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HYPERSPACE FUNCTOR IN THE COARSE CATEGORY

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We consider the hyperspace monad in the category of topological coarse spaces and equivalence classes of coarse maps. It is proved that the G-symmetric power functor acting on the category of topological spaces can be naturally defined also for the category of topological coarse spaces and, on this category, it can be extended to the Kleisli category of the hyperspace monad.

Key words: coarse space, coarse map.

1. The coarse category was first introduced by Higson, Pedersen, and Roe [1]. Methods of coarse topology (geometry) found numerous applications in different areas of topology and analysis (see, e. g. [1–5]). The present paper is devoted to the hyperspace functor and hyperspace monad in the coarse category.

The paper is organized as follows. Section 2 contains necessary definitions. In Section 3 the hyperspace functor acting in the coarse category is defined and we prove in Section 4 that the hyperspace functor determines a monad in the coarse category. In Section 5 we consider the problem of extension of functors onto the Kleisli category of the hyperspace monad.

2.1. PRELIMINARIES. Coarse structures. Let X be a set and $M, M \subset X \times X$. The *composition* of M and N is the set $MN = \{(x,y) \in X \times X \mid \text{ there exists } z \in X \text{ such that } (x,z) \in M, (z,y) \in N\}$, the *inverse* of M is the set $M^{-1} = \{(x,y) \in X \times X \mid (y,x) \in M\}$.

A *coarse structure* on a set X is a family $\mathcal{E}$ of subsets, which are called the *entourages*, in the product $X \times X$ that satisfies the following properties:

1) any finite union of entourages is contained in an entourage;
2) for every entourage M, its inverse M^{-1} is contained in an entourage;
3) for every entourages M, N, their composition MN is contained in an entourage;
4) $\cup\mathcal{E} = X \times X$.

A coarse structure on X is called *unital* if the diagonal Δ_X is contained in an entourage. A coarse structure on X is called *anti-discrete* if $X \times X$ is an entourage.

If $\mathcal{E}_1$, $\mathcal{E}_2$ are coarse structures on X, then $\mathcal{E}_1 \leqslant \mathcal{E}_2$ means that for every $M \in \mathcal{E}_1$ there is $N \in \mathcal{E}_2$ such that $M \subset N$.

Two coarse structures, $\mathcal{E}_1$ and $\mathcal{E}_2$, are said to be *equivalent* if $\mathcal{E}_1 \leqslant \mathcal{E}_2$ and $\mathcal{E}_2 \leqslant \mathcal{E}_1$. We usually identify coarse spaces with equivalent coarse structures.

If $\mathcal{E}$ is a coarse structure on a set X, then, obviously, the coarse structure $\mathcal{E}_1 = \{M \cup M^{-1} \mid M \in \mathcal{E}\}$ is equivalent to $\mathcal{E}$ and is *symmetric* in the sense that $N^{-1} \in \mathcal{E}_1$ for every $N \in \mathcal{E}_1$.

Given $M \in \mathcal{E}$ and $A \subset X$, we define the *M-neighborhood* $M(A)$ of A as follows: $M(A) = \{x \in X \mid (a, x) \in M$ for some $a \in A\}$. We use the notation $M(a)$ instead of $M(\{a\})$. A set $A \subset X$ is *bounded* if there exists $x \in X$ such that $A \subset M(x)$.

Let $(X_i, \mathcal{E}_i)$, $i = 1, 2$, be coarse spaces. A map $f: X_1 \to X_2$ is called *coarse* if the following two conditions hold:

1) for every $M \in \mathcal{E}_1$ there exists $N \in \mathcal{E}_2$ such that $(f \times f)(M) \subset N$;
2) for any bounded subset A of X_2 the set $f^{-1}(A)$ is bounded.

It is easy to see that the coarse spaces and coarse maps form a category. We denote it by CS.

Definition 2.1. A subset A of X is called *coarsely dense* in X if there exists $M \in \mathcal{E}$ such that $M(A) = X$.

Lemma 2.2. *A subset A in X is coarsely dense in X iff the class $[i]$ of the inclusion map $i: A \to X$ is an isomorphism in $\mathcal{E}$.*

Proof. Suppose that A is coarsely dense in X, then there is $M \in \mathcal{E}$ such that $X = M(A)$.

Define a map $g: X \to A$ as follows: for any $x \in X$, $g(x)$ is an arbitrary point af A with $x \in M(g(x))$. Obviously, g is coarse.

Then,

$$(gi(x), x) = (g(x), x) \in M,$$

$$(ig(x), x) = (g(x), x) \in M,$$

i.e. $gi \sim 1_A, ig \sim 1_X$, which means that $[g][i] = [1_A], [i][g] = [1_X]$. □

If $[i]$ is an isomorphism, then there exists a coarse map $g: X \to A$ such that $[i][g] = [ig] = [g] = [1_X]$. That means that $g \sim 1_X$, i.e. there is $M \in \mathcal{E}$ such that $(g(x), x) \in M$, for every $x \in X$.

Proposition 2.3. *Let $f, g: (X, \mathcal{E}) \to (X', \mathcal{E}')$ be a coarse maps. If $f|_A \sim g|_A$ on some coarsely dense subset A of X, then $f \sim g$.*

Proof. Let $i: A \to X$ denote the inclusion map. Then $fi \sim gi$ and therefore $[f][i] = [fi] = [gi] = [g][i]$. Since $[i]$ is an isomorphism (by previous lemma), we obtain that $[f] = [g]$. □

2.2. PRELIMINARIES. Topological coarse structures. Now suppose that X is a Hausdorff topological space. A coarse structure $\mathcal{E}$ on X is called *topological* if the following conditions are satisfied:

1) every entourage is open in $X \times X$;
2) every bounded set is precompact.

Note that if a space X can be endowed with coarse structure, then X is necessarily locally compact.

Proposition 2.4. *In a coarse topological space, every dense subset is coarsely dense.*

2.3. PRELIMINARIES. Coarse categories. We denote it by CTS (respectively CTS) the category of coarse topological spaces and coarse maps (respectively, of coarse topological spaces and proper continuous maps).

We will need one more category related to the coarse structures. In order to define it, we introduce the following notion.

Let $f, g\colon (X,\mathcal{E}) \to (X',\mathcal{E}')$ be coarse maps. We say that f and g are *equivalent* (and write $f \sim g$) if there exists $M \in \mathcal{E}'$ such that $(f(x), g(x)) \in M$ for every $x \in X$. It is easy to verify that $\sim$ is an equivalence relation and we denote by $[f]$ the equivalence class of f.

Lemma 2.5. *Let $f_1, f_2\colon (X,\mathcal{E}) \to (X',\mathcal{E}')$, $g_1, g_2\colon (X',\mathcal{E}') \to (X'',\mathcal{E}'')$ be coarse maps. If $f_1 \sim f_2$ and $g_1 \sim g_2$, then $g_1 f_1 \sim g_2 f_2$.*

Proof. Since $f_1 \sim f_2$, there is $M' \in \mathcal{E}'$ such that $(f_1(x), f_2(x)) \in M'$ for every $x \in X$. Since g_1 is coarse, there is $M'' \in \mathcal{E}''$ such that $(g_1 \times g_1)(M') \subset M''$. We see that $(g_1 f_1(x), g_1 f_2(x)) \in M''$, for every $x \in X$, i. e. $g_1 f_1 \sim g_1 f_2$. Obviously, $g_1 f_2 \sim g_2 f_2$ and the result follows from the transitivity of $\sim$. □

Lemma 2.5 allows us to define a composition of the equivalence classes as $[gf] = [g][f]$. We define the category $CTS/\sim$ as the category whose objects are as in CTS and the morphisms are the equivalence classes of the morphisms in CTS with respect to the equivalence relation $\sim$.

3. Hyperspaces of coarse spaces. Given a Hausdorff topological space endowed with a topological coarse structure $\mathcal{E}$, denote by $\exp X$ the set of all nonempty compact subsets in X. A base for the *Vietoris topology* on $\exp X$ is formed by the sets

$$\langle U_1, \dots, U_k\rangle = \{A \in \exp X \mid A \subset \bigcup_{i=1}^{k} U_i,\ A \cap U_i \neq \emptyset \text{ for all } i = 1, \dots, k\},$$

where $U_1, \dots, U_k$ run over the topology of X. For every $M \in \mathcal{E}$ let $M_H = \{(A, B) \in \exp X \times \exp X \mid$ for every $a \in A$ there exists $b \in B$ with $(a, b) \in M$ and for every $b \in B$ there exists $a \in A$ with $(a, b) \in M\}$.

Proposition 3.1. *The family $\mathcal{E}_H = \{M_H \mid M \in \mathcal{E}\}$ is a topological coarse structure on $\exp X$. $\mathcal{E}_H$ is unital if so is $\mathcal{E}$.*

Proof. Obviously, if $M \subset N$, then $M_H \subset N_H$. Show that for every $M, N \in \mathcal{E}$ we have

$$M_H N_H = (MN)_H. \qquad (3.1)$$

Indeed, suppose that $(A, B) \in M_H N_H$. Then there exists $C \in \exp X$ such that $(A, C) \in M_H$, $(C, B) \in N_H$.

Given $a \in A$, there is $c \in C$ with $(a, c) \in M$ and there is $b \in B$ with $(c, b) \in N$. Therefore, $(a, b) \in MN$.

Similarly, we show that for every $b \in B$ there is $a \in A$ with $(a, b) \in MN$. This shows that $(A, B) \in (MN)_H$.

Using (3.1) we conclude that the product of entourages in $\mathcal{E}_H$ is contained in an entourage. Besides, if $M, N \in \mathcal{E}$, then $M_H \subset (M \cup N)_H$, $N_H \subset (M \cup N)_H$, i.e. $M_H \cup N_H \subset (M \cup N)_H$, which implies that the union of two entourages is contained in an entourage.

Finally, show that $\cup\mathcal{E}_H = \exp X \times \exp X$. Given $(A, B) \in \exp X \times \exp X$, find, for each $a \in A$, $b \in B$, an entourage $M_{ab} \in \mathcal{E}$ such that $(a, b) \in M_{ab}$. The cover $\{M_{ab} \mid a \in A,\ b \in B\}$ contains a finite subcover $\{M_{a_i b_i} \mid i = 1, \dots, k\}$ of $A \times B$.

There exists $M \in \mathcal{E}$ such that $\cup_{i=1}^{k} M_{a_i b_i} \subset M$. Then $A \times B \subset M$ and, obviously, $(A, B) \in M_H$.

Now suppose that $\mathcal{E}$ is a unital coarse structure on X. There exists $M \in \mathcal{E}$ with $\Delta_X \subset M$. Then, obviously, $\Delta_{\exp X} \subset M_H$.

Show that $\mathcal{E}_H$ is a topological coarse structure on $\exp X$.

First, show that every set M_H is open in $\exp X \times \exp X$, for every $M \in \mathcal{E}_1$. Indeed, suppose the opposite and let $(A, B) \in M_H$ be a non-interior point of M_H. Then there is a net $(A_\gamma, B_\gamma)_{\gamma \in \Gamma}$ converging to (A, B) such that $(A_\gamma, B_\gamma) \notin M_H$ for every $\gamma \in \Gamma$. Without loss of generality, we may assume that, for every $\gamma \in \Gamma$, there exists $b_\gamma \in B_\gamma \setminus M(A_\gamma)$.

There exists a subnet (b_{γ_i}) of (b_γ) converging to $b \in B$ (see the definition of the limit in the Vietoris topology [6]). Show that $b \notin M(A)$. Indeed, otherwise we would have a net (a_{γ_i}) converging to $a \in A$ such that $a_{\gamma_i} \in A_{\gamma_i}$. Since the net $(a_{\gamma_i}, b_{\gamma_i})$ converges to $(a, b) \in M$, there exists $i(0)$ such that $(a_{\gamma_{i(0)}}, b_{\gamma_{i(0)}}) \in M$, i. e. $b_{\gamma_{i(0)}} \in M(A_{\gamma_{i(0)}})$, a contradiction.

Now show that every subset of the form $M_H(A)$ is relatively compact. Note that the set $M(A)$ is bounded and, therefore, $\overline{M(A)}$ is compact. Obviously, $M_H(A) \subset \exp(\overline{M(A)})$, and therefore the closure of $M_H(A)$ is compact. □

The coarse structure $\mathcal{E}_H$ is called the *Vietoris* coarse structure on $\exp X$. In the sequel, we always endow the hyperspace of a coarse topological space with the Vietoris coarse structure.

Let $f\colon (X, \mathcal{E}) \to (X', \mathcal{E}')$ be a coarse map between coarse topological spaces. Define the map $\exp f\colon \exp X \to \exp X'$ by the formula $\exp f(A) = \overline{f(A)}$. Note that $\exp f$ is well-defined as the set $f(A)$ is obviously bounded for every compact subset $A \subset X$ and therefore the set $\overline{f(A)}$ is compact.

Proposition 3.2. *The map* $\exp f\colon (\exp X, \mathcal{E}_H) \to (\exp X', \mathcal{E}'_H)$ *is coarse.*

Proof. Indeed, suppose that $M \in \mathcal{E}$. Then there is $M' \in \mathcal{E}'$ such that $(f \times f)(M) \subset M'$. Then it is easy to see that $(\exp f \times \exp f)(M_H) \subset M'_H$, this shows that $\exp f$ is coarsely uniform.

Show that $\exp f$ is coarsely proper. It suffices to show that the preimage under the map $\exp f$ of every set of the form $M'_H(\{x'\})$ is bounded. Since f is coarsely proper, there exist $M \in \mathcal{E}$ and $x \in X$ such that $f^{-1}(M'(x')) \subset M(x)$. It is easy to see that then $(\exp f)^{-1}(M'_H(\{x'\}) \subset M_H(\{x\})$. □

It is not difficult to construct two coarse maps f, g such that $\exp(gf) \neq \exp g \exp f$. Indeed, consider the real line $\mathbb{R}$ with the *bounded* coarse structure, i. e. the coarse structure

$$\mathcal{E} = \{\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x - y| \leqslant C\} \mid C > 0\}.$$

Define $f, g\colon \mathbb{R} \to \mathbb{R}$ as follows: $f(x) = x$ whenever $x \leqslant 0$ and $f(x) = x + 1$ otherwise, $g(x) = x$ whenever $x \leqslant 1$ and $g(x) = x + 1$ otherwise. Let $A = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$, then $\exp f(A) = \{0\} \cup \{1\} \cup \{1 + 1/n \mid n \in \mathbb{N}\}$ and

$$\exp g \exp f(A) = \{0\} \cup \{1\} \cup \{2\} \cup \{2 + 1/n \mid n \in \mathbb{N}\},$$

while

$$\exp(gf)(A) = \{0\} \cup \{2\} \cup \{2 + 1/n \mid n \in \mathbb{N}\}.$$

This example can be regarded as a motivation of introducing the category $CTS/\sim$.

Lemma 3.3. *If A is a subset of a coarse topological space $(X, \mathcal{E})$, then for every $M \in \mathcal{E}$ we have $\bar{A} \subset M(A)$.*

Proof. Suppose $x \in \bar{A}$, then there is $a \in A \cap M^{-1}(x)$. This means that $x \in M(A)$. □

Proposition 3.4. *Let $f_1, f_2: (X, \mathcal{E}) \to (X', \mathcal{E}')$ be coarse maps. If $f_1 \sim f_2$, then* $\exp f_1 \sim \exp f_2$.

Proof. There exists $M' \in \mathcal{E}'$ such that $(f_1(x), f_2(x)) \in M'$ for every $x \in X$. Without loss of generality, we may assume that $M' = (M')^{-1}$. If $A \in \exp X$, then $f_1(A) \subset M'(f_2(A))$ and $f_2(A) \subset M'(f_1(A))$. By Lemma 3:3,

$$\overline{f_1(A)} \subset M'(f_1(A)),\ \overline{f_2(A)} \subset M'(f_2(A))$$

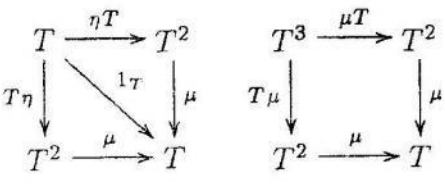
and we obtain

$$\overline{f_1(A)} \subset M'M'(f_2(A)),\ \overline{f_2(A)} \subset M'M'(f_1(A)).$$

The latter means that $(\exp f_1(A), \exp f_2(A)) \in M_H$. □

Proposition 3.4 allows us to define the hyperspace functor exp in the category $CTS/\sim$ as follows. Given a morphism $f: X \to Y$ in CTS, we define $\exp[f]: X \to Y$ in $CTS/\sim$ as $\exp[f] = [\exp f]$.

4. Hyperspace monad in the coarse category. Recall that a monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$ consisting of an endofunctor $T: \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta: 1_{\mathcal{C}} \to T$ (unit), $\mu: T^2 \to T$ (multiplication) making the diagrams



commutative (see [7] for details).

Theorem 4.1. *The triple $\mathbb{H} = (\exp, s, u)$ is a monad on the category $CTS/\sim$.*

Proof. First show that the map $u_x: (\exp^2 X, \mathcal{E}_{HH}) \to (expX, \mathcal{E}_H)$ is coarse. Let $(\mathcal{A}, \mathcal{B}) \in M_{HH}$, for some $M \in \mathcal{E}$. Show that $(\cup\mathcal{A}, \cup\mathcal{B}) \in M_H$. Indeed, if $a \in \cup\mathcal{A}$, then there is $A \in \mathcal{A}$ with $a \in A$. By the definition of M_{HH}, there is $B \in \mathcal{B}$ with $(A, B) \in M_H$. Then there is $b \in \cup\mathcal{B}$ with $(a, b) \in M$.

We can similarly prove that for every $b \in \cup\mathcal{B}$ there is $a \in \cup\mathcal{A}$ with $(a, b) \in M$. Together this means that $(\cup\mathcal{A}, \cup\mathcal{B}) \in M_H$.

To this end, we have to show that $u_x^{-1}(M_H(A))$ is bounded for every $A \in \exp X$ and every $M \in \mathcal{E}$. If $\mathcal{B} \in u_x^{-1}(M_H(A))$, then $(\cup\mathcal{A}, \cup\mathcal{B}) \in M_H$.

There exists an entourage $N \in \mathcal{E}$ such that $\overline{M(A)} \subset N(x)$, for some $x \in X$.

Suppose that $\mathcal{B} \in u_x^{-1}(M_H(A))$ and $B \in \mathcal{B}$. Then $B \subset M(A)$.

Given $b \in B$ we see that $b \in N(x)$, whence $x \in N^{-1}(b)$ and $A \subset NN^{-1}(b) \subset NN^{-1}(B)$. Let $L \in \mathcal{E}$ an entourage containing $M \cup (NN^{-1})$. Then for any $\mathcal{B} \in u_x^{-1}(M_H(A))$ and any $B \in \mathcal{B}$ we have $B \subset M(A) \subset L(A)$ and $A \subset NN^{-1}(B) \subset L(B)$. This means that $\mathcal{B} \in L_H(A)$.

Now we are going to show that $([u_X])$ is a natural transformation of $\exp^2$ into exp.

Given a coarse map $f: X \to Y$ we have to show that the diagram

$$\begin{array}{ccc} \exp^2 X & \xrightarrow{\exp^2[f]} & \exp^2 Y \\ {\scriptstyle [u_X]}\downarrow & & \downarrow{\scriptstyle [u_Y]} \\ \exp X & \xrightarrow[\exp[f]]{} & \exp Y \end{array}$$

is commutative.

We first start with finite sets.

Let $\mathcal{A} \in \exp^2 X$, then

$$\exp^2[f](\mathcal{A}) = \overline{\{\exp[f](A) | A \in \mathcal{A}\}} = \overline{\{\overline{f(A)} | A \in \mathcal{A}\}},$$

$$[u_Y](\exp^2[f](\mathcal{A})) = \cup \overline{\{\overline{f(A)} | A \in \mathcal{A}\}}.$$

On the other hand,

$$\exp[f]([u_X](\mathcal{A})) = \overline{f(\cup\mathcal{A})}.$$

Let $\mathcal{A} = \{A_1, \dots, A_n\}$. Then

$$[u_Y](\exp^2[f](\mathcal{A})) = \cup \overline{\{\overline{f(A_i)} | i = 1, \dots, n\}} =$$

$$= \cup \{\overline{f(A_i)} | i = 1, \dots, n\} = \overline{\cup\{f(A_i) | i = 1, \dots, n\}} = \overline{f(\cup\mathcal{A})}.$$

The set $\{\mathcal{A} \in \exp^2 X | |\mathcal{A}| < \infty\}$ is dense in $\exp^2 X$ and therefore this set is coarsely dense in $\exp^2 X$. According to the Proposition 2.3 we conclude that the diagram is commutative for each $\mathcal{A} \in \exp^2 X$.

Show that the diagram

$$\begin{array}{ccc} \exp^3 X & \xrightarrow{\exp[u_X]} & \exp^2 X \\ {\scriptstyle [u_{\exp X}]}\downarrow & & \downarrow{\scriptstyle [u_X]} \\ \exp^2 X & \xrightarrow[{[u_X]}]{} & \exp X \end{array}$$

is commutative.

Similarly as above, we consider the set

$$F = \{\mathfrak{A} \in \exp^3 X \mid |u_X(u_{\exp X}(\mathfrak{A}))| < \infty\}.$$

It is well-known that F is dense in $\exp^3 X$ and the restriction of the above diagram on F is commutative. The result follows from Proposition 2.3. □

5. Coarse structures on symmetric powers. Let $\mathbb{T}$ be a monad on a category $\mathcal{C}$. The *Kleisli category of the monad* $\mathbb{T}$ is the category $\mathcal{C}_{\mathbb{T}}$ defined as follows: $|\mathcal{C}_{\mathbb{T}}| = |\mathcal{C}|$, $\mathcal{C}_{\mathbb{T}}(X, Y) = \mathcal{C}(X, TY)$, and the composition $g * f$ of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$, $g \in \mathcal{C}_{\mathbb{T}}(Y, Z)$ is given by $g * f = \mu Z \circ Tg \circ f$ (see[7]).

Note that the category $\mathcal{C}_{\mathbb{T}}$ can be embedded into $\mathcal{C}^{\mathbb{T}}$ as a full subcategory by means of the functor Φ

$$\Phi X = (TX, \mu X), \quad \Phi f = \mu Y \circ Tf, \ f \in \mathcal{C}_{\mathbb{T}}(X, Y).$$

A functor $\overline{F}: \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ called an *extension of the functor* $F: \mathcal{C} \to \mathcal{C}$ *on the Kleisli category* $\mathcal{C}_{\mathbb{T}}$ if $IF = \overline{F}I$.

The following theorem is a criterion of extension of functors onto the Kleisli category; see [8–12] for the proof.

Theorem 5.1. *There exists a bijective correspondence between extensions of functor F onto the Kleisli category $\mathcal{C}_{\mathbb{T}}$ of monad $\mathbb{T}$ and natural transformations $\xi: FT \to TF$ satisfying*

1) $\xi \circ F\eta = \eta F$;
2) $\mu F \circ T\xi \circ \xi T = \xi \circ F\mu$.

For any X, as usual, X^n denotes its nth cartesian power. Given a coarse structure $\mathcal{E}$ on X, define the coarse structure $\mathcal{E}^n$ on $\mathcal{X}^n$ as $\mathcal{E}^n = \{M^n \mid M \in \mathcal{E}\}$.

Let G be a subgroup of the symmetric group S_n (the group of bijections of the set $\{1, \dots, n\}$. Recall that the G-symmetric power functor is defined as follows. Define an equivalence relation $\sim_G$ on X^n by the condition: $(x_1, \dots, x_n) \sim_G (y_1, \dots, y_n)$ if and only if there exists $\sigma \in G$ such that $x_i = y_{\sigma(i)}$ for all $i = 1, \dots, n$. We denote by $[x_1, \dots, x_n]_G$ the equivalence class that contains $(x_1, \dots, x_n)$. By the definition, the G-symmetric power of X is $SP^n_G X = X^n / \sim_G$.

Given a map $f: X \to Y$, we define a map $SP^n_G f: SP^n_G X \to SP^n_G Y$ by the formula

$$SP^n_G f([x_1, \dots, x_n]_G) = [f(x_1), \dots, f(x_n)]_G.$$

Now suppose that $(X, \mathcal{E})$ is a coarse space. For any $M \in \mathcal{E}$ let

$$\hat{M} = \{([x_1, \dots, x_n]_G, [y_1, \dots, y_n]_G) \in SP^n_G X \times SP^n_G X$$
$$\mid \text{there is } \sigma \in G \text{ such that } (x_i, y_{\sigma(i)}) \in M \text{ for every } i = 1, \dots, n\}.$$

If X is a topological space, then $SP^n X$ is endowed with the quotient topology of X^n. A base of this topology is formed by the sets of the form

$$[U_1, \dots, U_n]_G = \{[x_1, \dots, x_n]_G \mid x_i \in U_i,\ i = 1, \dots, n\}.$$

Proposition 5.2. *The family $\hat{\mathcal{E}} = \{\hat{M} \mid M \in \mathcal{E}\}$ is a coarse structure on $SP^n_G X$. If $\mathcal{E}$ is topological (unital), then so is $\hat{\mathcal{E}}$.*

Proof. The fact that $\hat{\mathcal{E}}$ is a coarse structure easily follows from the equalities $(MN)\hat{} = \hat{M}\hat{N}$ and $(M^{-1})\hat{} = (\hat{M})^{-1}$.

Suppose now that $\mathcal{E}$ is topological and $([a_1, \dots, a_n]_G, [b_1, \dots, b_n]_G) \in \hat{M}$ for some $M \in \mathcal{E}$. Then there exists $\sigma \in G$ such that $(a_i, b_{\sigma(i)}) \in M$, for all $i = 1, \dots, n$. There exist open sets U_i and $V_{\sigma(i)}$ in X such that $(a_i, b_{\sigma(i)}) \in U_i \times V_{\sigma(i)} \subset M$. Then obviously

$$([a_1, \dots, a_n]_G, [b_1, \dots, b_n]_G) \in ([U_1, \dots, U_n]_G, [V_1, \dots, V_n]_G) \in \hat{M}.$$

□

Proposition 5.3. *SP^n_G is an endofunctor in the category CS (respectively CTS).*

Proof. We only consider the case of the category CTS. It is sufficient to verify that the map $SP^n_G f$ is coarse, for every coarse map $f: (X, \mathcal{E}) \to (X', \mathcal{E}')$. Given $M \in \mathcal{E}$ we can find $M' \in \mathcal{E}'$ such that $(f \times f)(M) \subset M'$. Then it can be immediately verified that $(SP^n_G f \times SP^n_G f)(\hat{M}) \subset \hat{M}'$.

Besides, we have to prove that for every $[a_1, \dots, a_n]_G \in SP^n_G X'$ and every $M' \in \mathcal{E}'$ the set $K = (SP^n_G f)^{-1}(\hat{M}'([a_1, \dots, a_n]_G))$ is relatively compact. It is easy to see that K is contained in the closure of the set $\cup_{i=1}^n f^{-1}(M'(a_i))$; the latter is relatively compact, because f is coarse. $\square$

Theorem 5.4. *There exists an extension of the functor SP^n_G onto the Kleisli category $(CTS/\sim)_H$.*

Proof. We exploit an idea from [13]. For every coarse topological space X define a map $d_X : SP^n_G \exp X \to \exp SP^n_G$ by the formula

$$d_X([A_1, \dots, A_n]_G) = \{[a_1, \dots, a_n]_G \mid a_i \in A_i,\ i = 1, \dots, n\}.$$

It is easy to verify and we leave it to the reader that d_X is a coarse map for every X.

That $d = (d_X)$ is a natural transformation of the functor $SP^n_G \exp$ into the functor $\exp SP^n_G$ follows from the facts that

$$d_Y SP^n \exp f([A_1, \dots, A_n]_G) = \exp SP^n f d_X([A_1, \dots, A_n]_G)$$

for every finite $A_1, \dots, A_n$ (see [13]), that the set $\{[A_1, \dots, A_n]_G \mid A_i \text{ is finite}, i = 1, \dots, n\}$ is dense in $SP^n_G \exp X$, and Proposition 2.3.

Similarly, one can prove the equalities $d_X \circ SP^n_G s_X = s_{SP^n_G X}$ and $u_{SP^n_G X} \circ \exp d_X \circ d_{\exp X} = d_X \circ SP^n_G u_X$ which are known to be true for finite X (see [13]). Again, by Proposition 2.3, this shows that the conditions of Theorem 5.1. hold. Applying Theorem 5.1 we complete the proof. $\square$

6. Remarks. The importance of the hyperspace monad in the category of compact Hausdorff spaces is closely related to the fact that the category of algebras for this monad (see [7] for the definition) can be described as the category of compact continuous semilattices [14]. A natural question arises whether a counterpart of this result exists in the coarse category.

Besides, in [13] the symmetric power functors are characterized as the normal functors of finite degree that have extensions to the Kleisli category of the hyperspace monad. In the forthcoming publication we are going to extend this result (at least partially) to the coarse category.

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ФУНКТОР ГІПЕРПРОСТОРУ В ГРУБІЙ КАТЕГОРІЇ

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Розглянуто монаду гіперпростору в категорії грубих топологічних просторів і класів еквівалентності грубих відображень. Доведено, що функтор G-симетричного степеня, що діє в категорії топологічних просторів, може бути природно означений і для категорії грубих топологічних просторів, і на цій останній категорії він може бути продовжений на категорію Клейслі монади гіперпростору.

Ключові слова: грубий простір, грубе відображення.

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