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TOPOLOGICAL BRANDT λ-EXTENSIONS OF ABSOLUTELY H-CLOSED TOPOLOGICAL INVERSE SEMIGROUPS

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Some properties of homomorphic images of Brandt λ-extensions of algebraic semigroups are established. It is proved that for every cardinal $\lambda \geqslant 2$ any topological Brandt λ-extension of an absolutely H-closed topological inverse semigroup is absolutely H-closed in the class of topological inverse semigroups.

Key words: topological inverse semigroup, Brandt λ-extension, topological Brandt λ-extension, H-closed topological semigroup, absolutely H-closed topological semigroup, algebraically h-closed semigroup, topological semilattice, topological semigroup.

In this paper all spaces are Hausdorff.

A *topological* (*inverse*) *semigroup* is a topological space together with a continuous multiplication (and an inversion, respectively).

We follow the terminology of [2, 3, 7].

If S is a semigroup, then by $E(S)$ we denote the band (the subset of idempotents) of S, and by S^1 we denote the semigroup S with the adjoined unit (see: [3]). By ω we denote the first infinite ordinal. Further, we identify all cardinals with their corresponding initial ordinals. If Y is a subspace of a topological space X, and $A \subseteq Y$, then by $\mathrm{cl}_Y(A)$ we denote the topological closure of A in Y.

Let S be a semigroup and I_λ be a set of cardinality $\lambda \geqslant 2$. On the set $B_\lambda(S) = I_\lambda \times S^1 \times I_\lambda \bigcup\{0\}$ we define the semigroup operation " $\cdot$ " as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma, \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$ for $\alpha, \beta, \gamma, \delta \in I_\lambda$, $a, b \in S^1$. The semigroup $B_\lambda(S)$ is called the *Brandt-Howie semigroup of the weight λ over S* [8] or the *Brandt λ-extension of the semigroup S* [9]. Obviously $B_\lambda(S)$ is the Rees matrix semigroup $M^0(S^1; I_\lambda, I_\lambda, \mathcal{M})$, where $\mathcal{M}$ is the $I_\lambda \times I_\lambda$-identity matrix. Further, if $A \subseteq S^1$ then we shall denote $A_{\alpha\beta} = \{(\alpha, s, \beta) \mid s \in A\}$ for $\alpha, \beta \in I_\lambda$. If a semigroup S is trivial (i.e. S contains only one element), then $B_\lambda(S)$ is *the semigroup of $I_\lambda \times I_\lambda$ matrix units* [3] and we shall denote it by B_λ.

Further, by $\mathcal{S}$ we denote some class of topological semigroups.

Definition 1 [9]. Let λ be a cardinal $\geqslant 2$, and $(S,\tau) \in \mathcal{S}$. Let τ_B be a topology on $B_\lambda(S)$ such that

a) $(B_\lambda(S), \tau_B) \in \mathcal{S}$;

b) $\tau_B|_{(\alpha,S^1,\alpha)} = \tau$ for some $\alpha \in I_\lambda$.

Then $(B_\lambda(S), \tau_B)$ is called a *topological Brandt λ-extension of (S,τ) in $\mathcal{S}$*. If $\mathcal{S}$ coincides with the class of all topological semigroups, then $(B_\lambda(S), \tau_B)$ is called a *topological Brandt λ-extension of (S,τ)*.

A semigroup $S \in \mathcal{S}$ is called *H-closed in $\mathcal{S}$*, if S is a closed subsemigroup of any topological semigroup $T \in \mathcal{S}$ which contains S as a subsemigroup. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *H-closed*. H-closed topological semigroups were introduced by J. W. Stepp in [12], and there they were called *maximal semigroups*.

Definition 2 [10, 13]. A topological semigroup $S \in \mathcal{S}$ is called *absolutely H-closed in the class $\mathcal{S}$*, if any continuous homomorphic image of S into $T \in \mathcal{S}$ is H-closed in $\mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *absolutely H-closed*.

An algebraic semigroup S is called *algebraically h-closed in $\mathcal{S}$*, if S with discrete topology $\mathfrak{d}$ is absolutely H-closed in $\mathcal{S}$ and $(S,\mathfrak{d}) \in \mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *algebraically h-closed*.

Absolutely H-closed topological semigroups and algebraically h-closed semigroups were introduced by J. W. Stepp in [13], and there they were called *absolutely maximal* and *algebraic maximal*, respectively.

Obviously, any algebraically h-closed semigroup (in a class $\mathcal{S}$) is absolutely H-closed (in a class $\mathcal{S}$), and every absolutely H-closed topological semigroup (in a class $\mathcal{S}$) is H-closed (in a class $\mathcal{S}$). Further we shall show that the converse statements do not hold.

Recall [1], a topological group G is called *absolutely closed* if G is a closed subgroup of any topological group which contains G as a subgroup. In our terminology such topological groups are called H-closed in the class of topological groups. In [11] D. A. Raikov proved that a topological group G is absolutely closed if and only if it is Raikov complete, i.e. G is complete with respect to the two-sided uniformity.

A topological group G is called *h-complete* if for every continuous homomorphism $h: G \to H$ the subgroup $f(G)$ of H is closed [5]. The h-completeness is preserved under taking products and closed central subgroups [5].

For any $\lambda \geqslant 2$ the semigroup of $I_\lambda \times I_\lambda$ matrix units is a Brandt λ-extension of the trivial semigroup. The semigroup of $I_\lambda \times I_\lambda$ matrix units is algebraically h-closed in the class of topological inverse semigroups for each $\lambda \geqslant 2$ [10]. In [9] it is proved that for every $\lambda \geqslant 2$ any topological Brandt λ-extension of an H-closed topological inverse semigroup is H-closed in the class of topological inverse semigroups. In this paper we show that a similar statements hold for absolutely H-closed topological inverse semigroups and that any Brandt λ-extension of an algebraically h-closed inverse semigroup is algebraically h-closed in the class of topological inverse semigroups.

Proposition 3. *Let $h: B_\lambda(S) \to T$ be a homomorphism, such that $h((\alpha,x,\beta)) = h(0)$ for some $x \in S^1$, $\alpha,\beta \in I_\lambda$. Then $h((\gamma,y,\delta)) = h(0)$ for all $y \in S^1xS^1$, $\gamma,\delta \in I_\lambda$.*

Proof. Assume that $y \in S^1xS^1$. Then $y = axb$ for some $a, b \in S^1$. Therefore

$$h((\gamma, y, \delta)) = h((\gamma, a, \alpha) \cdot (\alpha, x, \beta) \cdot (\beta, b, \delta)) = h((\gamma, a, \alpha)) \cdot h((\alpha, x, \beta)) \cdot h((\beta, b, \delta)) =$$

$$h((\gamma, a, \alpha)) \cdot h(0) \cdot h((\beta, b, \delta)) = h((\gamma, a, \alpha) \cdot 0 \cdot (\beta, b, \delta)) = h(0).$$

A semigroup homomorphism $h\colon S \to T$ is called ***annihilating*** if there exists $c \in T$ such that $h(a) = c$ for all $a \in S$.

Corollary 4. *A homomorphism $h\colon B_\lambda(S) \to T$ is annihilating if and only if the homomorphism $h|_{B_\lambda}\colon B_\lambda = B_\lambda(1) \to T$ is annihilating.*

Proposition 5. *Let $h\colon B_\lambda(S) \to T$ be a homomorphism and $h((\alpha_1, a, \beta_1)) = h((\alpha_2, b, \beta_2))$ for some $a, b \in S^1$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I_\lambda$. If $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$ then $h((\alpha_1, a, \beta_1)) = h(0)$.*

Proof. Assume that $\alpha_1 \neq \alpha_2$. Then

$$h((\alpha_1, a, \beta_1)) = h((\alpha_1, 1, \alpha_1)(\alpha_1, a, \beta_1)) = h((\alpha_1, 1, \alpha_1)) \cdot h((\alpha_1, a, \beta_1)) =$$

$$h((\alpha_1, 1, \alpha_1)) \cdot h((\alpha_2, b, \beta_2)) = h((\alpha_1, 1, \alpha_1) \cdot (\alpha_2, b, \beta_2)) = h(0).$$

The proof of the case $\beta_1 \neq \beta_2$ is similar.

Lemma 6. *Let $\lambda \geqslant 2$ and $B_\lambda(S)$ be a topological λ-extension of a topological semigroup S. Let T be a topological semigroup and $h\colon B_\lambda(S) \to T$ be a continuous homomorphism. Then the sets $h(A_{\alpha\beta})$ and $h(A_{\gamma\delta})$ are homeomorphic in T for all $\alpha, \beta, \gamma, \delta \in I_\lambda$, and $A \subseteq S^1$.*

Proof. If h is an annihilating homomorphism, then the statement of the lemma is trivial.

In the other case we fix $\alpha, \beta, \gamma, \delta \in I_\lambda$. Define the maps $\varphi^{\gamma\delta}_{\alpha\beta}\colon T \to T$ and $\varphi^{\alpha\beta}_{\gamma\delta}\colon T \to T$ by the formulae $\varphi^{\gamma\delta}_{\alpha\beta}(s) = h((\gamma, 1, \alpha)) \cdot s \cdot h((\beta, 1, \delta))$ and $\varphi^{\alpha\beta}_{\gamma\delta}(s) = h((\alpha, 1, \gamma)) \cdot s \cdot h((\delta, 1, \beta))$, $s \in T$. Obviously $\varphi^{\alpha\beta}_{\gamma\delta}\left(\varphi^{\gamma\delta}_{\alpha\beta}\left(h\left((\alpha, x, \beta)\right)\right)\right) = h\left((\alpha, x, \beta)\right)$, $\varphi^{\gamma\delta}_{\alpha\beta}\left(\varphi^{\alpha\beta}_{\gamma\delta}\left(h\left((\gamma, x, \delta)\right)\right)\right) = h\left((\gamma, x, \delta)\right)$, for all $\alpha, \beta, \gamma, \delta \in I_\lambda$, $x \in S^1$, and hence $\varphi^{\gamma\delta}_{\alpha\beta}|_{A_{\alpha\beta}} = (\varphi^{\alpha\beta}_{\gamma\delta})^{-1}|_{A_{\alpha\beta}}$. Since the maps $\varphi^{\gamma\delta}_{\alpha\beta}$ and $\varphi^{\alpha\beta}_{\gamma\delta}$ are continuous on T, then $\varphi^{\gamma\delta}_{\alpha\beta}|_{h(A_{\alpha\beta})}\colon h(A_{\alpha\beta}) \to h(A_{\gamma\delta})$ is a homeomorphism.

Proposition 7. *Let $\lambda \geqslant 2$ and $B_\lambda(S)$ be a topological λ-extension of a topological semigroup S. Let T be a topological semigroup and $h\colon B_\lambda(S) \to T$ be a continuous homomorphism, $A \subseteq h(B_\lambda(S))$, and the set A intersects at least two subsets of the type $h(S_{\alpha\beta})$. Then $h(0) \in A \cdot A$.*

Proof. The case $h(0) \in A$ is trivial. Assume that $h(0) \notin A$, $A \cap h(S_{\alpha_1\alpha_2}) \neq \varnothing$ and $A \cap h(S_{\beta_1\beta_2}) \neq \varnothing$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I_\lambda$, i.e. there exist $x, y \in S^1$ such that $h((\alpha_1, x, \alpha_2)) \in A$ and $h((\beta_1, y, \beta_2)) \in A$. If $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$, then $h(0) = h((\alpha_1, x, \alpha_2)) \cdot h((\alpha_1, x, \alpha_2)) \in A \cdot A$ or $h(0) = h((\beta_1, y, \beta_2)) \cdot h((\beta_1, y, \beta_2)) \in A \cdot A$. If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then $\alpha_2 \neq \beta_1$, and hence $h(0) = h((\alpha_1, x, \alpha_2)) \cdot h((\beta_1, y, \beta_2)) \in A \cdot A$.

Lemma 8. *Let $\lambda \geqslant 2$, $B_\lambda(S)$ and T be topological semigroups and $h: B_\lambda(S) \to T$ be a continuous homomorphism. Let $h(B_\lambda(S))$ be a dense subsemigroup of T and $h(S_{\alpha\beta})$ be a closed subset in T for some $\alpha, \beta \in I_\lambda$. Then $a \cdot a = h(0)$ for all $a \in T \backslash h(B_\lambda(S))$, and $h(0)$ is the zero of T.*

Proof. Since $h(B_\lambda(S))$ is a dense subsemigroup of T, then by Proposition 2 [9], $h(0)$ is the zero of T.

Assume that $a \cdot a = b \neq h(0)$ for some $a \in T \backslash h(B_\lambda(S))$. Then for any open neighbourhood $U(b) \not\ni h(0)$ there exists an open neighbourhood $V(a) \not\ni h(0)$ such that $V(a) \cdot V(a) \subseteq U(b)$. By Lemma 6 the set $h(S_{\gamma\delta})$ is closed for each $\gamma, \delta \in I_\lambda$. Therefore the neighbourhood $V(a)$ intersects infinitely many sets of the type $h(S_{\alpha\beta})$ $(\alpha, \beta \in I_\lambda)$. Then by Proposition 7 we have $h(0) \in V(a) \cdot V(a) \subseteq U(b)$, a contradiction with the choice of $U(b)$.

Proposition 9. *Let $\lambda \geqslant 2$, S and T be algebraic semigroups. Let $h: B_\lambda(S) \to T$ be a homomorphism, A and B be disjunctive subsets of $h(B_\lambda(S))$. If the sets A and B intersect at least two subsets of the type $h(S_{\alpha\beta})$ $(\alpha, \beta \in I_\lambda)$, then $h(0) \in A \cdot B$ or $h(0) \in B \cdot A$.*

Proof. The cases $h(0) \in A$ or $h(0) \in B$ are trivial. In the other case for $i = 1, 2, 3, 4$ we fix $\alpha_i, \beta_i \in I_\lambda$ such that $A \bigcap h(S_{\alpha_1\beta_1}) \neq \varnothing$, $A \bigcap h(S_{\alpha_2\beta_2}) \neq \varnothing$, $B \bigcap h(S_{\alpha_3\beta_3}) \neq \varnothing$ and $B \bigcap h(S_{\alpha_4\beta_4}) \neq \varnothing$. By Proposition 5 the sets $h(S_{\alpha_1\beta_1}) \backslash h(0)$ and $h(S_{\alpha_2\beta_2}) \backslash h(0)$ are disjunctive in $h(B_\lambda(S))$, hence $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. Let x_1, x_2, x_3, x_4 be elements of the semigroup S^1 such that $h((\alpha_1, x_1, \beta_1))$, $h((\alpha_2, x_2, \beta_2)) \in A$ and $h((\alpha_3, x_3, \beta_3)), h((\alpha_4, x_4, \beta_4)) \in B$. If $\alpha_1 \neq \alpha_2$, then $\alpha_1 \neq \beta_3$ or $\alpha_2 \neq \beta_3$, and hence

$$h(0) = h((\alpha_3, x_3, \beta_3) \cdot (\alpha_1, x_1, \beta_1)) = h((\alpha_3, x_3, \beta_3)) \cdot h((\alpha_1, x_1, \beta_1)) \in B \cdot A,$$

or

$$h(0) = h((\alpha_3, x_3, \beta_3) \cdot (\alpha_2, x_2, \beta_2)) = h((\alpha_3, x_3, \beta_3)) \cdot h((\alpha_2, x_2, \beta_2)) \in B \cdot A.$$

If $\beta_1 \neq \beta_2$ then $\beta_1 \neq \alpha_3$ or $\beta_2 \neq \alpha_3$, and hence

$$h(0) = h((\alpha_1, x_1, \beta_1) \cdot (\alpha_3, x_3, \beta_3)) = h((\alpha_1, x_1, \beta_1)) \cdot h((\alpha_3, x_3, \beta_3)) \in A \cdot B,$$

or

$$h(0) = h((\alpha_2, x_2, \beta_2) \cdot (\alpha_3, x_3, \beta_3)) = h((\alpha_1, x_1, \beta_1)) \cdot h((\alpha_3, x_3, \beta_3)) \in A \cdot B.$$

Theorem 10. *Let $\lambda \geqslant 2$, $B_\lambda(S)$ and T be topological inverse semigroups, $h: B_\lambda(S) \to T$ be a continuous homomorphism such that the set $h(S_{\alpha\beta})$ be a closed in T for some $\alpha, \beta \in I_\lambda$. Then $h(B_\lambda(S))$ is a closed subsemigroup of T.*

Proof. In the case $2 \leqslant \lambda < \omega$ the statement of the lemma follows from Lemma 6.

Let $\lambda \geqslant \omega$. We denote $G = \mathrm{cl}_T(h(B_\lambda(S)))$. By Proposition II.2 [6], G is a topological inverse semigroup. Let $b \in G \backslash h(B_\lambda(S))$. Then by Lemma 8, $b, b^{-1} \in G \backslash E(G)$. We remark that $b \cdot b^{-1} \neq h(0)$ and $b^{-1} \cdot b \neq h(0)$. Suppose contrary: $b \cdot b^{-1} = h(0)$ or $b^{-1} \cdot b = h(0)$. Since $h(0)$ is the zero of G, then $b = b \cdot b^{-1} \cdot b = h(0) \cdot b = h(0)$ or $b^{-1} = b^{-1} \cdot b \cdot b^{-1} = h(0) \cdot b^{-1} = h(0)$, a contradiction with $b \in G \backslash h(B_\lambda(S))$.

Therefore there exist $e, f \in E(G) = E(h(B_\lambda(S)))$ such that $b \cdot b^{-1} = e$ and $b^{-1} \cdot b = f$. At first we consider the case $e \neq f$. Let $W(e) \not\ni h(0)$ and $W(f) \not\ni h(0)$ be disjunctive open neighbourhoods of e and f in T, respectively. Then there exist disjunctive open neighbourhoods $U(b) \not\ni h(0)$ and $U(b^{-1}) \not\ni h(0)$ in T such that $U(b) \cdot U(b^{-1}) \subseteq W(e)$ and $U(b^{-1}) \cdot U(b) \subseteq W(f)$. By Lemma 6 the set $h(S_{\alpha\beta})$ is closed in T for each $\alpha, \beta \in I_\lambda$, and hence the sets $U(b)$ and $U(b^{-1})$ intersect infinitely many sets of the type $h(S_{\gamma\delta})\setminus h(0)$ $(\gamma, \delta \in I_\lambda)$. thus by Proposition 9 we get $h(0) \in U(b) \cdot U(b^{-1}) \subseteq W(e)$ or $h(0) \in U(b^{-1}) \cdot U(b) \subseteq W(f)$, a contradiction with the choice of the neighbourhoods $W(e)$ and $W(f)$.

In the case $e = f$ we similarly obtain a contradiction.

The obtained contradictions imply the statement of the theorem.

The proof of the following proposition is trivial.

Proposition 11. *If S as absolutely H-closed topological semigroup (in the class of topological semigroups $\mathcal{S}$), then so is S^1 (if $S^1 \in \mathcal{S}$).*

Propositions 6 and 11, and Theorem 10 imply

Theorem 12. *For any cardinal $\lambda \geqslant 2$, every topological Brandt λ-extension $B_\lambda(S)$ of an absolutely H-closed topological inverse semigroup S in the class of topological inverse semigroups, is absolutely H-closed in the class of topological inverse semigroups.*

Corollary 13. *For any cardinal $\lambda \geqslant 2$, every topological Brandt λ-extension $B_\lambda(S)$ of a compact topological inverse semigroup S in the class of topological inverse semigroups, is absolutely H-closed in the class of topological inverse semigroups.*

Theorem 14. *Let S be a topological inverse semigroup. Then the following conditions are equivalent:*

(i) S is an absolutely H-closed semigroup in the class of topological inverse semigroups;

(ii) there exists a cardinal $\lambda \geqslant 2$ such that any topological Brandt λ-extension $B_\lambda(S)$ of the semigroup S is absolutely H-closed in the class of topological inverse semigroups;

(iii) for each cardinal $\lambda \geqslant 2$ any topological Brandt λ-extension $B_\lambda(S)$ of the semigroup S is absolutely H-closed in the class of topological inverse semigroups.

Proof. The implication (iii)⇒(ii) is trivial, and Theorem 12 implies the implications (i)⇒(ii) and (i)⇒(iii).

We shall show that the implication (ii)⇒(i) holds. Suppose contrary: there exists non absolutely H-closed topological inverse semigroup S in the class of topological inverse semigroups, and for some cardinal $\lambda_0 \geqslant 2$ every topological Brandt λ_0-extension $B_{\lambda_0}(S)$ is absolutely H-closed in the class of topological inverse semigroups. Then there exist a topological inverse semigroup T and a continuous homomorphism "into" $h: S \to T$ such that $h(S)$ is not closed subsemigroup of T.

Let τ_S and τ_T be direct sum topologies on $B_{\lambda_0}(S)$ and $B_{\lambda_0}(T)$, respectively (see: [8, p. 129]). Then $(B_{\lambda_0}(S), \tau_S)$ and $(B_{\lambda_0}(T), \tau_T)$ are topological inverse semigroups, S^1 and T^1 are homeomorphic to $S_{\alpha\beta}$ and $T_{\alpha\beta}$, for all $\alpha, \beta \in I_\lambda$ (see: [8, p. 129]). We define the map $\widetilde{h}: B_{\lambda_0}(S) \to B_{\lambda_0}(T)$ as follows: $\widetilde{h}(0) = 0$ and $\widetilde{h}((\alpha, s, \beta)) = (\alpha, h(s), \beta)$ for all $\alpha, \beta \in I_\lambda$, $s \in S^1$. Obviously, the homomorphism $\widetilde{h}: (B_{\lambda_0}(S), \tau_S) \to (B_{\lambda_0}(T), \tau_T)$ is continuous and $\widetilde{h}(B_{\lambda_0}(S))$ is not a closed subgroup of $(B_{\lambda_0}(T), \tau_T)$. Therefore, there exists a topological Brandt λ_0-extension $B_{\lambda_0}(S), \tau_S)$, which is not absolutely H-closed in the class of topological inverse semigroups.

The obtained contradiction implies the statement of the theorem,

The following example shows that there exists an absolutely H-closed topological Brandt λ-extension $B_\lambda(S)$ in the class of topological inverse semigroups of a topological inverse semigroup S, such that S is not absolutely H-closed in the class of topological inverse semigtoups.

Example 15. Obviously, $S = (\mathbb{N}, \max)$ with the discrete topology is a topological semigroup. We define a topology τ_B on $B_2(S)$ as follows:

a) (α, x, β) is an isolated point in $B_2(S)$ for all $\alpha, \beta = 1, 2$, $x \in S$;

b) the family $\mathcal{B}(0) = \{\{\{0\} \bigcup \{(\alpha, x, \beta) \mid \alpha, \beta = 1, 2, x \geqslant k\}\} \mid k \in \mathbb{N}\}$ is a base of the topology τ_B at the point $0 \in B_2(S)$.

It is easy to see that $(B_2(S), \tau_B)$ is a compact topological inverse semigroup, and hence it is absolutely H-closed. But S is not H-closed in the class of topological inverse semigroups.

Theorem 12 implies

Theorem 16. *For each cardinal $\lambda \geqslant 2$, every topological Brandt λ-extension $B_\lambda(S)$ of an algebraically h-closed inverse semigroup S in the class of topological inverse semigroups, is algebraically h-closed in the class of topological inverse semigroups.*

Theorem 14 implies

Theorem 17. *For an inverse semigroup S the following conditions are equivalent:*

(i) S is an algebraically h-closed semigroup in the class of topological inverse semigroups;

(ii) $B_\lambda(S)$ is algebraically h-closed in the class of topological inverse semigroups for some cardinal $\lambda \geqslant 2$;

(iii) $B_\lambda(S)$ is algebraically h-closed in the class of topological inverse semigroups for any cardinal $\lambda \geqslant 2$.

Since the band of a topological semigroup is a closed subset of it, then we have

Proposition 18. *If L is a subsemigroup of the band of a topological semigroup S, the so is $\mathrm{cl}_S(L)$.*

The closure of an Abelian subsemigroup of a topological semigroup is an Abelian semigroup [2, Vol. 1, pp. 9–10], then Proposition 18 implies

Corollary 19. *The closure of a topological semilattice in a topological semigroup is a semilattice.*

Therefore we get

Proposition 20. *A topological semilattice is H-closed if and only if it is H-closed in the class of topological semilattices.*

Since a homomorphic image of a semilattice is a semilattice, then Corollary 19 implies

Proposition 21. *A topological semilattice is absolutely H-closed if and only if it is absolutely H-closed in the class of topological semilattices.*

In [13] J. W. Stepp proved that a semilattice is algebraically h-closed if and only if any chain of it is finite.

Since a maximal subgroup of a topological inverse semigroup is a closed subset, then we have

Proposition 22. *A topological group is [absolutely] H-closed in the class of topological inverse semigroups if and only if it is [absolutely] H-closed in the class of topological groups.*

Absolutely H-closed topological groups in the theory of topological group are called *h-complete* [4]. Complete minimal topologically simple groups and locally compact totally minimal groups are h-complete [4]. There exist non-compact non-Abelian h-complete topological groups, but an h-complete Abelian topological group is compact [4, Example 3.8]. Every locally compact topological group is H-closed in the class of topological groups. Therefore in the class of topological groups the notions a compact group, an absolutely H-closed topological group, and an H-closed topological group, and hence in the class of topological inverse semigroups, are different. We also remark that there exists absolutely H-closed non-compact Abelian Clifford topological inverse semigroup, such as algebraically H-closed infinite semilattices [13].

The following example shows that there exists a Clifford topological inverse semigroup S with a compact band and finite maximal subgroups, such that S is not H-closed in the class of Clifford topological inverse semigroups.

Example 23. Let be $\mathcal{J} = \{0\} \bigcup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ with the usual topology, and operation "max". Then $(\mathcal{J}, \max)$ is a compact semilattice. Let $G = \{e, a\}$ be the two-elements group. Then $S = \mathcal{J} \times G$ with the product topology is a Clifford compact topological inverse semigroup. Obviously $T = S \setminus \{(0, e)\}$ is a subsemigroup of S, and T is not closed subset of S.

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Add to proof. The authors wish to thank the referee for the many important helpful suggestions.

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ТОПОЛОГІЧНІ λ-РОЗШИРЕННЯ БРАНДТА АБСОЛЮТНО H-ЗАМКНЕНИХ ТОПОЛОГІЧНИХ ІНВЕРСНИХ НАПІВГРУП

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Вивчено властивості гомоморфних образів λ-розширень Брандта алгебричних напівгруп. Доведено, що для кожного кардинала $\lambda \geqslant 2$ довільне топологічне λ-розширення Брандта абсолютно H-замкненої топологічно інверсної напівгрупи є абсолютно H-замкненим у класі топологічних інверсних напівгруп.

Ключові слова: топологічна напівгрупа, топологічна інверсна напівгрупа, λ-розширення Брандта, H-замкнена топологічна група, абсолютно H-замкнена топологічна напівгрупа, алгебрично h-замкнена напівгрупа, топологічна напівгратка.

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