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REDUCTION OF A PAIR OF MATRICES OVER AN ADEQUATE DUO-RING TO A SPECIFIC TRIANGULAR FORM BY IDENTICAL UNILATERAL TRANSFORMATIONS

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It is proved that a pair of matrices over an adequate duo-ring can be reduced to a specific triangular form by means of identical unilateral transformations.

Key words: adequate ring, duo-ring, elementary divisor ring.

Throughout this paper, all rings are associative adequate duo-rings with identity. A ring is said to be a *duo-ring* if every its left or right ideal is two-sided. A ring is a *Bezout ring* if every its finitely generated right and left ideal is principal.

Matrices A and B over ring R are *equivalent* ($A \sim B$), if there exist invertible matrices P and Q over R such that $A = PBQ$.

An $m \times n$ matrix A *admits diagonal reduction* if A is equivalent to a diagonal matrix $[\epsilon_{ij}]$ (i.e. $\epsilon_{ij} = 0$ whenever $i \neq j$) with the property $R\epsilon_{i+1,i+1}R \subseteq R\epsilon_{i,i} \cap \epsilon_{i,i}R$ (in the case of a duo-ring we can write: $\epsilon_{i+1,i+1}R \subseteq \epsilon_{i,i}R$). If every matrix over R admits diagonal reduction, then R is an *elementary divisor ring*. The elements $\epsilon_{11}, \epsilon_{22}, \dots, \epsilon_{rr}$ are called the *invariant factors* of the matrix A.

A ring R is called *right adequate* if R is a Bezout ring without zero divisors and for $a, b \in R$ with $a \neq 0$, there exist $r, s \in R$ such that $a = rs, rR + bR = R$, and $s'R + bR \neq R$ for any nonunit $s' : sR \subset s'R$.

Using left principal ideals by analogy we can define left adequate rings. In the class of duo-rings these notions are equivalent and we will use the term adequate ring.

Commutative adequate rings were considered in [1-3].

V.Petrychkovych investigated the reducibility of pairs of matrices by means of the generalized equivalent transformations to the diagonal form [4].

Let R be an adequate duo-ring.

Lemma 1. *Let $a, b, c \in R$ and $a \neq 0, c \neq 0$. Then there exists an element $r \in R$ such that $(a+rb)R+rcR = aR+bR+cR$ and if $aR+bR+cR = R$ then $rR+aR+bR+cR = R$.*

Proof. Let $aR+bR+cR = R$. Assume that $(a+rb)R+rcR = hR$, and $h \notin U(R)$. Then we obtain:

1) $rc = rrs, rR + hR = h'R$ and $(a + rb)R \subseteq h'R$.

It follows that $a \in h'R$ and $aR \subset h'R$. We obtain a contradiction with

$$R = rR + aR \subset h'R + aR \neq R.$$

2) If $rR + hR = R$, then

$$r^2R + hR = R, r^2u + hv = 1$$

$$r^2us + hvs = s, r^2su' + hsv' = s$$

$$u, v, u', v' \in R.$$

We have

$$sR \subset hR \text{ and } hR + aR = h'R,$$

where $h' \notin U(R)$. Thus,

$$(a + rb)R \subset h'R, rbR \subset h'R, R = rR + aR =' rR + h'R,$$

$$aR \subset h'R, bR \subset h'R, cR \subset h'R.'$$

This yields

$$R = aR + bR + cR \subset h'R,$$

and we have $h' \in U(R)$.

If $aR + bR + cR = dR, a = da_0, b = db_0, c = dc_0$ we provide the proof similarly for elements a_0, b_0, c_0.

Lemma 2. *Let $A_i, i = 1, 2$ be $2 \times k_i$ matrices over a ring R, and at least one of them is not a right zero divisor. Then there exist invertible matrices P and $Q_i, i = 1, 2$ over R such that*

$$PA_iQ_i = \begin{pmatrix} \varepsilon_1^{(i)} & 0 & 0 & \dots & 0 \\ * & \varepsilon_2^{(i)} & 0 & \dots & 0 \end{pmatrix},$$

where $\epsilon_j^{(i)}$ are invariant factors of matrices $A_i, i = 1, 2$.

Proof. We may assume that A_2 is not a right zero divisor, so that $k_1 \geqslant 1, k_2 \geqslant 2$. Since R is an elementary divisor ring [5], there exist invertible matrices S, M_1, M_2 over R such that

$$SA_1M_1 = \begin{pmatrix} \varepsilon_1^1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon_2^1 & 0 & \dots & 0 \end{pmatrix}, SA_2M_2 = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ b & c & 0 & \dots & 0 \end{pmatrix},$$

$\epsilon_2^1 R \subseteq \epsilon_1^1 R$ and $a \neq 0, c \neq 0$.

By Lemma 1 for elements $a, b, c \in R$ there exists an element $r \in R$ such that $(a + rb)R + rcR = aR + bR + cR$. Consider the matrix

$$T = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

It is easy to verify that matrices TSA_iM_i can be reduced to the form

$$\begin{pmatrix} \varepsilon_1^{(i)} & 0 & 0 & \dots & 0 \\ * & \varepsilon_2^{(i)} & 0 & \dots & 0 \end{pmatrix}.$$

using right-side multiplication by invertible matrices.

The proof is complete.

Theorem 1. *Let $A_i, i = 1, 2$ be $m \times k_i$ matrices over a ring R, and at least one of them is not a right zero divisor.*

Then there exist invertible matrices P and $Q_i, i = 1, 2$ over R such that

$$\begin{pmatrix} \epsilon_1^{(i)} & 0 & \dots & 0 & \dots & 0 \\ & \epsilon_2^{(i)} & \dots & 0 & \dots & 0 \\ & * & \ddots & & & \\ & & & \epsilon_m^{(i)} & \dots & 0 \end{pmatrix}$$

where $\epsilon_j^{(i)}$ are invariant factors of the matrices $A_i, i = 1, 2$.

Proof. Assume that A_2 is not a right zero divisor. We shall prove the theorem by induction on number m of rows of the matrices. If $m = 2$, the Theorem is true by Lemma 2. Suppose that the theorem is true for matrices with the number of rows $m - 1$. Thus R is an adequate duo-ring and there exist invertible matrices $S, Q_i, i = 1, 2$, such that

$$SA_1Q_1 = \begin{pmatrix} \epsilon_1^{(1)} & 0 & \dots & 0 & \dots & 0 \\ 0 & \epsilon_2^{(1)} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \\ 0 & 0 & \dots & \epsilon_m^{(1)} & \dots & 0 \end{pmatrix} = B_1,$$

$$SA_2Q_2 = \begin{pmatrix} a_{11} & 0 & \dots & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \\ a_{m1} & a_{m2} & \dots & a_m & \dots & 0 \end{pmatrix} = B_2,$$

where $\epsilon_i^{(1)}$ are invariant factors of the matrix A_1. Consider submatrices $B_i', i = 1, 2$, of the matrices B_i obtained by crossing off the last rows of the matrices B_i. For them by the induction hypothesis there exist invertible matrices M, N_i such that

$$MB_1'N_1 = \begin{pmatrix} \epsilon_1^{(1)} & 0 & \dots & 0 & \dots & 0 \\ & \epsilon_2^{(1)} & \dots & 0 & \dots & 0 \\ & * & \ddots & & & \\ & & & \epsilon_{m-1}^{(1)} & \dots & 0 \end{pmatrix},$$

$$MB_2'N_2 = \begin{pmatrix} \varphi_1^{(2)} & 0 & \dots & 0 & \dots & 0 \\ & \varphi_2^{(2)} & \dots & 0 & \dots & 0 \\ & * & \ddots & & & \\ & & & \varphi_{m-1}^{(2)} & \dots & 0 \end{pmatrix},$$

where $\varphi_j^{(2)}$ are invariant factors of the matrix B_2'. Then

$$C_1 = \begin{pmatrix} & & & 0 \\ & M & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} B_1 \begin{pmatrix} & & & 0 \\ & N_1 & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \epsilon_1^{(1)} & 0 & \ldots & 0 & \ldots & 0 \\ & \epsilon_2^{(1)} & \ldots & 0 & \ldots & 0 \\ & * & \ddots & & & \\ 0 & \ldots & 0 & \epsilon_m^{(1)} & \ldots & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} & & & 0 \\ & M & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} B_2 \begin{pmatrix} & & & 0 \\ & N_2 & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \varphi_1^{(2)} & 0 & \ldots & 0 & \ldots & 0 \\ * & \ddots & & & & \\ & & \varphi_{m-1}^{(2)} & 0 & \ldots & 0 \\ a'_{m1} & \ldots & \ldots & a'_{mm} & \ldots & 0 \end{pmatrix}.$$

Let $\varphi_1^{(2)} R + a'_{m1} R + a'_{mm} R = R$. By Lemma 1, there exist $r \in R$ such that for the elements $\varphi_1^{(2)}, a'_{m1}, a'_{mm}$ we obtain

$$(\varphi_1^{(2)} + r a'_{m1}) R + r a'_{mm} R = \varphi_1^{(2)} R + a'_{m1} R + a'_{mm} R,$$

$$r R + \varphi_1^{(2)} R + a'_{m1} R + a'_{mm} R = R. \tag{1}$$

Consider an $m \times m$ matrix of the form

$$T = \begin{pmatrix} 1 & 0 & \ldots & r \\ 0 & 1 & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Multiply the matrices $C_i, i = 1, 2$, on the left by this invertible matrix T:

$$TC_1 = \begin{pmatrix} \varepsilon_1^{(1)} & 0 & \ldots & r\varepsilon_m^{(1)} & 0 & \ldots & 0 \\ & & & * & & & \end{pmatrix},$$

$$TC_2 = \begin{pmatrix} \varphi_1^{(2)} + r a'_{m1}, & r a'_{m2}, & \ldots & r a'_{mm}, & 0 & \ldots & 0 \\ & & & * & & & \end{pmatrix}.$$

Using condition (1) we obtain that the greatest common right divisor of the elements of the first row of the matrix TC_2 is the greatest common right divisor of all elements of the matrix C. Since $\epsilon_m^{(m)} R \subset \epsilon_1^{(1)} R$, we have a similar situation for the elements of the first row of the matrix TC_1. Thus by multiplication on the right by the matrices $L_i, i = 1, 2$ the matrices TC_i can be reduced to the form

$$TC_i L_i = \begin{pmatrix} \epsilon_1^{(i)} & 0 & \ldots & 0 & \ldots & 0 \\ & b_{22}^{(i)} & \ldots & 0 & \ldots & 0 \\ & * & \ddots & & & \\ & & & b_{mm}^{(i)} & \ldots & 0 \end{pmatrix},$$

where $\epsilon_1^{(i)}$ is the first invariant factor of the matrix A_i.

Consider the submatrices of the matrices TC_iL_i obtained by crossing off the first rows and columns. They have the number of rows $m-1$, satisfy the condition of the theorem and for them by induction hypothesis the theorem is true.

If $\varphi_1^{(2)}R + a'_{m1}R + a'_{mm}R = dR$ we can represent the matrix $C_2 = DC'_2$, where $D = diag[d, d, \ldots, d]$ is a diagonal matrix and repeat the same arguments for the matrix C'_2. The proof is complete.

Theorem 2. *Let $C = AB$, where A, B are matrices over R which are not right and left zero divisors. Then the elementary divisors of the matrix C are divizible on coresponding elementary divisors of matrices A and B.*

The same result was obtained for other classes of rings in [2], [3], [6].

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1. *Helmer O.* The elementary divisor theorem for certain rings without chain conditions // Bull. Amer. Math. Soc. - 1943. - 49. - P. 225-236.
2. *Kaplansky J.* Elementary divisors and modules // Trans. Amer. Maht. Soc. - 1949. - 66. - P. 464-491.
3. *Zabavs'ky B. V., Kazimirs'ky P. S.* Reduction of a pair of matrices over an adequate ring to a specific triangular form by means of idential unilateral transformations // Ukrain. Mat. Zh. - 1984. - 36. - P. 256-258.
4. *Petrychkovych V.* Generalized equivalence of pairs of matrices // Linear and Multilinear Algebra. - 2000. - 48. - P. 179-188.
5. *Gatalevich A. I.* On adequate and general adequate duo-rings and elementary divisor duo-rings // Matem. Studii. - 1998. - 49. - P. 10-15.
6. *Newman M.* On the Smith normal form // J. Res. Bur. Stand. Sect. - 1971. - 75. - P. 81-84.

ЗВЕДЕННЯ ПАРИ МАТРИЦЬ НАД АДЕКВАТНИМ ДУО-КІЛЬЦЕМ ДО СПЕЦІАЛЬНОГО ТРИКУТНОГО ВИГЛЯДУ ШЛЯХОМ ІДЕНТИЧНИХ ОДНОБІЧНИХ ПЕРЕТВОРЕНЬ

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Доведено, що пара матриць над адекватним дуо-кільцем зводиться до спеціального трикутного вигляду шляхом ідентичних однобічних перетворень.

Ключові слова: адекватне кільце, дуо-кільце, кільце елементарних дільників.

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