ВІСНИК ЛЬВІВ. УН-ТУ
Серія мех.-мат.2003.Вип.61.С.92-97

VISNYK LVIV UNIV.
Ser.Mech-Math.2003.Vol.61.P.92-97

УДК 519.116

COMBINATORIAL SIZE OF SUBSETS OF SEMIGROUPS AND ORGRAPHS

Andriy Gnatenko, Igor Protasov

Kyiv Taras Shevchenko National University,
64 Volodymyrska Str. 01033 Kyiv, Ukraine

A triple $\mathbf{B} = (X, P, B)$ is called a balls structure if X, P are nonempty sets and, for all $x \in X$, $\alpha \in P$, $B(x, \alpha) \ni x$ is a subset of X, called a ball of radius α around x. We classify subsets of X by their sizes with respect to the ball's structure $\mathbf{B}$ and apply this classification to semigroups and oriented graphs.

Key words: ball's structure, large and small subsets.

1. Ball's structures. Let X, P be nonempty sets and let, for any $x \in X$, $\alpha \in P$, $B(x, \alpha) \ni x$ be a subset of X, which is called *the ball of radius α around x*. Following [1], a triple $\mathbf{B} = (X, P, B)$ is called a *ball's structure*.

For any $x \in X$, $\alpha \in P$, put $B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}$. A ball's structure $\mathbf{B}^* = (X, P, B^*)$ is called dual to $\mathbf{B}$. Observe that $B^{**}(x, \alpha) = B(x, \alpha)$ for all x, α and thus $\mathbf{B}^{**} = \mathbf{B}$.

Define a preordering $\leqslant$ on the set P by the rule: $\alpha \leqslant \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset P' of P is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ with $\alpha \leqslant \beta$. A ball's structure $\mathbf{B}$ is called *symmetric* if there exists a cofinal subset $P' \subseteq P$ such that $B(x, \beta) = B^*(x, \beta)$ for all $x \in X$, $\beta \in P'$.

Given any subset $A \subseteq X$ and $\alpha \in P$, put

$$B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha), \quad Int(A, \alpha) = \{x \in X : B^*(x, \alpha) \subseteq A\}.$$

A ball's structure $\mathbf{B} = (X, P, B)$ is called *multiplicative* if, for any $\alpha, \beta \in P$ there exists $\gamma(\alpha, \beta) \in P$ such that $B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$ for every $x \in X$. Since $B^*(B^*(x, \alpha), \beta) \subseteq B^*(x, \gamma(\beta, \alpha))$, $\mathbf{B}$ is multiplicative if and only if $\mathbf{B}^*$ is multiplicative.

Example 1. Let $Gr = (V, E)$ be an oriented graph where V is the set of vertices of Gr and $E \subset V \times V$ is the set of its edges. For every $x \in V$, put $d(x, x) = 0$. If for distinct $x, y \in V$ there exists an oriented path from x to y, then let $d(x, y)$ be the length of the shortest oriented path from x to y. Otherwise, put $d(x, y) = \infty$. Given any $x \in V$ and $n \in \omega$, put $B(x, n) = \{y \in V : d(x, y) \leqslant n\}$. The ball's structure (V, ω, B) will be denoted by $\mathbf{B}(Gr)$. Taking into account that $B(B(x, n), m) \subseteq B(x, n + m)$ we conclude that $\mathbf{B}(Gr)$ is multiplicative. Note also that $\mathbf{B}^*(Gr)$ coincides with

$\mathbf{B}(Gr^*)$, where $Gr^* = (V, E^{-1})$, $E^{-1} = \{(y,x) : (x,y) \in E\}$. If $E = E^{-1}$, then $\mathbf{B}(Gr) = \mathbf{B}^*(Gr)$.

Example 2. Let S be a semigroup with the identity e and let Fin be the family of all finite subsets of S containing e. Given any $s \in S$ and $F \in Fin$, put

$$B_l(x,F) = Fx \text{ and } B_r(x,F) = xF.$$

The balls's structures (S, Fin, B_l) and (S, Fin, B_r) will be denoted by $\mathbf{B}_l(S)$ and $\mathbf{B}_r(S)$. If $x \in S$ and $F,\ F' \in Fin$, then

$$B_l(B_l(x,F),F') \subseteq B_l(x,F'F) \text{ and } B_r(B_r(x,F),F') \subseteq B_r(x,FF').$$

Hence, $\mathbf{B}_l(S)$ and $\mathbf{B}_r(S)$ are multiplicative. If S is a group, then $\mathbf{B}_l(S)$ and $\mathbf{B}_r(S)$ are symmetric [1, Example 2].

2. Classification of subsets by their sizes. Fix a ball's structure $\mathbf{B} = (X, P, B)$. A subset $A \subseteq X$ is called

- *large* if there exists $\alpha \in P$ such that $X = B(A,\alpha)$;
- *small* if $X \backslash B(A,\alpha)$ is large for every $\alpha \in P$;
- *extralarge* if $Int(A,\alpha)$ is large for every $\alpha \in P$;
- *piecewise large* if there exists $\beta \in P$ such that $Int(B(A,\beta),\alpha) \neq \emptyset$ for every $\alpha \in P$.

Observe that for a multiplicative ball's structure $\mathbf{B} = (X, P, B)$ a subset $A \subset X$ is large if and only if $B(A,\alpha)$ is large for some $\alpha \in P$.

Lemma 1. *Let $\mathbf{B} = (X, P, B)$ be a ball's structure, $A \subseteq X$, $\alpha \in P$. Then* $Int(X\backslash A,\alpha) = X\backslash B(A,\alpha)$.

Proof. Let $x \in Int(X\backslash A,\alpha)$. Then $B^*(x,\alpha) \cap A = \emptyset$, so $x \notin B(a,\alpha)$ for every $a \in A$. Hence, $x \in X\backslash B(A,\alpha)$.

Let $x \in X\backslash B(A,\alpha)$. Then $x \notin B(a,\alpha)$ for every $a \in A$. Hence, $a \notin B^*(x,\alpha)$ for every $a \in A$, so $B^*(x,\alpha) \subseteq X\backslash A$ and $x \in Int(X\backslash A,\alpha)$. □

The following statement is a refinement of Theorem 1 from [1].

Theorem 1. *Let $\mathbf{B} = (X, P, B)$ be a ball's structure and let $S \subseteq X$. Then the following statements are equivalent:*

1) S is small;

2) S is not piecewise large;

3) $X\backslash S$ is extralarge.

If, moreover, $\mathbf{B}$ is multiplicative, then the statements 1)-3) are equivalent to

4) $(X\backslash S) \cap L$ is large for every large subset L of X.

Proof. 1) $\Rightarrow$ 2). For every $\alpha \in P$, pick $\beta(\alpha) \in P$ such that $B(X\backslash B(S,\alpha),\beta(\alpha)) = X$. Take any $x \in X$ and choose $y \in X\backslash B(S,\alpha)$ with $x \in B(y,\beta(\alpha))$. Then $y \in B^*(x,\beta(\alpha))$ and $B^*(x,\beta(\alpha)) \cap (X\backslash B(S,\alpha)) \neq \emptyset$. Hence, $Int(B(S,\alpha),\beta(\alpha)) = \emptyset$ and S is not piecewise large.

2) $\Rightarrow$ 3). For every $\alpha \in P$, pick $\beta(\alpha) \in P$ such that $Int(B(S,\alpha),\beta(\alpha)) = \emptyset$. Then $B^*(x,\beta(\alpha)) \cap (X\backslash B(S,\alpha)) \neq \emptyset$ for every $x \in X$. By Lemma 1,

$$B^*(x,\beta(\alpha)) \cap (Int(X\backslash S,\alpha)) \neq \emptyset$$

for every $x \in X$. Hence, $X = B(Int(X\backslash S,\alpha),\beta(\alpha))$ and $X\backslash S$ is extralarge.

3) $\Rightarrow$ 1). For every $\alpha \in P$, pick $\beta(\alpha) \in P$ such that $B(Int(X\backslash S, \alpha), \beta(\alpha)) = X$. By Lemma 1, $B(X\backslash B(S, \alpha), \beta(\alpha)) = X$. Hence, S is small.

3) $\Rightarrow$ 4). Put $Y = X\backslash S$ and take any large subset L. Choose $\alpha \in P$ such that $X = B(L, \alpha)$. For every $x \in Int(Y, \alpha)$, choose $y(x) \in L$ with $x \in B(y(x), \alpha)$, equivalently, $y(x) \in B^*(x, \alpha)$. Put $Y' = \{y(x) : x \in Int(Y, \alpha)\}$ and note that $Y' \subseteq Y \cap L$. Since $Int(Y, \alpha) \subseteq B(Y', \alpha)$ and $Int(Y, \alpha)$ is large, by the multiplicativity of $\mathbf{B}$, Y' is large. Since $Y' \subseteq Y \cap L$, we get that $Y \cap L$ is large.

4) $\Rightarrow$ 3). Put $Y = X\backslash S$. Since $Y \cap X = Y$ and X is large, Y is large too. Fix any $\alpha \in P$ and show that $Int(Y, \alpha)$ is large. For every $x \in Y\backslash Int(Y, \alpha)$, pick $y(x) \in B^*(x, \alpha)\backslash Y$. Put $Y' = \{y(x) : x \in Y\backslash Int(Y, \alpha)\}$, $L = Y' \cup Int(Y, \alpha)$. Note that $Y \subseteq B(L, \alpha)$. Since Y is large, $B(L, \alpha)$ is large. By the multiplicativity of $\mathbf{B}$, L is large. By the assumption, $Y \cap L$ is large. Since $Y \cap L = Int(Y, \alpha)$, $Int(Y, \alpha)$ is large. □

Theorem 2. *Let $\mathbf{B} = (X, P, B)$ be a multiplicative ball's structure. If subsets $X_1, X_2, \dots, X_n$ of X are extralarge, then $X_1 \cap X_2 \cap \dots \cap X_n$ is extralarge. If subsets $S_1, S_2, \dots, S_n$ of X are small, then $S_1 \cup S_2 \cup \dots \cup S_n$ is small. If a piecewise large subset A of X finitely partitioned $A = A_1 \cup A_2 \cup \dots \cup A_n$, then at least one cell A_i of the partition is piecewise large. In particular, X can not be partitioned into finitely many small subsets.*

Proof. Take any large subset L of X. By equivalence $3 \Leftrightarrow 4$ Theorem 1, $X_n \cap L$ is large. Since $(X_1 \cap X_2 \cap \dots \cap X_n) \cap L = (X_1 \cap X_2 \cap \dots \cap X_{n-1}) \cap (X_n \cap L)$, by induction, $(X_1 \cap X_2 \cap \dots \cap X_n) \cap L$ is large. By equivalence $3 \Leftrightarrow 4$ of Theorem 1, $X_1 \cap X_2 \cap \dots \cap X_n$ is extralarge. The second statement follows from the first one and the equivalence $1 \Leftrightarrow 3$ of Theorem 1. The third statement follows from the second statement and the equivalence $1 \Leftrightarrow 2$ of Theorem 1. □

By Theorem 2, the family $\varphi(\mathbf{B})$ of all extralarge subsets of X is a filter on X.

Theorem 3. *Let $\mathbf{B} = (X, P, B)$ be a multiplicative ball's structure and let ψ be an ultrafilter on X. Then $\varphi(\mathbf{B}) \subseteq \psi$ if and only if every subset $A \in \psi$ is piecewise large.*

Proof. Suppose that $\varphi(\mathbf{B}) \subseteq \psi$ and take any subset $A \in \psi$. Assume that A is not piecewise large. By equivalence $1 \Leftrightarrow 2$ of Theorem 1, A is small. By equivalence $1 \Leftrightarrow 3$ of Theorem 1, $X\backslash A$ is extralarge. Hence, $X\backslash A \in \varphi(\mathbf{B})$, a contradiction with $A, X\backslash A \in \psi$.

Suppose that every subset $A \in \psi$ is piecewise large, but $\varphi(\mathbf{B}) \not\subseteq \psi$. Choose any subset $Y \in \varphi(\mathbf{B}), Y \notin \psi$. Since ψ is an ultrafilter, then $X\backslash Y \in \psi$. By equivalence $1 \Leftrightarrow 3$ of Theorem 1, $X\backslash Y$ is small, a contradiction with equivalence $1 \Leftrightarrow 2$ of Theorem 1. □

3. Resolvability of ball's structures. Let $\mathbf{B} = (X, P, B)$ be a ball's structure and let $\mathcal{L}$ be the family of all large subsets of X. A subset $A \subseteq X$ is called *$\mathcal{L}$-dense* if $A \cap L \neq \emptyset$ for every large subset L of X. A ball's structure $\mathbf{B}$ is called *ω-resolvable* if X can be partitioned into countably many $\mathcal{L}$-dense subsets.

Lemma 2. *Let $\mathbf{B} = (X, P, B)$ be a ball's structure. Suppose that there exists a cofinal linearly ordered sequence $\langle \alpha_n \rangle_{n \in \omega}$ of elements of P and a sequence $\langle x_n \rangle_{n \in \omega}$*

of elements of X such that the family $\{B^*(x_n, \alpha_n) : n \in \omega\}$ is disjoint. Then $\mathbf{B}$ is ω-resolvable.

Proof. Let $\omega = \bigcup_{k\in\omega} W_k$ be a partition of ω into countably many infinite subsets. It suffices to show that, for every $k \in \omega$, the subset $A_k = \bigcup_{n\in W_k} B^*(x_n, \alpha_n)$ is $\mathcal{L}$-dense. Take any large subset L of X and pick $\alpha \in P$ such that $X = B(L, \alpha)$. Choose $n \in W_k$ such that $\alpha_n > \alpha$. Since $X = B(L, \alpha_n)$, we get $x_n \in B(L, \alpha_n)$ and $B^*(x_n, \alpha_n) \cap L \neq \emptyset$. Hence, $L \cap A_k \neq \emptyset$. □

The following statement is a generalization of Theorem 5.31 from [2] concerning a resolvability of the ball's structures of groups.

Theorem 4. *Let $\mathbf{B} = (X, P, B)$ be a ball's structure such that the balls $B(x, \alpha)$, $B^*(x, \alpha)$ are finite for all $x \in X$, $\alpha \in P$. If there exists a cofinal linearly ordered sequence $\langle \alpha_n \rangle_{n\in\omega}$ of elements of P, then $\mathbf{B}$ is ω-resolvable.*

Proof. Using the assumptions, construct inductively a sequence $\langle x_n \rangle_{n\in\omega}$ of elements of X such that the family $\{B^*(x_n, \alpha_n)\}$ is disjoint. Then apply Lemma 2. □

4. Applications to semigroups. Let S be a semigroup with the identity e and let Fin be the family of all finite subsets of S containing e. Given any subsets $A, B \subseteq S$, put

$$A^{-1}B = \{s \in S : As \cap B \neq \emptyset\}, \quad AB^{-1} = \{s \in S : sB \cap A \neq \emptyset\}.$$

For every element $s \in S$ and every subset $A \subseteq S$, we write $A^{-1}s$ and sA^{-1} instead of $A^{-1}\{s\}$ and $\{s\}A^{-1}$.

A subset $A \subseteq S$ is called

- *left (right) large* if there exists $F \in Fin$ such that $S = FA$ ($S = AF$);
- *left (right) small* if the subset $S \setminus FA$ ($S \setminus AF$) is left(right) large for every subset $F \in Fin$;
- *left (right) extralarge* if $S \setminus A$ is left(right) small;
- *left (right) piecewise large* if there exists $F \in Fin$ such that, for every subset $H \in Fin$, there exists $x \in S$ with $H^{-1}x \subseteq FA$ ($xH^{-1} \subseteq AF$).

Note that a left (right) size of subset A of semigroup S is exactly a size of A in the ball's structure $\mathbf{B}_l(S)$ ($\mathbf{B}_r(S)$).

A subset $A \subseteq S$ is called

- *left* (right*) large* if there exists $F \in Fin$ such that $S = F^{-1}A$ ($S = AF^{-1}$);
- *left* (right*) small* if $S \setminus F^{-1}A$ ($S \setminus AF^{-1}$) is left* (right*) large for every subset $F \in Fin$;
- *left* (right*) extralarge* if $S \setminus A$ is left* (right*) small;
- *left* (right*) piecewise large* if there exists $F \in Fin$ such that, for every subset $H \in Fin$, there exists $x \in S$ with $Hx \subseteq F^{-1}A$ ($xH \subseteq AF^{-1}$).

In topological dynamics [3], left* (right*) large subsets are called *left* (right*) syndetic* while left* (right*) piecewise large subsets are called *left (right) syndetic.*

Note that a left* (right*) size of subset $A \subseteq S$ is exactly a size of A in the ball's structure $\mathbf{B}_l^*(S)(\mathbf{B}_r^*(S))$.

Theorem 5. *For every finite partition of semigroup S, among the cells of the partition there exist a left piecewise large subset, a right piecewise large subset, a left* piecewise large subset, and a right* piecewise large subset.*

Proof. Apply Theorem 2 to the ball's structures $\mathbf{B}_l(S)$, $\mathbf{B}_r(S)$, $\mathbf{B}_l^*(S)$, $\mathbf{B}_r^*(S)$ respectively. □

Theorem 6. *Let S be a countable semigroup such that the subsets $F^{-1}x$ and xF^{-1} are finite for every subset $F \in Fin$. Then the ball's structures $\mathbf{B}_l(S)$, $\mathbf{B}_r(S)$, $\mathbf{B}_l^*(S)$, $\mathbf{B}_r^*(S)$ are ω-resolvable.*

Proof. Apply Theorem 4. □

Remark 1. By Theorem 6, every countable group G can be partitioned $G = \bigcup_{n\in\omega} A_n$ so that each subset $G\setminus A_n$ is not right large. In particular, there exist a partition $G = B_1 \cup B_2$ such that B_1, B_2 are not right large. Let X be an infinite set of cardinality γ and let $S = S(X)$ be the semigroup of all mappings $X \to X$. A.Ravsky [4] proved that, for every partition $S = \bigcup_{\alpha<\gamma} S_\alpha$, there exist $\alpha < \gamma$ and $s \in S$ such that $S = S_\alpha s$, i.e. at least one cell of the partition is right large. A countable counterpart of this statement was proved in [5]. There exist a countable semigroup S such that, for every finite partition $S = A_1 \cup A_2 \cup \ldots \cup A_n$, there exist $i \leqslant n$ and $s \in S$ such that $S = A_i s$. Obviously, the ball's structure $\mathbf{B}_r(S)$ is not resolvable, i.e. S can not be partitioned into two $\mathcal{L}$-dense subset, where $\mathcal{L}$ is a family of all right large subsets of S.

Remark 2. By [1], every infinite group can be partitioned into countably many subsets such that each of them is left and right small. Ravsky's results concerning $S(X)$ shows that this statement is not valid for all semigroups.

Question [5]. Does there exist an infinite semigroup S such that, for every partition $S = A_1 \cup A_2$, one of the cells A_1, A_2 is left and right large.

5. Application to orgraphs. Let $Gr = (V, E)$ be an oriented graph. By Theorem 2, for every finite partition of V, at least one cell of the partition is piecewise large with respect to the ball's structure $\mathbf{B}(Gr)$. In particular, if V is finite, then there exists a vertex $v \in V$ such that the subset $\{v\}$ is piecewise large. Let us illustrate the last observation.

Let $Gr = (V, E)$ be an arbitrary oriented graph. For every $v \in V$, denote by $St(v)$ (resp. $St^*(v)$) the set of all $x \in V$ such that there exists an oriented path from v to x (resp. from x to v). Define a preordering $\leqslant$ on V by the rule: $v_1 \leqslant v_2$ if and only if $St(v_1) \subseteq St(v_2)$.

Theorem 7. *Let $Gr = (V, E)$ be a finite orgraph and let $v \in V$. Then v is $\leqslant$-maximal if and only if $\{v\}$ is a piecewise large in the ball's structure $\mathbf{B}(Gr)$.*

Proof. Suppose that v is $\leqslant$-maximal. Since V is finite, it suffices to show that $St^*(v) \subseteq St(v)$. Take any $x \in St^*(v)$. Then $v \in St(x)$. By maximality of v, $x \in St(v)$. Hence, $St^*(v) \subseteq St(v)$.

Assume that $\{v\}$ is piecewise large. Since V is finite, then there exists $x \in V$ such that $St^*(x) \subseteq St(v)$. Take any element y with $v \in St(y)$. Then $y \in St^*(x)$, so $y \in St(v)$. Hence, v is $\leqslant$-maximal. □

An orgraph $Gr = (V, E)$ is called *locally finite* if the set $\{y \in V : (x, y) \in E\} \cup \{y \in V : (y, x) \in E\}$ is finite for every $x \in V$

Theorem 8. *Let $Gr = (V, E)$ be an infinite locally finite orgraph. Then the ball's structure $\mathbf{B}(Gr)$ is ω-resolvable.*

Proof. Apply Theorem 4. □

---

1. *Protasov I. V.* Combinatorial size of subsets of groups and graphs // manuscript.
2. *Protasov I., Zelenyuk E.* Topologies on Groups Determined by Sequences // Mathem. Studii: Monogr. Series. – 1999. – Vol. 4. Lviv.
3. *Hindman N., Strauss D.* Algebra in the Stone-Čech Compactification. Theory and Applications, Walter de Grueter. Berlin; New York, 1998.
4. *Ravsky A.* Personal communication.
5. *Protasov I. V.* On Ravsky's Theorem // Third Internation Algebraic Conference in Ukraine, Sumy, July 2-8, 2001. – P. 95-96.

КОМБІНАТОРНИЙ РОЗМІР ПІДМНОЖИН У НАПІВГРУПАХ НА ОРІЄНТОВАНИХ ГРАФАХ

А. Гнатенко, І. Протасов

Київський національний університет імені Тараса Шевченка, вул. Володимирська, 64 01033 Київ, Україна

Трійка $\mathbf{B} = (X, P, B)$ називається кульовою структурою, якщо X, P – непорожні множини і для довільних $x \in X$ та $\alpha \in P$ в X зафіксовано підмножину $B(x, \alpha) \ni x$, яка називається кулею радіуса α навколо x. Класифікуємо підмножини X за їх розміром щодо кульової структури $\mathbf{B}$, застосовуємо отримані результати до проблеми розкладності напівгруп та орієнтованих графів.

Ключові слова: кульова структура, великі та малі підмножини.

Стаття надійшла до редколегії 01.04.2002

Прийнята до друку 14.03.2003