begin{pmatrix} L\Gamma_k + M\pi^{2^k}\Gamma_k & L\pi^{2^k}\Gamma_k + M\Gamma_k \\ N\Gamma_k + T\pi^{2^k}\Gamma_k & N\pi^{2^k}\Gamma_k + T\Gamma_k \end{pmatrix} = I_{k+1},$$

$$\Gamma_{k+1}I_{k+1} = \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} \begin{pmatrix} L & M \\ N & T \end{pmatrix} =$$

$$= \begin{pmatrix} \Gamma_k L + \pi^{2^k}\Gamma_k N & \Gamma_k M + \pi^{2^k}\Gamma_k T \\ \pi^{2^k}\Gamma_k L + \Gamma_k N & \pi^{2^k}\Gamma_k M + \Gamma_k T \end{pmatrix} = I_{k+1}.$$

We obtain the following equalities: $L\Gamma_k + M\pi^{2^k}\Gamma_k = L$, $L\pi^{2^k}\Gamma_k + M\Gamma_k = M$, $N\Gamma_k + T\pi^{2^k}\Gamma_k = N$, $N\pi^{2^k}\Gamma_k + T\Gamma_k = T$, $\Gamma_k L + \pi^{2^k}\Gamma_k N = L$, $\Gamma_k M + \pi^{2^k}\Gamma_k T = M$, $\pi^{2^k}\Gamma_k L + \Gamma_k N = N$, $\pi^{2^k}\Gamma_k M + \Gamma_k T = T$.

Thus, L, M, N, T are ideals of the ring Γ_k. Taking into account these equalities we have

$$\hat{I}_{k+1}\Gamma_{k+1} = \begin{pmatrix} N & T \\ L & M \end{pmatrix} \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} =$$

$$= \begin{pmatrix} N\Gamma_k + T\pi^{2^k}\Gamma_k & N\pi^{2^k}\Gamma_k + T\Gamma_k \\ L\Gamma_k + M\pi^{2^k}\Gamma_k & L\pi^{2^k}\Gamma_k + M\Gamma_k \end{pmatrix} = \begin{pmatrix} N & T \\ L & M \end{pmatrix} = \hat{I}_{k+1},$$

$$\Gamma_{k+1}\hat{I}_{k+1} = \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} \begin{pmatrix} N & T \\ L & M \end{pmatrix} = \begin{pmatrix} \Gamma_k N + \pi^{2^k}\Gamma_k L & \Gamma_k T + \pi^{2^k}\Gamma_k M \\ \pi^{2^k}\Gamma_k N + \Gamma_k L & \pi^{2^k}\Gamma_k T + \Gamma_k M \end{pmatrix} =$$

$$= \begin{pmatrix} N & T \\ L & M \end{pmatrix} = \hat{I}_{k+1}.$$

Therefore $\hat{I}_{k+1}$ is an ideal of the ring Γ_{k+1}. Analogously, it can be proved that $\check{I}_{k+1}$ is an ideal of the ring Γ_{k+1}. The proposition is proved.

3.2. Proposition. *Let I_{k+1} be an ideal of the ring Γ_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is quasi-Frobenius. Then $I_{k+1} = \begin{pmatrix} I_k & \pi^{2^k}I_k \\ \pi^{2^k}I_k & I_k \end{pmatrix}$ or*

$I_{k+1} = \begin{pmatrix} \pi^{2^k}I_k & I_k \\ I_k & \pi^{2^k}I_k, \end{pmatrix}$ where I_k is an ideal of the ring Γ_k such that the factor ring Γ_k/I_k is quasi-Frobenius.

Proof. The entries of the rows of the matrix $\mathcal{E}(\Gamma_{k+1})$ have such a property: if $i, l \leqslant 2^k$ or $i, l > 2^k$ then $|\alpha_{ij} - \alpha_{lj}| < 2^k$ for any j; if $i \leqslant 2^k, l > 2^k$ or $i > 2^k, l \leqslant 2^k$ then there exist at the least two values j and s such that $\alpha_{ij} - \alpha_{lj} \geqslant 2^k$ and $\alpha_{is} - \alpha_{ls} \leqslant 2^k$. The elements of the columns of the matrix $\mathcal{E}(\Gamma_{k+1})$ possess this property, too .

Let I_{k+1} be an ideal of the ring Γ_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is quasi-Frobenius, $1 = e_1 + e_2 + \cdots + e_{2^{k+1}}$ be a decomposition of the identity of the ring into a sum of local pairwise orthogonal idempotents. Hence by [2] for any i there exists t such that the right module $e_i I_{k+1}$ is isomorphic to the indecomposable projective right module $e_t\Gamma_{k+1} = P_t$. Analogously, for any j there exists m such that the left module $I_{k+1}e_j$ is isomorphic to the indecomposable projective left module $\Gamma_{k+1}e_m = Q_m$. Note that if $e_i I_{k+1} \simeq P_t$, $I_{k+1}e_j \simeq Q_m$ and $\mathcal{E}(I_{k+1}) = (\kappa_{uv})$ so by [1] $\kappa_{iv} = \alpha_{tv} + a_i$ for all v, $\kappa_{uj} = \alpha_{um} + b_j$ for all u (a_i, b_j are some integers).

Let $e_i I_{k+1} \simeq P_t$ and $e_j I_{k+1} \simeq P_m$. Assume that $i, j \leqslant 2^k, t \leqslant 2^k$ and $m > 2^k$. Then $\kappa_{iv} = \alpha_{tv} + a_i, \kappa_{jv} = \alpha_{mv} + a_j$. Let $a_i \geqslant a_j$. By the property of the rows of the matrix $\mathcal{E}(\Gamma_{k+1})$ there exists w such that $\alpha_{tw} - \alpha_{mw} \geqslant 2^k$. Hence $\kappa_{iw} - \kappa_{jw} = \alpha_{tw} + a_i - (\alpha_{mw} + a_j) \geqslant 2^k$. Since $0 \leqslant \alpha_{lw} < 2^k$ for all $0 < l \leqslant 2^k$, or $2^k \leqslant \alpha_{lw} < 2^{k+1}$for all $0 < l \leqslant 2^k$ then if $i, j \leqslant 2^k$ we have $\kappa_{iw} - \kappa_{jw} = \alpha_{iz} + b_w - (\alpha_{jz} + b_w) < 2^k$, where z satisfies the condition $I_{k+1}e_w \simeq Q_z$. We obtain contradiction.

Analogously, the case $t > 2^k$ and $m \leqslant 2^k$ is impossible for $i, j \leqslant 2^k$.

So for $i, j \leqslant 2^k$ two cases are possible : a) $t, m \leqslant 2^k$, b) $t, m > 2^k$.

By Proposition 3.1, if $I_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix}$ is an ideal of the order Γ_{k+1} then L, M, N, T are ideals of the order Γ_k.

In case a) for every $i \leqslant 2^k$ there exists $t \leqslant 2^k$ such that $e_i I_{k+1} \simeq P_t$.

Since $\alpha_{t,2^k+v} = \alpha_{t,v} + 2^k$ for $v \leqslant 2^k$, we have $\kappa_{i,2^k+v} = \alpha_{t,2^k+v} + a_i = \alpha_{tv} + 2^k + a_i = \kappa_{iv} + 2^k$, that is we have $M = \pi^{2^k} L$.

For every $j > 2^k$ there exists $m > 2^k$ such that $e_j I_{k+1} \simeq P_m$. Since $\alpha_{m,v} = \alpha_{m,2^k+v} + 2^k$ for $v \leqslant 2^k$, hence $\kappa_{jv} = \alpha_{mv} + a_j = \alpha_{m,2^k+v} + 2^k + a_j = \kappa_{j,2^k+v} + 2^k$, i.e. $N = \pi^{2^k} T$.

In case b) we obtain analogously $L = \pi^{2^k} M$, $T = \pi^{2^k} N$.

Now let us examine the entries of the columns of the matrix $\mathcal{E}(\Gamma_{k+1})$. Let $I_{k+1}e_i \simeq Q_p$ and $I_{k+1}e_j \simeq Q_r$, $i, j \leqslant 2^k$. For these entries as well as for the elements of the rows of this matrix there are two possible cases: c) $p, r \leqslant 2^k$, d) $p, r > 2^k$.

Suppose that in case a) for the entries of the rows we have case d) for the elements of the columns. Then we obtain $L = \pi^{2^k} N$ and $T = \pi^{2^k} M$.

Since $M = \pi^{2^k} L$ and $N = \pi^{2^k} T$, we obtain

$$L = \pi^{2^k} N = \pi^{2^k} \pi^{2^k} T = \pi^{2^{k+1}} T \text{ and } T = \pi^{2^k} M = \pi^{2^k} \pi^{2^k} L = \pi^{2^{k+1}} L.$$

Thus, $L = T = N = M = 0$ and $I_{k+1} = 0$.

In case c) for the entries of the columns we obtain $N = \pi^{2^k} L$, $M = \pi^{2^k} T$. As $M = \pi^{2^k} L$ and $N = \pi^{2^k} T$ so $L = T$, and therefore $M = N$.

Denote $L = I_k$, then $T = I_k$, $M = N = \pi^{2^k} I_k$. Thus, in case a) we have

$$I_{k+1} = \begin{pmatrix} I_k & \pi^{2^k} I_k \\ \pi^{2^k} I_k & I_k \end{pmatrix}.$$

Analogously in case b)

$$I_{k+1} = \begin{pmatrix} \pi^{2^k} I_k & I_k \\ I_k & \pi^{2^k} I_k \end{pmatrix}.$$

It remains to prove only the factor ring Γ_k / I_k is quasi-Frobenius. In case a) we have

$$\begin{aligned} \kappa_{iv} &= \alpha_{tv} + a_i \quad \text{for all } v = 1, \ldots, 2^{k+1}, \\ \kappa_{uj} &= \alpha_{um} + b_j \quad \text{for all } u = 1, \ldots, 2^{k+1}. \end{aligned} \tag{6}$$

Since $\mathcal{E}(I_k) = (\kappa_{uv})$, where $u, v \leqslant 2^k$, and $\mathcal{E}(\Gamma_k) = (\alpha_{ij})$, $i, j \leqslant 2^k$, the equalities (6) are true for the entries of the matrices $\mathcal{E}(I_k)$ and $\mathcal{E}(\Gamma_k)$. Therefore by [1] $e_i I_k \simeq e_t \Gamma_k$ and $I_k e_j \simeq \Gamma_k e_m$. In view of [2] we obtain that the factor ring Γ_k / I_k is quasi-Frobenius.

This completes the proof of the proposition.

3.3. Proposition. *There exist 2^{k+1} essentially different ideals I_{k+1} of the ring Γ_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is quasi-Frobenius.*

Proof. By Proposition 3.2, it follows that on the basis of any ideal I_k such that Γ_k/I_k is a QF-ring one can construct two ideals of the ring Γ_{k+1} such that Γ_{k+1}/I_{k+1} is a QF-ring. So the number of the ideals of the ring Γ_{k+1} is twice as many as the number of the ideals of the ring Γ_k (with the property Γ_k/I_k being a QF-ring). As $\Gamma_0 = \mathcal{O}$ has the unique non-isomorphic ideal, by induction it is easily to obtain the proposition.

Let I_{k+1} be an ideal of the ring Γ_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is quasi-Frobenius. Therefore by Proposition 3.2 the transformation $I_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix} \to \hat{I}_{k+1} = \begin{pmatrix} N & T \\ L & M \end{pmatrix}$ is equivalent to the transformation $I_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix} \to \check{I}_{k+1} = \begin{pmatrix} M & L \\ T & N \end{pmatrix}$. By such transformations we obtain again an ideal $\overline{I}_{k+1} = \hat{I}_{k+1} = \check{I}_{k+1}$ such that the factor ring $\Gamma_{k+1}/\overline{I}_{k+1}$ is quasi-Frobenius.

3.4. Proposition. *By the above-indicated transformations any ideal I_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is a QF-ring can be obtained from the principal ideal $\tilde{I}_{k+1} = \pi^p\Gamma_{k+1}$.*

Proof. The transformation $I_{k+1} \to \overline{I}_{k+1}$ is invertible, hence it is sufficient to show that any ideal I_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is a QF-ring can be reduced to the principal ideal $\tilde{I}_{k+1} = \pi^p\Gamma_{k+1}$.

Suppose that an ideal I_{s+1} has the form $I_{s+1} = \begin{pmatrix} \pi^{2^k} I_s & I_s \\ I_s & \pi^{2^k} I_s \end{pmatrix}$ for some $s \leqslant k$. Then $\overline{I}_{s+1} = \begin{pmatrix} I_s & \pi^{2^k} I_s \\ \pi^{2^s} I_k & I_s \end{pmatrix}$. Along with this, the ideal I_{k+1} can be transformed into an ideal I'_{k+1} but the factor ring Γ_{k+1}/I'_{k+1} still remains quasi-Frobenius. Therefore by the indicated transformations the ideal I_{k+1} is reduced to the ideal $\tilde{I}_{k+1}$ for which $\tilde{I}_{s+1} = \begin{pmatrix} \tilde{I}_s & \pi^{2^k} \tilde{I}_s \\ \pi^{2^s} \tilde{I}_k & \tilde{I}_s \end{pmatrix}$ for all $s \leqslant k$.

Let $\mathcal{E}(\tilde{I}_{k+1}) = (\tilde{\kappa}_{uv})$ and $\tilde{\kappa}_{11} = p$. Then $\tilde{I}_1 = \begin{pmatrix} \tilde{I}_0 & \pi^{2^0}\tilde{I}_0 \\ \pi^{2^0}\tilde{I}_0 & \tilde{I}_0 \end{pmatrix} = \begin{pmatrix} \tilde{I}_0 & \pi\tilde{I}_0 \\ \pi\tilde{I}_0 & \tilde{I}_0 \end{pmatrix} = \begin{pmatrix} \pi^p\mathcal{O} & \pi^{p+1}\mathcal{O} \\ \pi^{p+1}\mathcal{O} & \pi^p\mathcal{O} \end{pmatrix} = \pi^p \begin{pmatrix} \mathcal{O} & \pi\mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix} = \pi^p\Gamma_1$.

Assume that $\tilde{I}_k = \pi^p\Gamma_k$. Hence

$$\tilde{I}_{k+1} = \begin{pmatrix} \tilde{I}_k & \pi^{2^k}\tilde{I}_k \\ \pi^{2^k}\tilde{I}_k & \tilde{I}_k \end{pmatrix} = \begin{pmatrix} \pi^p\Gamma_k & \pi^{2^k}\pi^p\Gamma_k \\ \pi^{2^k}\pi^p\Gamma_k & \pi^p\Gamma_k \end{pmatrix} = \pi^p \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} = \pi^p\Gamma_{k+1}.$$

By induction we have proved that $\tilde{I}_{k+1} = \pi^p\Gamma_{k+1}$.

Obviously, $e_i\tilde{I}_{k+1} \simeq P_i$, $\tilde{I}_{k+1}e_j \simeq Q_j$. We note that by the indicated transformations the first column of the matrix $\mathcal{E}(\Gamma_k)$ (and the others, too) always turns into another column of this matrix. If the first column turns into l-th column then for any $i \leqslant 2^k$ there exists an integer r such that $\alpha_{i1} = \alpha_{lr}$. So by such transformations every principal ideal turns into an ideal I_k for which $e_lI_k \simeq P_i$, $I_ke_l \simeq Q_i$. As

$\alpha_{i1} = i - 1 = \alpha_{lr}$ so $i = \alpha_{lr} + 1$. Therefore

$$e_l I_k \simeq P_{1+\alpha_{lr}}, \ I_k e_l \simeq Q_{1+\alpha_{lr}}. \tag{7w}$$

The exponent matrix $\mathcal{E}(\Gamma_k)$ possesses 2^k columns. Let us enumerate 2^k ideals in such a way that every ideal for which (3) holds has number r. Thus, we have proved the following theorem.

3.5. Theorem. *There exist exactly 2^k essentially different ideals I_{kr}, $r = 1, 2, \ldots, 2^k$, such that the factor rings Γ_k / I_{kr} are quasi-Frobenius. Besides, $e_l I_k \simeq P_{1+\alpha_{lr}}$, $I_k e_l \simeq Q_{1+\alpha_{lr}}$.*

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ІДЕАЛИ ГОРЕНШТЕЙНОВИХ ЧЕРЕПИЧНИХ ПОРЯДКІВ, ФАКТОР-КІЛЬЦЯ ЗА ЯКИМИ КВАЗІФРОБЕНІУСОВІ

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Вивчено горенштейновий порядок, матриця показників якого є таблицею Келі скінченної групи. Описано ідеали цього порядку, фактор-кільця за якими є квазіфробеніусовими.

Ключові слова: горенштейновий черепичний порядок, матриця показників, матриця суміжності, квазіфробеніусове кільце.

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