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FROBENIUS RINGS

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We prove that a finite dimensional algebra A is a weakly symmetric if and only if when every algebra C which is Morita equivalent to a Frobenius algebra A is Frobenius. We give a description of serial rings the square of Jacobson radical of which is zero.

Key words: quasi-Frobenius ring, Frobenius ring, serial ring.

1. Let A be a two-sided artinian ring and R be its Jacobson radical. For a (right) A-module M we denote by M^n the direct sum of n copies of M and we set $M^0 = 0$. Then A can be represented as a direct sum of right ideals: $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$, where $P_1, \ldots, P_s$ are pairwise non-isomorphic indecomposable right A-modules, which are called the *principal right A-modules.* Set $U_i = P_i/P_iR, i = 1, \ldots, s$. It is well-known that $P_1, \ldots, P_s$ represent up to isomorphism all indecomposable projective A-modules, while $U_1, \ldots, U_s$ form a representative set of isomorphism classes of all simple right A-modules. Let M be a right A-module and N be a left A-module. We set $top\ M = M/MR$ and $top\ N = N/RN$. We denote by $soc\ M$ (respectively $soc\ N$) the largest semisimple right (respectively left) submodule of M (respectively N). Since A is artinian, *soc* exists for all A-modules. Let $1 = f_1 + \ldots + f_s$ be a decomposition of the identity element of A into a sum of idempotents such that $f_iA = P_i^{n_i}$ $(i = 1, \ldots, s)$. Then $Af_i = Q_i^{n_i}$, where $Q_1, \ldots, Q_s$ are the pairwise non-isomorphic indecomposable projective left A-modules (the *principal left A-modules*). Set $A_{ij} = f_iAf_j$ $(i, j = 1, \ldots, s)$. Then A has the following *canonical Peirce decomposition*

$$A = \bigoplus_{i,j=1}^{s} A_{ij}. \tag{1}$$

Denote by R_i the radical of A_{ii}, $(i = 1, \ldots, s)$. Obviously, A_{ii} is artinian. Since $Hom(P_j^{n_j}, P_i^{n_i}) \cong A_{ij}$, then $A_{ij} \subset R$ if $i \neq j$. The radical R of A has the following Peirce decomposition:

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$$R = \bigoplus_{i,j=1}^{s} f_i R f_j, \tag{2}$$

where $f_i R f_i = R_i$ and $f_i R f_j = A_{ij}, i \neq j \, (i = 1, \dots, s)$.

Observe that two principal A-modules P and P' are isomorphic if and only if $top\, P \simeq top\, P'$.

We recall now the classical definition of Frobenius and quasi-Frobenius rings as given by Tadasi Nakayama (see [13, p.8], [9, Section 13.4]).

Definition 1.1. A two sided artinian ring A is called *quasi-Frobenius*, if there exists a permutation ν of $\{1, 2, \dots, s\}$ such that for each $k = 1, \dots, s$ we have

(qf1) $soc\ P_k \cong top\ P_{\nu(k)}$,

(qf2) $soc\ Q_{\nu(k)} \cong top\ Q_k$.

A quasi-Frobenius ring A is called *Frobenius*, if $n_{\nu(i)} = n_i$ for all $i = 1, \dots, s$. This permutation ν is called the *Nakayama permutation* of A. Clearly, ν is determined up to conjugation in the symmetric group on s letters, and conjugations correspond to renumberings of the principal modules $P_1, \dots, P_s$.

We construct now some examples of quasi-Frobenius rings. Recall that a local ring $\mathcal{O}$ with non-zero unique maximal right ideal $\mathcal{M}$ is called a discrete valuation ring, if it has no zero divisors, the right ideals of $\mathcal{O}$ form the unique chain:

$$\mathcal{O} \supset \mathcal{M} \supset \mathcal{M}^2 \supset \dots \supset \mathcal{M}^n \supset \dots,$$

and, moreover, this chain is also the unique chain of left ideals of A. Then, obviously, $\mathcal{O}$ is noetherian, but not artinian, all powers of $\mathcal{M}$ are distinct and $\bigcap_{k=1}^{\infty} \mathcal{M}^k = 0$. Moreover, $\mathcal{M}$ is principal as a right (left) ideal.

Example. Denote by $H_s(\mathcal{O})$ the ring of all $s \times s$ matrices of the following form:

$$H = H_s(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \end{pmatrix}.$$

It is easily seen that the radical R of $H_s(\mathcal{O})$ is

$$R = \begin{pmatrix} \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{M} \end{pmatrix} \text{ and } R^2 = \begin{pmatrix} \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}^2 & \mathcal{M} & \cdots & \mathcal{M} \end{pmatrix}.$$

The principal right modules of H are the "row-ideals" of H and the submodules of each of them form a chain. In particular, the submodules of the "first-row-ideal" form the following chain:

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \begin{pmatrix} \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \begin{pmatrix} \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \dots$$

It is easy to see that each other row-ideal of H is isomorphic to a submodule of the above module. In a similar fashion, the principal left H-modules are the column-ideals, whose submodules form corresponding chains. Thus, H is a serial ring in the sense of [5, p. 224]. Let $P_1, \dots, P_s$ be the principal right modules of the quotient ring $A = H_s(\mathcal{O})/R^2$ and $Q_1, \dots, Q_s$ be the principal left A-modules numbered such that $P_i = e_{ii}A, Q_i = Ae_{ii}, (i = 1, \dots, s)$, where e_{ij} denote the $s \times s$ matrix whose (i, j)'s entry is 1 and all other entries are zero. Then the submodules of every P_i and Q_i form finite chains, and a direct verification show that

$$soc\ P_1 \cong top\ P_2, soc\ P_2 \cong top\ P_3, \dots, soc\ P_s \cong top\ P_1$$

and

$$top\ Q_1 \cong soc\ Q_2, top\ Q_2 \cong soc\ Q_3, \dots, top\ Q_s \cong soc\ Q_1.$$

Moreover, each of these modules is a one-dimensional vector space over $\mathcal{O}/\mathcal{M}$. Hence, A is a quasi-Frobenius ring whose Nakayama permutation is $(1, 2, ..., s)$.

More in general, the quotient ring $A = H_s(\mathcal{O})/R^m$ $(m \geqslant 2)$ is a quasi-Frobenius ring whose Nakayama permutation is $(1, 2, ..., s)^{m-1}$. It follows, in particular, that the Nakayama permutation of A is identical if and only if $m \equiv 1 (mod\ s)$.

We shall use the next two results.

Lemma 1.1. [4, Lemma 6.3.12.]. *Let $1 = e_1 + \dots + e_m = h_1 + \dots + h_n$ be two decompositions of $1 \in A$ into a sum of pairwise orthogonal primitive idempotents. Then $m = n$ and there exists an invertible element $a \in A$ and a permutation $i \to \sigma(i)$ such that $e_i = ah_{\sigma(i)}a^{-1}$ for each $i = 1, \dots, n$.*

Lemma 1.2. *For every simple right A-module U_i and for each f_j we have $U_i f_j = \delta_{ij} U_i$, $(i, j = 1, \dots, s)$. Similarly, for every simple left A-module V_i and for each f_j, $f_j V_i = \delta_{ij} V_i$, $(i, j = 1, \dots, s)$.*

Proof. Go modulo R and apply the Wedderburn-Artin Theorem.

This lemma will be a useful tool in our further considerations and we shall refer to it as to *Lemma on annihilation of simple modules.* An idempotent $f \in A$, which is central modulo R, shall be called *minimal modulo R* if f can not be decomposed into a sum of two orthogonal idempotents, which are central modulo R. For two idempotents e and g of A we shall write $e \in g$, if $g = e + e'$, where $ee' = e'e = 0$. Clearly, e' is also an idempotent in A.

Theorem 1.3. *Let $1 = f_1 + \ldots + f_s = g_1 + \ldots + g_t$ be two decompositions of $1 \in A$ into a sum of pairwise orthogonal idempotents, which are minimal central modulo R. Then $s = t$ and there exist an invertible element $a \in A$ and a permutation $i \to \tau(i)$ of $\{1, \ldots, s\}$ such that $f_i = ag_{\tau(i)}a^{-1}$ for each $i = 1, \ldots, s$.*

Proof. Applying the Wedderburn-Artin Theorem to $\bar{A} = A/R$, we get immediately that $s = t$. Let $f_i = e_1^{(i)} + \ldots + e_{n_i}^{(i)}$ be a decomposition of f_i into a sum of pairwise orthogonal local idempotents. Then, obviously, $U_i e_k^{(i)} \neq 0$ for $k = 1, \ldots, n_i$. It follows from the Lemma on annihilation of simple modules that $U_i g_{\sigma(i)} = U_i$ for some $g_{\sigma(i)}$ and, moreover, $U_i g_j = 0$ if $j \neq \sigma(i)$. Renumber the idempotents $g_1, \ldots g_s$ such that $U_i g_i = U_i$ $(i = 1, \ldots, s)$. Take a decomposition $g_i = h_1^{(i)} + \ldots + h_{n_i}^{(i)}$ into a sum of pairwise orthogonal local idempotents. Then we obtain two decompositions of $1 \in A$, which satisfy the assumptions of Lemma 1.1. Hence, there exits a conjugating element $a \in A$ which transforms one decomposition into the other, up to a permutation. It follows from our numeration of idempotents $g_1, \ldots g_s$ that $a\{h_1^{(i)}, \ldots, h_{n_i}^{(i)}\}a^{-1} = \{e_1^{(i)}, \ldots, e_{n_i}^{(i)}\}$ for each $i = 1, \ldots, s$ and, consequently, $ag_i a^{-1} = f_i$ $(i = 1, \ldots, s)$.

Set $A_{ij} = f_i A f_j$. Then

$$A = \bigoplus_{i,j=1}^{s} A_{ij}, \quad R = \bigoplus_{i,j=1}^{s} R_{ij},$$

where $R_{ij} = f_i R f_j = A_{ij}$ for $i \neq j$ and R_{ii} is the Jacobson radical of A_{ii} $(i, j = 1, \ldots, s)$.

Such two-sided Peirce decompositions of A and R shall be called *canonical.* It follows from Theorem 1.3. that every other canonical Peirce decomposition of A can be obtained from

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix}$$

by a simultaneous permutation of lines and columns and the substitution of all Peirce components A_{ij} by $aA_{ij}a^{-1}$.

2. MONOMIAL IDEALS. Let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents. By an ideal we mean a two-sided ideal. For an ideal I of A the abelian group $e_i I e_j$ $(i, j = 1, \ldots, n)$ obviously lies in I, and $I = \oplus_{i,j=1}^{n} I_{ij}$ is a decomposition of I into a direct sum of abelian subgroups. Such decomposition is called the *two-sided Peirce decomposition* of I corresponding to $1 = e_1 + \ldots + e_n$. It has a natural matrix form:

$$I = \begin{pmatrix} I_{11} & I_{12} & \cdots & I_{1n} \\ I_{21} & I_{22} & \cdots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n1} & I_{n2} & \cdots & I_{nn} \end{pmatrix}.$$

If $J = \oplus_{i,j=1}^{n} J_{ij}$ is also an ideal, then

$$I + J = \begin{pmatrix} I_{11}+J_{11} & I_{12}+J_{12} & \cdots & I_{1n}+J_{1n} \\ I_{21}+J_{21} & I_{22}+J_{22} & \cdots & I_{2n}+J_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n1}+J_{n1} & I_{n2}+J_{n2} & \cdots & I_{nn}+J_{nn} \end{pmatrix},$$

and each Peirce component $(IJ)_{ij}$ of the product IJ is given by
$(IJ)_{ij} = \sum_{k=1}^{n} I_{ik}J_{kj} \quad (i,j = 1,\dots,n)$,
so that addition and multiplication of elements from I and J can be done by the addition and multiplication of corresponding matrices.

Let A be a two-sided artinian ring and $1 = f_1+\ldots+f_s$ be a canonical decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents. Then $I = \oplus_{i,j=1}^{s} I_{ij}$ with $I_{ij} = f_iIf_j$ $(i,j = 1,\dots,s)$ is called the *canonical Peirce decomposition* of I. As above, it is easily seen that one canonical Peirce decomposition of I can be obtained from another one by a simultaneous permutation of lines and columns and the substitution of each Peirce component I_{ij} by $aI_{ij}a^{-1}$.

Definition 2.1. An ideal I of a artinian ring A shall be called *monomial* if each line and each column of a canonical Peirce decomposition of I contains exactly one non-zero Peirce component.

If I is a monomial ideal, then there exists a permutation $\nu \to \nu(i)$ of $\{1,\dots,s\}$ such that $I_{i\nu(i)} \neq 0$. Clearly, ν is determined up to conjugation in the symmetric group on s letters. We denote this permutation by $\nu(I)$.

Lemma 2.1. *Let A be a artinian ring. If I is a monomial ideal of A then each canonical Peirce component of I is an ideal in A.*

Proof. Let $1 = f_1+\ldots+f_s$ be a canonical decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents. Write $\nu = \nu(I)$, then $I = \oplus_{i,j=1}^{s} f_iIf_\nu(i)$. Obviously $f_iIf_{\nu(i)}f_kAf_l = 0$ if $k \neq \nu(i)$. Moreover, $f_iIf_{\nu(i)}f_{\nu(i)}Af_l \subseteq f_iIf_l$ which is non-zero if and only if $l = \nu(i)$, as I is monomial. Similarly, $f_kAf_lf_iIf_{\nu(i)} \neq 0$ if and only if $k = l = i$. It follows that $f_iIf_{\nu(i)}$ is an ideal in A for each $i = 1,\dots,n$.

Lemma 2.2. *Let A be a artinian ring. Then soc A_A coincides with the left annihilator $l(R)$ of $R = R(A)$, whereas soc ${}_AA$ coincides with the right annihilator $r(R)$. In particular, soc${}_AA$ and soc A_A are two-sided ideals.*

Proof. If U is a simple right A-module, then, obviously, $UR = 0$ and, consequently, *soc* $A_A \subseteq l(R)$. On the other hand, the equality $l(R)R = 0$ implies that $l(R)$ is a semisimple right A-module, so it has to be contained in the right socle of A, hence, $l(R) =$ *soc* A_A. Similarly, $r(R) =$ *soc* ${}_AA$.

The first statement of the next theorem is well known (see [1]), however, we include a proof in order to show that the whole result is a consequence of the Lemma on annihilation of simple modules.

Theorem 2.3. *Let A be a quasi-Frobenius ring. Then $soc\ {}_AA = soc\ A_A$. Moreover, $\mathcal{Z} = soc\ {}_AA$ is a monomial ideal and $\nu(\mathcal{Z})$ coincides with the Nakayama permutation $\nu(A)$ of A.*

Proof. Denote by $\mathcal{Z}_l$ (respectively $\mathcal{Z}_r$) the left (respectively right) socle of A. It follows from the definition of quasi-Frobenius rings and from the Lemma on annihilation of simple modules that $f_i\mathcal{Z}_l \neq 0$ for each $i = 1, \ldots, s$. Then for every local idempotent $e \in f_i$ the set $ef_i\mathcal{Z}_l = e\mathcal{Z}_l$ is different from 0. Therefore, the right ideal $e\mathcal{Z}_l$ is a nonzero submodule of the principal module P_i and, consequently, $e\mathcal{Z}_l$ contains $soc\ P_i$, which implies that $\mathcal{Z}_l \supseteq \mathcal{Z}_r$. Since the Nakayama's definition of quasi-Frobenius rings is left-right symmetric, it follows that $\mathcal{Z}_r \supseteq \mathcal{Z}_l$, and thus, $\mathcal{Z}_l = \mathcal{Z}_r = \mathcal{Z}$.

It remains to show that $\mathcal{Z}$ is monomial and $\nu(\mathcal{Z}) = \nu(A)$. Write $\nu = \nu(A)$ and consider the canonical Peirce decomposition of $\mathcal{Z}$: $\mathcal{Z} = \oplus_{i,j=1}^{s} f_i\mathcal{Z}f_j$. Since $A_A = \oplus_{i=1}^{s} f_iA = \oplus_{i=1}^{s} P_i^{n_i}$, we have that $\mathcal{Z} = \oplus_{i=1}^{s} soc\ f_iA$ and $f_i\mathcal{Z} = soc\ f_iA = soc\ P_i^{n_i}$. It follows from Definition 1.1. that $soc\ P_i^{n_i} \cong U_{\nu(i)}^{n_i}$, so $f_i\mathcal{Z} \cong U_{\nu(i)}^{n_i}$, and the Lemma on annihilation of simple modules implies that $f_i\mathcal{Z}f_j = 0$ if and only if $j \neq \nu(i)$. Hence, $\mathcal{Z}$ is monomial and $\nu(\mathcal{Z})$ coincides with $\nu(A)$.

3. FROBENIUS RINGS. In [9] a ring A was called Frobenius if it is quasi-Frobenius and $soc\ A_A \cong top\ A_A$, $soc\ {}_AA \cong top\ {}_AA$. We want to point out that one of these isomorphisms can be ommited, namely:

Proposition 3.1. *A quasi-Frobenius ring A is Frobenius if and only if*

$$soc\ A_A \cong top\ A_A.$$

Proof. Suppose that $soc\ A_A \cong top\ A_A$. Since $top\ A_A \cong \oplus_{k=1}^{s} U_{\nu(k)}^{n_{\nu(k)}}$ and $soc\ A_A \cong \oplus_{k=1}^{s} U_{\nu(k)}^{n_k}$, it follows from the Jordan-Hölder Theorem that $n_k = n_{\nu(k)}$ for all k.

We have that $top\ A_A \cong \oplus_{k=1}^{s} top\ P_{\nu(k)}^{n_{\nu(k)}} \cong \oplus_{k=1}^{s} U_{\nu(k)}^{n_{\nu(k)}} \cong soc\ A_A$. Then $top\ {}_AA \cong \oplus_{k=1}^{s} top\ Q_{\nu(k)}^{n_{\nu(k)}} \cong \oplus_{k=1}^{s} V_{\nu(k)}^{n_{\nu(k)}} \cong soc\ {}_AA$.

Proposition 3.2. *A reduced QF-ring is Frobenius.*

Proof. Immediately follows from Definition 1.1.

Lemma 3.3. *If A is a Frobenius ring and $\nu(A)$ is a cycle then $A = M_n(B)$, where B is a reduced Frobenius ring with cyclic Nakayama permutation.*

Proof. Let $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition of a Frobenius ring A into a direct sum of principal A-modules. We can suppose that $\nu(A) = (1 \ldots s)$. Then by Definition 1.1. $n_1 = n_2 = \ldots = n_s$ and $A = (P_1 \oplus P_2 \oplus \ldots \oplus P_s)^n$ which yields $A = M_n(B)$ where $B = E(P_1 \oplus \ldots \oplus P_s)$ and $\nu(B) = (1 \ldots s)$.

Recall that a ring A is indecomposable if A cannot be decomposed into a direct product of two rings.

Proposition 3.4. *If A is a QF-ring and $\nu(A)$ is a cycle, then A is indecomposable.*

Proof. We can obviously suppose that $\nu(A) = (12\ldots s)$. Then $\mathcal{Z} = soc\ A$ is a monomial ideal and $\nu(A) = \nu(\mathcal{Z})$. Therefore the canonical Peirce components $A_{ii+1} (i = 1, \ldots, s-1)$ and A_{s1} are different from zero, by implies that A is indecomposable.

Definition 3.1. [2] A ring A is called *weakly prime* if the product of any two ideals that are not in the Jacobson radical R of A is non-zero.

Obviously, any prime ring is weakly prime.

Proposition 3.5. [2] *Let $1 = e_1 + \ldots + e_n$ be a decomposition of the identity of semi-perfect ring A into a sum of mutually orthogonal local idempotents and $A_{ij} = e_iAe_j (i, j = 1, \ldots, n)$. Then A is weakly prime if and only if $A_{ij} \neq 0$ for all i, j.*

In [14] QF-rings A are considered which satisfy the following conditions:

a) A is reduced;

b) $\nu(A)$ is a cycle;

c) for any non-trivial idempotent $e \in A$ eAe is a QF-ring and $\nu(eAe)$ is a cycle.

Proposition 3.6. *If a Frobenius ring A satisfies conditions (a), (b), (c) then A is weakly prime and every local ring e_iAe_i is Frobenius.*

Proof. Since A is reduced, the local idempotents coincide with the canonical idempotents. Let $A_{ij} = f_iAf_j$ for $i = 1, 2 \ldots s$. If $A_{ij} = 0$ then $eAe (e = f_i + f_j)$ is a QF-ring. Obviously, $eAe = \begin{pmatrix} A_{ii} & 0 \\ A_{ji} & A_{jj} \end{pmatrix}$ and $\nu(eAe)$ is a cycle.

By Proposition 3.4., eAe is an indecomposable ring. Let $\mathcal{Z} = soc\, eAe$. The local ring e_iAe_i are Frobenius by condition (c).

Let $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition of an artinian ring A into a direct sum of principal right A-modules and let $1 = f_1 + \ldots + f_s$ be the corresponding decomposition of identity of the ring A into a sum of pairwise orthogonal idempotents, i.e., $f_iA = P_i^{n_i}$.

Definition 3.2. An idempotent $g \in A$ will be called *standard* if $g = f_{i_1} + \ldots + f_{i_k}$, where $n_{i_1} = \ldots = n_{i_k}$, in particular, if $f \in \{f_1, \cdots, f_s\}$ and $fA = P^{n_{i_1}}$ then $f \in \{f_{i_1}, \cdots, f_{i_k}\}$.

Definition 3.3. Let $1 = g_1 + \ldots + g_m$ be a decomposition of $1 \in A$ into a sum of pairwise orthogonal standard idempotents. Put $A_{ij} = g_iAg_j\ (i, j = 1, \ldots, m)$. The decomposition $A = \oplus_{i,j=1}^{m} A_{ij}$ will be called *the standard Peirce decomposition* of artinian ring A.

Theorem 3.7. *Let* $A = \bigoplus_{i,j=1}^{m} A_{ij}$ *be a standard Peirce decomposition of a Frobenius ring* A, *then* $A_{ii} = M_{k_i}(B_i)$ *where all rings* B_i *are reduced Frobenius rings.*

Proof. By Lemma on annihilation of simple modules the socle of arbitrary principal A-module $P = eA$ with $e \in g_i$ is annihilated by every standard idempotent $g_j \neq g_i$. Hence, $g_i \mathcal{Z} g_i$ is the socle of $M_{n_i}(B_i)$ and for each local idempotent $e \in g_i$ the A_{ii}-module $e\mathcal{Z}q_i$ is simple as a simple left A_{ii}-module and it is also simple as a right A_{ii}-module. Therefore A_{ii} is quasi-Frobenius.

The multiplicities of all principal A_{ii}-modules are k_i and consequently A_{ii} is Frobenius for all $i = 1 \ldots, s$.

Theorem 3.8. *Let* A *be a* QF*-ring and the Nakayama permutation* $\nu(A)$ *of* A *is identical. Then* A *is Frobenius and every ring* C *which is Morita equivalent to* A *is also Frobenius. Conversely, if every ring* C *which is Morita equivalent to a Frobenius ring* A *is Frobenius, then* $\nu(A)$ *is identity.*

Proof. By Definition 1.1. every QF-ring with identical Nakayama permutation is automatically Frobenius. Clearly, every ring which is Morita equivalent to a Frobenius ring with identical Nakayama permutation is Frobenius.

Let A is a Frobenius ring and $\nu(A)$ is not identity. Then we can assume that $soc\ P_1 = top\ P_2$. Let $A = P_1^{n_1} \oplus P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}$ be a canonical decomposition of A into a direct sum of principal A-modules. It follows from the Definition 1.1., $n_2 = n_1$. Set $P = P_1^2 \oplus P_2 \oplus \ldots \oplus P_s$. Then $C = End_A P$ is a QF-ring, and $\nu(A) = \nu(C)$, and multiplicity of the first principal C-module is 2 and does not coincide with the multiplicity of the second principal C-module. Therefore, C is not Frobenius.

Finite-dimensional Frobenius algebras with identical Nakayama permutation were called by [11, p. 444] weakly symmetric algebras. So from Theorem 3.8. we have such theorem.

Theorem 3.9. *Let* A *be a weakly symmetric algebra. Then* A *is Frobenius and every algebra* C *which is Morita equivalent to* A *is also Frobenius. Conversely, if every finite dimensional algebra* C *which is Morita equivalent to a Frobenius algebra* A *is Frobenius, then* A *is a weakly symmetric algebra.*

4. SERIAL QUASI-FROBENIUS RINGS.

Definition 4.1. A module is called *uniserial* if the lattice of its submodules is a chain, i.e. the set if all its submodules is linearly ordered by inclusion. A module is said to be serial of it is a finite direct sum of uniserial submodules.

Definition 4.2. A ring A is called *right* (resp. *left*) *uniserial* if A_A (resp. ${}_AA$) is an uniserial A-module. A ring which is right and left uniserial is called *uniserial*. A ring A is *right* (resp. *left*) *serial* if A_A (resp. ${}_AA$) is a serial A-module. A right serial and left serial ring shall be called *serial*.

Theorem 4.1. [10, Theorem 2.1] *The quiver* $Q(A)$ *of a serial two-sided Noetherian ring* A *is a disconnected union of cycles and chaines (i.e. of quivers corresponding to finite lineary ordered sets).*

Proposition 4.2. *Let $Q(A)$ be a quiver of a quasi-Frobenius ring A. If there is a vertex $i \in Q(A)$ which is either a sink (i is not the tail of any arrow) or a source (i is not the head of any arrow) then $A \simeq A_1 \times A_2$, where $A_1 \simeq M_n(\mathcal{D})$ with a division algebra $\mathcal{D}$.*

Proof. Let i is a sink. Then indecomposable projective A-module P_i is simple. Therefore $\nu(i) = i$, where $\nu = \nu(A)$ is Nakayama permutation of a ring A, and $A_{ij} \neq 0$ for $j = 1, \dots, i-1, i+1, \dots, n$.

Now we shall show that $A_{ki} = 0$ for $k = 1, \dots, i-1, i+1, \dots, s$. Let $A_{ki} \neq 0$. Then, because $A_{ki} \simeq Hom(P_i^{n_i}, P_k^{n_k})$ we obtain by the Lemma of Shur that the simple module U_i appears in a direct decomposition of $soc\, P_k$. So $\nu(k) = \nu(i) = i$ and $A_{ki} = 0$. Analogously, if i is source, then the left indecomposable projective A-module Q_i is simple and $A_{ki} = 0$, $A_{ij} = 0$ for $j, k = 1, \dots, i-1, i+1, \dots, s$.

As a corollary of this result and Theorem 4.1 we obtain the description of the quivers of serial QF-rings.

Theorem 4.3. *The quiver $Q(A)$ of a serial QF-ring A is a disconnected union of cycles and one-point quivers without arrows.*

Definition 4.3. A local serial (=uniserial) ring is called *a Köthe ring.*

Proposition 4.5. *A Köthe ring is Frobenius.*

Proof. Immediatly follows from Definition 1.1.

Let A be a Köthe ring. Then the length $l(A_A)$ of the right regular A-module coincides with the length $l({}_AA)$ of the left regular A-module. Then $l = l(A_A) = l({}_AA)$ shall be called the length of a Köthe ring A and denoted by $l(A)$.

A Köthe ring of length 1 is a division ring.

A Köthe ring of length m has a unique chain of ideals (right, left, two-sided):

$$A \supset R \supset R^2 \supset \dots \supset R^{m-1} \supset 0.$$

Lemma 4.6. *A local Frobenius ring A with $R^2 = 0$ is either a division ring or a Köthe ring of length 2. In the second case $Q(A)$ is a loop.*

The proof follows from Definition 1.1.

We give the description of serial reduced rings, the square of Jacobson radical of which is zero. Such rings are two-sided artinian, since the length of every right and every left principal module is less or equal than 2.

Lemma 4.7. *If A is an artinian indecomposable reduced serial non-local ring with $R^2 = 0$ then there is a subring A_0 in A such that $A = A_0 \oplus R$ (direct sum of abelian groups).*

Proof. Obviously, $Q(A)$ has more than one vertex. We have two cases:

a) $Q(A) = \{1 \to 2 \to \dots \to s-1 \to s\}$ is a chain;

b) $Q(A) = \{1 \to 2 \to \dots \to s-1 \to s \to 1\}$ is a cycle.

Suppose (a). By [6, p.287] $A \simeq T_s(\mathcal{D})/I$, where I is two-sided ideal of the ring $T_s(\mathcal{D})$ of all upper-triangular $s \times s$-matrices over $\mathcal{D}$.

Clearly, we can take A_0 be equal to the subring of all diagonal $s \times s$-matrices over $\mathcal{D}$.

In case (b) suppose first that $s = 2$. Then $1 = e_1 + e_2$. Put $A_i = e_i A e_i$, R_i is the Jacobson radical of $A_i (i = 1, 2), X = e_1 A e_2$ and $Y = e_2 A e_1$.

By formula 2 (see §1) we have $R = \begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix}$.

Clearly,

$$R^2 = \begin{pmatrix} R_1^2 + XY & R_1 X + X R_2 \\ Y R_1 + R_2 Y & R_2^2 + YX \end{pmatrix}.$$

Since $Q(A)$ is two-pointed cycle by the Lemma on annihilation of simple modules we have that $XY = R_1$ and $YX = R_2$, which implies that $XYX = XR_2 = R_1 X$ and $YXY = YR_1 = R_2 Y$. Since $R^2 = 0$ it follows that $R_1 = 0$ and $R_2 = 0$. Hence, $A_1 = \mathcal{D}_1$ and $A_2 = \mathcal{D}_2$ are division rings and $A_0 = \begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix}, R = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$, i.e. $A = A_0 \oplus R$.

Let $Q(A) = \{1 \to 2 \to \ldots \to s-1 \to s \to 1\}$ be a cycle which contains at least three vertices. Let $1 = e_1 + \ldots + e_s$ be a decomposition of $1 \in A$ in a sum of mutually orthogonal idempotents.

Set $A_i = e_i A e_i$ where R_i is the Jacobson radical of $A_i (i = 1, \ldots, s)$. Let $A_{ij} = e_i A e_j (i \neq j; i, j = 1, \ldots, s)$. By the definition of $Q(A)$ we have that $A_{ii+1} \neq 0$ for $i = 1, \ldots, s-1$ and $A_{s1} \neq 0$.

We show that $R_i = 0$ for all i. Then by formula 2 of §1 we obtain that $A = A_0 \oplus R$ where

$$A_0 = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{pmatrix}.$$

In fact, let $R_k \neq 0$ for some $1 \leqslant k \leqslant s$. applying a cyclic renumbering of principal modules we may assume that $R_1 \neq 0$.

Consider the ring $B = (e_1 + e_2) A (e_1 + e_2)$. By [3] B is a serial ring. Clearly, B is reduced and $(R(B))^2 = 0$ where $R(B)$ is radical of B. Since $A_{12} \neq 0$, then if $A_{21} \neq 0$ we obtain that $Q(B)$ is a two-pointed cycle. But then it follows from the above arguments that $R_1 = 0$.

If $A_{21} = 0$ then $Q(B) = \{1 \to 2\}$ and $B \simeq T_2(\mathcal{D})$. Again $R_1 = 0$. Lemma is proved.

Let $\mathcal{O} = \mathcal{D}[[x, \sigma]]$ be an augmented Ore domain (see [18], Chapter VII, §14). The ring $\mathcal{D}[[x, \sigma]]$ is the set of formal power series $\sum\limits_0^\infty \alpha_i x^i, \alpha_i \in \mathcal{D}$; σ is an automorphism of the division ring $\mathcal{D}$. Addition and equality is defined in the usual way. Multiplication is defined by the formula $\alpha x = x \alpha^\sigma$ and its consequences. Then $\mathcal{O}$ is a discrete valuation ring with unique maximal ideal $\mathcal{M} = x\mathcal{O} = \mathcal{O}x$. Denoted by $H_s^{(m)}(\mathcal{O})$ the quotient ring $H_s(\mathcal{O})/R^m$, where R is Jacobson radical of $H_s(\mathcal{O})$ (see Example from §1).

It follows from Lemma 4.7. and [10, §4] that every serial non-local reduced ring whose quiver is a cycle, the square of whose Jacobson radical is zero, is isomorphic to $H_s^{(2)}(\mathcal{O})$.

Set $T_s^{(2)}(\mathcal{D}) = T_s(\mathcal{D})/R^2$ where R is radical of $T_s(\mathcal{D})$. It follows from Lemma 4.7. and [6] that every serial reduced non-local ring A with $R(A)^2 = 0$ whose quiver is a chain is isomorphic to $T_s^{(2)}(\mathcal{D})$.

Since every serial A with one-point quiver and $R(A)^2 = 0$ is either a division ring or a Köthe ring of length 2, we obtain the following theorem.

Theorem 4.8. *Every indecomposable serial reduced ring A with $R(A)^2 = 0$ is isomorphic to one of the following:*

a) a division ring;

b) a Köthe ring of length 2;

c) $H_s^{(2)}(\mathcal{O})$;

d) $T_s^{(2)}(\mathcal{D})$.

In the cases (c) and (d) we have $s \geqslant 2$. Conversely, all these rings are indecomposable serial reduced rings, the square of the Jacobson radical of which is zero.

Remark. The rings of types (a), (b), (c) are Frobenius. In cases (a) and (b) the Nakayama permutation is identity and in case (c) it is a cycle $(1, 2, \dots, s)$.

Remark. If the quiver $Q(A)$ of serial ring A is a chain then, there is a subring A_0, such that $A = A_0 \oplus R$ (a direct sum of abelian groups). If $Q(A)$ is a cycle with s vertices and $R^s = o$ then there is a subring A_0 in A such that $A = A_0 \oplus R$ (a direct sum of abelian groups). In the last case if A is reduced then A is isomorphic to a quotient ring of the QF-ring $H_s^{(s)}(\mathcal{O})$, and the Nakayama permutation $\nu(H_s^{(s)}(\mathcal{O}))$ is equal to $(1, s, s-1, \dots, 2)$.

Proposition 4.9. *Let A be a serial ring, $P_1, \dots, P_s$ all pairwise non-isomorphic principal A-modules. If $l(P_i) = l_i$ then soc $P_i = U_k$, where $k = i + l_i - 1 \,(mod\, s)$.*

Proof. The proof immediately follows from the definition of $Q(A)$.

This implies the following well-known fact (see [12] and also [8]).

Corollary 4.10. *A serial artinian indecomposable ring A is a QF-ring if and only if the lengthes of all principal A-modules are equal.*

Proof. If the lengthes of all principal A-modules are equal to 1 . Then since A is indecomposable, by the Wedderburn-Artin Theorem A is isomorphic to $M_n(\mathcal{D})$ where $\mathcal{D}$ is a division ring, consequently, A is a QF-ring.

If the length of all principal A-modules are equal to $l \geqslant 2$ then $Q(A)$ is a cycle. The map

$$\nu : i \to \nu(i) = l + i - 1 \,(mod\, s)$$

is a permutation $\{1, \dots, s\}$.

By Definition 1.1. A is a QF-ring and there exists a principal A-module P which is simple. By Theorem 4.3. and Proposition 4.2. we obtain that $Q(A)$ is a one-point quiver without arrows. Therefore, $A = P^n$ and by Schur's Lemma $A \simeq M_n(E(P))$, where $E(P)$ is a division ring.

Thus, we can assume that if $l(P_i) \geqslant 2$ for all i. By Theorem 4.3. and Proposition 4.2. $Q(A)$ is a cycle.

Let $\varphi : P \to P_iR$ be an epimorphism of the principal A-module P on P_iR. If φ is an isomorphism then $soc\ P \simeq soc\ P_i$ which contradicts the Definition 1.1. Hence, $ker\,\varphi \neq 0$ and $l = l(P) \geqslant l_i = l(P_i)$. Let

$$Q(A) = \{1 \to 2 \to \dots s-1 \to s \to 1\}.$$

Then $P = P_{i+1}$ for $1 \leqslant i \leqslant s-1$ and $P = P_1$ for $i = s$. Thus $l_1 \leqslant l_2 \leqslant \dots \leqslant l_s \leqslant l_1$ as required.

Proposition 4.11. *Let A be an indecomposable serial artinian ring and $Q(A)$ is a cycle, J is a two-sided ideal with $J \subset R^2$. The quotient ring A/J is a QF-ring if and only if $J = R^l$ for some l.*

Proof. If $J = R^l$ then, obviously, the lengthes of all principal A/J-modules are equal and A/J is quasi-Frobenius.

Let A/J be a QF-ring. Then the lengthes of all principal modules are equal to $l \geqslant 2$. Thus, $[R(A/J)]^l = 0$ which implies that $J = R^l$.

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ФРОБЕНІУСОВІ КІЛЬЦЯ

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Доведено, що скінченновимірна алгебра A є слабо симетричною тоді і тільки тоді, коли кожна алгебра C , яка є Моріта еквівалентною фробеніусовій алгебрі A також фробеніусова. Наведено опис напівланцюгових кілець, квадрат радикала Джекобсона яких дорівнює нулю.

Ключові слова: квазі-фробеніусове кільце, фробеніусове кільце, напівланцюгове кільце.

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