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LATTICES OF SUBGROUPS OF FINITE GROUPS AND FORMATIONS

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The paper is devoted to the study of formations $\mathfrak{F}$, for which set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of all subgroups in any finite group. The review of the main results on formations with the given property obtained in Gomel algebraic school.

Key words: finite group, lattice of subgroups, subgroups functor, formation.

1. All groups considered are finite. Following Wielandt [1] we say that a subset $\mathfrak{L}$ of subgroups of a group G is a lattice if $A \cap B \in \mathfrak{L}$ and $< A, B > \in \mathfrak{L}$ for any A and B in $\mathfrak{L}$. By classical Wielandt theorem, the set of subnormal subgroups of G is a lattice. There are two generalizations of subnormality in the theory of formations. A formation is a class $\mathfrak{F}$ of groups which is closed under homomorphic images and is such that each group G has unique smallest normal subgroup $G^{\mathfrak{F}}$ (the $\mathfrak{F}$-residual of G) with factor group in $\mathfrak{F}$. Let $\mathfrak{F}$ be a non-empty formation. A subgroup H of G is called:

1) $\mathfrak{F}$-subnormal in G (Carter-Hawkes formation subnormality [2]) if either $H = G$ or there exists a chain

$$G = H_0 \supset H_1 \supset \ldots \supset H_n = H$$

such that H_i is a $\mathfrak{F}$-normal maximal subgroup in H_{i-1} for any $i \in 1, \ldots, n$;

2) $K\mathfrak{F}$-subnormal in G (Kegel formation subnormality [3]) if there exists a chain

$$G = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_n = H$$

such that for every $i \in 1, \ldots, n$ a subgroup H_i is either normal in H_{i-1} or $H_{i-1}^{\mathfrak{F}} \subseteq H_i$.

In 1978 L. A. Shemetkov ([4], problem 12; [5], problem 9.75) and O. Kegel [3] posed a problem of finding conditions under which the set of $\mathfrak{F}$-subnormal ($K\mathfrak{F}$-subnormal) subgroups of G is a lattice.

We will say that a formation $\mathfrak{F}$ has the lattice property (briefly, lattice formation) if the set of all $\mathfrak{F}$-subnormal subgroups is a lattice in every group.

2. Lattice formations. Saturated case. In 1992 at the conference of Byelorussian mathematicians (Grodno) S. F. Kamornikov, V. N. Semenchuk and A. F. Vasil'ev reported on the solution of problems of Kegel and Shemetkov for saturated formations (see [6]). The detailed article was published in the Kiev book dedicated to the memory of the algebraist S. N. Chernicov [7]. First of all, in [7] it was shown that Kegel's problem and Shemetkov's problem are equivalent. Remind that the formation $\mathfrak{F}$ is said to be saturated if the group G belongs to $\mathfrak{F}$ whenever the factor group $G/\Phi(G) \in \mathfrak{F}$.

2.1. Theorem ([7]). *Let $\mathfrak{F}$ be an S-closed (soluble S_n-closed) saturated formation. Then the following statements are equivalent:*

1) the set of all $K\mathfrak{F}$-subnormal subgroups is a lattice in every (soluble) group;

2) the set of all $\mathfrak{F}$-subnormal subgroups is a lattice in every (soluble) group.

Let $\{\mathfrak{F}_i : i \in I\}$ be a set of groups classes. Then $D_0(\cup_{i\in I}\mathfrak{F}_i)$ denotes a class of all groups G which is presented in the form $G = G_{i_1} \times \ldots \times G_{i_t}$ where $i_k \in I$ and $G_{i_k} \in \mathfrak{F}_{i_k}, k = 1, \ldots, t$. In this terminology we will formulate the results which is devoted to the Kegel-Shemetkov problem for saturated formations in the class of all (soluble) groups.

2.2. Theorem ([7]). *Let $\mathfrak{F}$ be an S-closed (soluble S_n-closed) saturated formation. Then the following statements are equivalent:*

1) the set of all $K\mathfrak{F}$-subnormal subgroups is a lattice in every group;

2) $\mathfrak{F}$ can be presented in the form $\mathfrak{F} = D_0(\cup_{i\in I}\mathfrak{S}_{\pi_i})$, where $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$ in I.

Recall that a class $\mathfrak{F}$ of groups is a Fitting class if $\mathfrak{F}$ is closed under taking normal subgroups and in every group G there is a unique normal subgroup that is maximal with respect to being in $\mathfrak{F}$; that subgroup, the $\mathfrak{F}$-radical, will be denoted by $G_{\mathfrak{F}}$.

2.3. Theorem ([7]). *Let $\mathfrak{F}$ be an S-closed saturated formation. Then the following statements are equivalent:*

1) the set of all $\mathfrak{F}$-subnormal subgroups is a lattice in every group;

2) $\mathfrak{F}$ can be presented in the form $\mathfrak{F} = D_0(\mathfrak{M} \cup \mathfrak{H})$, where $\mathfrak{M}$ and $\mathfrak{H}$ are S-closed local formations satisfying the following conditions:

a) $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \emptyset$;

b) $\mathfrak{H} = D_0(\cup_{i\in I}\mathfrak{S}_{\pi_i})$, where $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$ in I;

c) $\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M}$ is a Fitting class which is normal in $\mathfrak{M}\mathfrak{M}$;

d) every non-cyclic minimal non-$\mathfrak{M}$-group G has the following property: $G/\Phi(G)$ is monolithic, $Soc(G/\Phi(G)) = (G/\Phi(G))^{\mathfrak{M}}$ is non-abelian and $G/G^{\mathfrak{M}}\Phi(G)$ is a cyclic group of prime power order.

For the results in this direction also see [8].

3. Lattice formations. Nonsaturated case. The condition of saturation for a formation played an essential role in the proof of the results given above. In [9,10] we give another approach, different from the approach given in [7, 8], which allows to give a constructive description of soluble S-closed formations $\mathfrak{F}$ such that the set of all $\mathfrak{F}$-subnormal ($K\mathfrak{F}$-subnormal) subgroups is a lattice for every group.

3.1. Theorem ([9, 10]). *Let $\mathfrak{F}$ be a soluble S-closed formation. Then the following statements are equivalent:*

1) the set of all $K\mathfrak{F}$-subnormal subgroups in every group is a lattice;

2) the set of all $\mathfrak{F}$-subnormal subgroups in every group is a lattice;

3) there is a partition $\{\pi_i \mid i \in I\}$ *of* $\pi(\mathfrak{F})$ *such that* $\mathfrak{F} = D_0(\cup_{i\in I}\mathfrak{F}_{\pi_i})$, *where* $\mathfrak{F}_{\pi_i} = \mathfrak{F} \cap \mathfrak{S}_{\pi_i}$ *and* $\mathfrak{F}_{\pi_i} = \mathfrak{S}_{\pi_i}$ *if* $|\pi_i| > 1$.

A class of groups which is both a Fitting class and a formation is called a Fitting formation. There are Fitting formations which are neither subgroup-closed nor saturated (see [11]).

3.2. Theorem. *Let* $\mathfrak{F}$ *be a soluble Fitting formation. Then the following conditions are pairwise equivalent:*

1) the set of all $K\mathfrak{F}$*-subnormal subgroups in every (soluble) group is a lattice;*

2) the set of all $\mathfrak{F}$*-subnormal subgroups in every (soluble) group is a lattice;*

3) $\mathfrak{F}$ *can be presented in the form* $\mathfrak{F} = D_0(\cup_{i\in I}\mathfrak{S}_{\pi_i})$, *where* $\pi_i \cap \pi_j = \emptyset$ *for all* $i \neq j$ *in* I.

In order to prove our theorems, we need the following results about the formations with Shemetkov property. We say formation $\mathfrak{F}$ has the Shemetkov property if every minimal non-$\mathfrak{F}$-group is either a Schmidt group or a cyclic group of prime order.

3.3. Proposition ([12]). *Let* $\mathfrak{F}$ *be a soluble S-closed formation with the Shemetkov property. Then* $\mathfrak{F}$ *is saturated.*

3.4. Proposition ([13]). *A soluble S-closed saturated formation* $\mathfrak{F}$ *is a formation with the Shemetkov property if and only if* $\mathfrak{F} = LF(f)$ *and* f *satisfies the following conditions:*

1) $f(p) = \mathfrak{S}_{\pi(f(p))}$ *for each* $p \in \pi(\mathfrak{F})$;

2) $f(p) = \emptyset$ *for each* $p \notin \pi(\mathfrak{F})$;

3) $f(p) = \mathfrak{N}_p f(p)$ *for each prime* p.

4. Subgroups lattice functors. In spite of the completed results of [7, 8] the issue of the existence in finite groups of other natural lattices similar to the lattice of all subnormal subgroups is still under discussion. In [14] we introduce another (functor) approach to the development of Wielandt results.

Axiomatizing the main properties of subnormal subgroups (invariantness under homomorphism, transitivity, heredity in subgroups), we introduce the notion of the natural transitive lattice functor and describe all the lattices induced by such functors in finite soluble groups.

Let A, B be groups, $\phi : A \to B$ is an epimorphism, and let Ω and Σ be some systems of subgroups from A and B respectively. Further $\Omega^\phi = \{H^\phi | H \in \Omega\}$, and $\Sigma^{\phi^{-1}} = \{H^{\phi^{-1}} | H \in \Sigma\}$ is the full inverse image of all subgroups from Σ in A.

Let Θ be the map, which associates with every group G some non-empty system $\Theta(G)$ of its subgroups. It is reported in [15], that Θ is a group functor, if the condition of abstractness is

$$(\Theta(G))^\phi = \Theta(G^\phi)$$

for every isomorphism ϕ of every group G.

If H is a subgroup of group G, then we write $H \cap \Theta(G) = \{H \cap R | R \in \Theta(G)\}$.

Subgroup functor Θ will be called:

1) natural, if $(\Theta(A))^\phi \subseteq \Theta(B^\phi)$ and $(\Theta(B))^{\phi^{-1}} \subseteq \Theta(A)$ for any epimorphism $\phi : A \to B$, and also $H \cap \Theta(G) \subseteq \Theta(H)$ for any subgroup H of group G;

2) transitive, if $\Theta(H) \subseteq \Theta(G)$ for any subgroup $H \in \Theta(G)$;

3) lattice, if always from $H, K \in \Theta(G)$ follows, that $H \cap K \subseteq \Theta(G)$ and $< H, K > \subseteq \Theta(G)$.

The examples of natural transitive lattice functors (further called NTL-functors) are the functors, which in every finite group G take the set $S(G)$ of its all subgroups; the set $\{G\}$; the set $sn(G)$ of its all subnormal subgroups. Other examples of NTL-functors are given in [7].

There are posed the problem of finding all NTL-functors, given on groups from the given class of groups $\mathfrak{X}$.

The following theorem solves the problem for the case, when $\mathfrak{X}$ is the class of all soluble finite groups.

4.1. Теорема ([14, 15]). *Let Θ be a subgroup NTL-functor. Then*

1) class $\mathfrak{X}_\Theta = \{G | \Theta(G) = S(G)\}$ is an S-closed saturated formation;

2) there exists a partition $\{\pi_i | i \in I\}$ of set $\pi(\mathfrak{X}_\Theta)$, such that $\mathfrak{X}_\Theta = D_0(\cup_{i \in I} \mathfrak{S}_{\pi_i})$;

3) $\Theta(G) = sn_{\mathfrak{X}_\Theta}(G)$ for any group G.

5. Some characterizations and applications of lattice formations. From theorems 2 and 3 it follows that the saturated subgroup-closed lattice formations are Fitting formations, moreover a totally-saturated (primitive in soluble case) Fitting formations. These formations are generalizations of the class of all nilpotent groups in the sense that the groups in the lattice formation are the direct product of all Hall subgroups corresponding to pairwise disjoint sets of primes.

As applicatoins we will show that some well-known properties of the class of all nilpotent groups characterize lattice formations.

It is well-known that nilpotent radical $F(G)$ of a group G can be obtained as the join of the subnormal nilpotent subgroups of G.

5.1. Theorem ([7]). *Let $\mathfrak{F}$ be an S-closed saturated formation. Then the following statements are pairwise equivalent:*

1) $\mathfrak{F}$ is a Fitting class such that the $\mathfrak{F}$-radical $G_\mathfrak{F}$ of a group G is presented in the form

$G_\mathfrak{F} = < H \in \mathfrak{F} | H$ is $K\mathfrak{F}$-subnormal in $G >$;

2) if H and K are two $K\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of a group G, then $< H, K > \in \mathfrak{F}$;

3) $\mathfrak{F}$ is a lattice formation.

In [16] B. Amberg, B. Höfling and L. S. Kazarin studied subgroup-closed formations $\mathfrak{F}$ which are closed under taking groups, which are products of pairwise permutable $\mathfrak{F}$-subgroups. Earlier [17] R. Bryce and J. Cossey described subgroup-closed formations of soluble groups which are closed under taking products of normal $\mathfrak{F}$-subgroups. In this direction the next two theorems are obtained. We consider only soluble groups.

5.2. Theorem ([18]). *Let $\mathfrak{F}$ be a soluble S-closed saturated formation. Then the following statements are pairwise equivalent:*

1) if $G = AB$, where A and B are abnormal $\mathfrak{F}$-subgroups of G, then $G \in \mathfrak{F}$;

2) if the group G have abnormal $\mathfrak{F}$-subgroups A and B such that $(|G : A|, |G : B|) = 1$ than $G \in \mathfrak{F}$;

3) $\mathfrak{F}$ is a lattice formation.

5.3. Theorem ([19]). *Let $\mathfrak{F}$ be a soluble Fitting formation. Then the following statements are pairwise equivalent:*

1) if $G = AB$, where A and B are abnormal $\mathfrak{F}$-subgroups of G, then $G \in \mathfrak{F}$;

2) $\mathfrak{F}$ is S-closed saturated lattice formation.

Following [15, 20], will say that a function $\omega_{\mathfrak{F}} : G \to G^{\mathfrak{F}}$ is a Wielandt-Kegel operator if $< H, K >^{\mathfrak{F}} = < H^{\mathfrak{F}}, K^{\mathfrak{F}} >$ for any two $K\mathfrak{F}$-subnormal subgroups H and K of an arbitrary group G. S. F. Kamornikov investigated this operators in [15, 20–21].

5.4. Theorem ([15, 20]). *Let $\mathfrak{F}$ be an S-closed saturated lattice formation. Then $\omega_{\mathfrak{F}}$ is a Wielandt-Kegel operator.*

5.5. Theorem ([15, 20]). *Let $\mathfrak{F}$ be a soluble formation. Then $\omega_{\mathfrak{F}}$ is a Wielandt-Kegel operator if and only if $\mathfrak{F}$ is S-closed saturated lattice formation.*

In [22] V. S. Monakhov proved that $F(A) \cap F(B) \subseteq F(G)$ for every finite group $G = AB$ which is the product of two subgroups A and B. The result of Johnson [23] says that for every finite soluble group $G = AB$ and for every set of primes π, the maximal normal π-subgroups satisfy $O_\pi(A) \cap O_\pi(B) \subseteq O_\pi(G)$. B. Amberg and L. S. Kazarin in [24] showed that the result of Johnson cannot be extended to arbitrary finite groups and indicated conditions under which this problem has an affirmative solution.

Since classes of all nilpotent groups and all π-groups are Fitting classes it is natural to look for Fitting classes $\mathfrak{F}$ such that $A_{\mathfrak{F}} \cap B_{\mathfrak{F}} \subseteq G_{\mathfrak{F}}$ for every group $G = AB$.

5.6. Theorem ([25]). *For the universe of all soluble groups any two of the following statements about a Fitting formation $\mathfrak{F}$ are equivalent:*

1) if $G = AB$ then $A_{\mathfrak{F}} \cap B_{\mathfrak{F}} \subseteq G_{\mathfrak{F}}$;

2) if $G = AB$ and $A, B \in \mathfrak{F}$ then $A \cap B \subseteq G_{\mathfrak{F}}$;

3) $\mathfrak{F}$ is S-closed saturated lattice formation.

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ГРАТКИ ПІДГРУП СКІНЧЕННИХ ГРУП І ФОРМАЦІЇ

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Вивчено формації $\mathfrak{F}$, для яких множина всіх $\mathfrak{F}$-субнормальних ($\mathfrak{F}$-досяжних) підгруп утворює підгратку гратки всіх підгруп довільної скінченної групи. Розглянуто основні результати про формації з заданою властивістю, одержані в Гомельській алгебричній школі.

Ключові слова: скінченна група, гратка підгруп, підгруповий функтор, формація.

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