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ABSORBING SETS RELATED TO HAUSDORFF DIMENSION

Natalia MAZURENKO

Ivan Franko National University of Lviv, 1 Universitetska Str. 79000 Lviv, Ukraine

It is proved that the hyperspace of compact sets in the n-dimensional cube $\mathbb{I}^n$ of the Hausdorff dimension $> \alpha$, $0 < \alpha < n$, forms an $\mathcal{F}_\sigma$-absorber in the hyperspace $\exp(\mathbb{I}^n)$ homeomorphic to the Hilbert cube. Moreover, for arbitrary sequence (α_i), $0 < \alpha_1 < \alpha_2 < \ldots < n$, the sequence of hyperspaces of compact sets in $\mathbb{I}^n$ of the Hausdorff dimension $> \alpha_i$ forms an $\mathcal{F}_\sigma$-absorbing sequence in $\exp(\mathbb{I}^n)$.

Key words: hyperspace, Hausdorff dimension, Hilbert cube, absorbing system.

The classical result of West, Curtis, and Schori asserts that the hyperspace of any nondegenerate Peano continuum is homeomorphic to the Hilbert cube. This allows us to apply methods of infinite-dimensional topology to investigation of classes of sets with prescribed geometric properties.

In particular, in a series of papers [1],[2],[3],[5], the topology of the hyperspace of sets of given Lebesgue dimension (see also [1] for the case of cohomological dimension) is described. In this note we consider the case of the Hausdorff dimension.

PRELIMINARIES. A typical metric will be denoted by d. By $\mathrm{diam}(A)$ we denote the diameter of a subset A in a metric space. Given a cover $\mathcal{U}$ of a metric space, we define $\mathrm{mesh}(\mathcal{U})$ as $\sup\{\mathrm{diam}(U) | U \in \mathcal{U}\}$. For $x \in X$ and $\varepsilon > 0$ the set $O_\varepsilon(x) = \{y \in X | d(x,y) < \varepsilon\}$ is an *open ε-ball* centered at x.

By Q we denote the Hilbert cube, $Q = \prod_{i=1}^{\infty} [-1,1]_i$. The class of absolute neighborhood retracts is denoted by ANR. A closed subset A of $X \in \mathrm{ANR}$ is called a *Z-set* in X if for every continuous function $\varepsilon: X \longrightarrow (0,\infty)$ there exists a map $f: X \longrightarrow X \setminus A$ which is ε-close to the identity in the sense that $d(x, f(x)) < \varepsilon(x)$, for every $x \in X$. An embedding $g: Y \longrightarrow X$ is called a *Z-embedding* if its image $g(Y)$ is a Z-set in X. By $B(Q) = Q \setminus \prod_{i=1}^{\infty} (-1,1)_i$ we denote the ***pseudoboundary*** of Q.

HYPERSPACES. Let X be a metric space. The hyperspace of X is the space $\exp X$ of nonempty compact subsets of X endowed with the Vietoris topology. A base of this topology consists of the sets

$$\langle V_1, \ldots, V_n \rangle = \{A \in \exp X | A \subset \bigcup_{i=1}^{n} V_i \text{ and for every } i \in \{1,2,\ldots,n\}\ A \cap V_i \neq \emptyset\},$$

where $V_1, ..., V_n$ run over the topology of X. The Vietoris topology is generated by the Hausdorff metric d_H,

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\}.$$

For $n \in \mathbb{N}$, we denote by $\exp_n X$ the subspace of $\exp X$ consisting of sets of cardinality $\leqslant n$. Let $\exp_\omega X = \cup\{\exp_n X | n \in \mathbb{N}\}$.

HAUSDORFF DIMENSION. Let F be a subset of $\mathbb{R}^n$ for some n and s a non-negative number. For $\varepsilon > 0$ define

$$\mathcal{H}^s_\varepsilon(F) = \inf_{\mathcal{B}} \sum_{B \in \mathcal{B}} (\mathrm{diam} B)^s,$$

where the infimum is over all covers $\mathcal{B}$ of F with $\mathrm{mesh}(\mathcal{B}) < \varepsilon$.

Let $\mathcal{H}^s(F) = \lim\limits_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(F)$. There exists a unique number s_0, the *Hausdorff dimension* of F, such that $\mathcal{H}^s(F) = \infty$ whenever $0 \leqslant s < s_0$ and $\mathcal{H}^s(F) = 0$ whenever $s_0 < s < \infty$. We write $\dim_H F = s_0$.

Proposition. *For every $\alpha \geqslant 0$ the set $C_\alpha = \{A \in \exp \mathbb{R}^n \mid \dim_H(A) \leqslant \alpha\}$ is a G_δ-subset of* $\exp \mathbb{R}^n$.

Proof. For every $A \in C_\alpha$, by the definition of Hausdorff dimension, $\mathcal{H}^{\alpha+1/i}(A) = 0$, for every $i \in \mathbb{N}$. Therefore, for every $A \in C_\alpha$ there exist open sets $U_{m_1}, ..., U_{m_k}$, which are elements of a fixed countable base $\mathcal{U}$ of $\mathbb{R}^n$, such that

$$A \in \langle U_{m_1}, ..., U_{m_k} \rangle \text{ and } \sum_{j=1}^{k} (\mathrm{diam} U_{m_j})^{\alpha+1/i} < 1/i.$$

Let

$$V_i = \cup\{\langle U_{m_1}, ..., U_{m_k} \rangle \mid \sum_{j=1}^{k} (\mathrm{diam} U_{m_j})^{\alpha+1/i} < 1/i,\ U_{m_1}, ..., U_{m_k} \in \mathcal{U}\}.$$

We have just shown that $C_\alpha \subset \bigcap\limits_{i=1}^{\infty} V_i$. Prove the inclusion $\bigcap\limits_{i=1}^{\infty} V_i \subset C_\alpha$. Assuming the opposite, choose $B \in \exp \mathbb{R}^n$ such that $\dim_H B = s > \alpha$ and $B \in \bigcap\limits_{i=1}^{\infty} V_i$. Then there is $i_0 \in \mathbb{N}$ such that $\alpha + 1/i < s$ for every $i \in \mathbb{N}$, $i \geqslant i_0$.

We therefore have $\mathcal{H}^{\alpha+1/i}(B) = \infty$ for all $i \in \mathbb{N}$, $i \geqslant i_0$. Taking into account that $\mathcal{H}^{\alpha+1/i}_\varepsilon(B) > 0$, we conclude that $l(i) = \inf\{\mathcal{H}^{\alpha+1/i}_\varepsilon(B) | 0 < \varepsilon \leqslant 1\} > 0$.

The function $l(i)$ is an increasing function of i. Thus, there is $i_1 \in \mathbb{N}$ with $l(i_1) > 1/i_1$. Then obviously $B \notin V_{i_1} \supset \bigcap\limits_{i=1}^{\infty} V_i$ and we obtain a contradiction.

We have proven that $C_\alpha = \bigcap\limits_{i=1}^{\infty} V_i$. Since V_i are open in $\exp \mathbb{R}^n$, this completes the proof. □

ABSORBING SYSTEMS. We briefly recall some definitions from the theory of absorbing systems; see [3], [4] for details.

Let Γ be an ordered set and $\mathcal{M}_\gamma$ a class of metric spaces for $\gamma \in \Gamma$. Put $\mathcal{M}_\Gamma = (\mathcal{M}_\gamma)_{\gamma\in\Gamma}$. An $\mathcal{M}_\Gamma$-*system* in a space X is an order preserving indexed collection $(A_\gamma)_{\gamma\in\Gamma}$ of subsets of X such that $A_\gamma \in \mathcal{M}_\gamma$ for every γ.

An $\mathcal{M}_\Gamma$-system $\mathcal{X}$ in $X \in$ ANR is called *strongly* $\mathcal{M}_\Gamma$-*universal* in X if for every $\mathcal{M}_\Gamma$-system (A_γ) in Q, every map $f: Q \longrightarrow X$ that restricts to a Z-embedding on some compact subset K of Q can be approximated by a Z-embedding $g: Q \longrightarrow X$ such that $g|K = f|K$ and for every $\gamma \in \Gamma$ we have $g^{-1}(X_\gamma)\backslash K = A_\gamma \backslash K$.

An $\mathcal{M}_\Gamma$-system $\mathcal{X}$ is called $\mathcal{M}_\Gamma$-*absorbing* in X if the set $\bigcup_{\gamma\in\Gamma} X_\gamma$ is contained in a σ-compact σ-Z-set in X and $\mathcal{X}$ is strongly $\mathcal{M}_\Gamma$-universal in X.

By $\mathcal{F}_\sigma$ we denote the class of σ-compact spaces.

MAIN RESULT.

Lemma 1. *Let $n \in \mathbb{N}$. For every continuous function $f: Q \longrightarrow \exp(\mathbb{I}^n)$ that restricts to a Z-embedding on some compact subset K of Q and for every $\varepsilon > 0$ there is a Z-embedding $h: Q \longrightarrow \exp(\mathbb{I}^n)$ such that $h|K = f|K$, for every $x \in Q\backslash K$ $d_H(f(x), h(x)) < \varepsilon$ and $\dim_H(h(x)) = 0$.*

Proof. Consider a sequence of compact subsets $\{B_i\}_{i=1}^\infty$ in $\mathbb{I}^n$ defined as follows:

$$B_1 = \frac{1}{2} \cdot \mathbb{I}^n;$$

$$B_2 = \frac{1}{2^2} \cdot \mathbb{I}^n + \frac{1}{2} \cdot y_0;$$

$$\dots$$

$$B_k = \frac{1}{2^k} \cdot \mathbb{I}^n + \left(\frac{1}{2} + \dots + \frac{1}{2^{k-1}}\right) \cdot y_0,$$

$$\dots$$

where $y_0 = (1, 1, \dots, 1)$.

Let α_i be an embedding $[-1, 1]$ into B_i. For every $x \in Q$, $x = (x_i)_{i=1}^\infty$ let $\hat{x} \in Q$ be defined as follows:

$$\hat{x} = (x_1, x_1, x_2, x_1, x_2, x_3, x_1, x_2, x_3, x_4, \dots).$$

Let the map ξ be given by the formula

$$\xi(x) = \bigcup_{i=1}^{\infty} \alpha_i(\hat{x}_i) \cup \{y_0\}.$$

It is clear that for every $x \in Q$, $\xi(x)$ is a compact subset in $\mathbb{I}^n$. On the other hand, $\xi(x)$ is a countable subset of $\mathbb{I}^n$, therefore, $\dim_H(\xi(x)) = 0$.

Choose two points $x, x' \in Q$, $x = (x_i)_{i=1}^\infty$, $x' = (x'_i)_{i=1}^\infty$. If $x \neq x'$, then there is $i \in \mathbb{N}$ such that $x_i \neq x'_i$. In this case for some $j \in \mathbb{N}$, $\alpha_j(\hat{x}_j) \neq \alpha_j(\hat{x}'_j)$. Therefore, $\xi(x) \neq \xi(x')$. This implies that ξ is an injective map.

Let $\varepsilon > 0$. Let $f: Q \longrightarrow \exp(\mathbb{I}^n)$ be a map that restricts to a Z-embedding on some compact subset K of Q. Without loss of generality we may assume that f is a Z-embedding because $\exp(\mathbb{I}^n)$ is homeomorphic to the Hilbert cube (see [4]). Define $\mu: Q \longrightarrow [0, 1]$ by $\mu(x) = \frac{1}{3} \cdot \min\{\varepsilon, d_H(f(x), f[K])\}$. The set $\exp(\mathbb{I}^n)\backslash \exp_\omega(\mathbb{I}^n)$

is locally homotopy negligible in $\exp(\mathbb{I}^n)$(see [4]). Therefore, there is a homotopy $H\colon \exp(\mathbb{I}^n) \times \mathbb{I} \longrightarrow \exp(\mathbb{I}^n)$ such that

1) $H_0 = 1_{\exp(\mathbb{I}^n)}$;
2) for every $t \in (0,1]$, $H_t(\exp(\mathbb{I}^n)) \subseteq \exp_\omega(\mathbb{I}^n)$.

It is clear that we may additionally assume that

3) for every $t \in [0,1]$, $\hat{d}_H(H_t, 1_{\exp(\mathbb{I}^n)}) \leqslant 2t$;
4) for every $t \in (0,1]$, $H_t(\exp(\mathbb{I}^n)) \subseteq \exp_\omega([0,1-3t/4]^n)$.

For every $x \in Q$, let $F(x) = H(f(x), \mu(x))$. Then, if $\mu(x) > 0$, $F(x)$ is a finite approximation of $f(x)$.

Now define $h\colon Q \longrightarrow \exp(\mathbb{I}^n)$ as follows:

$$h(x) = F(x) \cup \bigcup_{y \in F(x)} [\mu(x)/4 \cdot \xi(x) + y].$$

CLAIM 1. The map h is well-defined, continuous and satisfies $h|K = f|K$. Moreover, for every $x \in Q$, $d_H(f(x), h(x)) \leqslant \frac{3}{4}\min\{\varepsilon, d(f(x), f[K])\}$ and for every $x \in Q\backslash K$, $\dim_H(h(x)) = 0$.

a) Let $x \in Q$. Then by (4), $F(x) \subseteq [0, 1 - 3\mu(x)/4]^n$. For every $y \in F(x)$, the diameter of the set $[\mu(x)/4 \cdot \xi(x) + y]$ does not exceed $\mu(x)/4$, which implies that $h(x) \subseteq [0, 1 - \mu(x)/2]^n$.

b) If $\mu(x) > 0$, then $h(x)$ is compact and non-empty, being a finite union of compact non-empty sets. If $\mu(x) = 0$, then $h(x) = f(x)$ which is also compact and non-empty. Therefore for every $x \in Q$, $h(x) \in \exp(\mathbb{I}^n)$.

c) That h is continuous follows from the continuity of the involved maps.

d) If $\mu(x) > 0$, then $h(x)$ is a finite union of countable sets. Therefore for every $x \in Q\backslash K$ $\dim_H(h(x)) = 0$.

e) Fix $x \in Q$. It is clear that $d_H(f(x), h(x)) \leqslant 2 \cdot \mu(x) + \mu(x)/4 = 9 \cdot \mu(x)/4$, from which it follows that $d_H(f(x), h(x)) \leqslant 3/4 \cdot \min\{\varepsilon, d_H(f(x), f[K])\}$. So we are done because this inequality implies that $h|K = f|K$.

CLAIM 2. The map h is injective.

Let us first observe that from Claim 1 and the fact that f is an embedding it follows that

$$h[Q\backslash K] \cap h[K] = \emptyset. \qquad (*)$$

Now fix $x, x' \in Q$. If both x and x' belong to K, then since $h|K = f|K$ and since f is an embedding, it is trivial that $h(x) = h(x')$ implies $x = x'$. If $x \notin K$ and $x' \in K$, then from (*) it follows that $h(x) \neq h(x')$. So without loss of generality we may assume that $x, x' \in Q\backslash K$.

Let $h(x) = h(x')$. Our task is to show that $x = x'$. We will first prove that $\mu(x) = \mu(x')$. Assume the contrary, e.g. assume $\mu(x) < \mu(x')$. For some $y \in F(x)$, consider in $\mathbb{I}^n$ the set $B_y = (\mu(x)/4) \cdot \mathbb{I}^n + y$. There exists a point $m \in h(x)$ such that $|m| \leqslant |p|$ for all $p \in h(x)$. Moreover, this point m is an element of $F(x) \cap F(x')$ (by construction of the map h and since $h(x) = h(x')$). For this m, we see that $B_m \cap h(x)$ is infinite while $B_m \cap h(x')$ is a finite set, being a finite union of finite sets. This contradiction establishes that $\mu(x) = \mu(x')$.

Again consider the point $\hat{m} = (m_1, \dots, m_n) \in h(x)$ such that $|p| \leqslant |\hat{m}|$ for every $p \in h(x)$. Since $\mu(x) = \mu(x')$, we have

$$m^* = (m_1 - \mu(x)/4, \dots, m_n - \mu(x)/4) \in F(x) \cap F(x').$$

Since $F(x)$ and $F(x')$ are finite, $\hat{m}$ is maximal, there are a neighborhood U of $\hat{m}$ and a $\delta \in (0, 1]$ such that

$$\begin{aligned} U \cap h(x) = m^* + \mu(x)/4(\xi(x) \cap O_\delta(y_0)) = \\ = m^* + \mu(x')/4(\xi(x') \cap O_\delta(y_0)). \end{aligned}$$

Since the coordinates of x appear infinitely often in the coordinates of $\hat{x}$, and the same is true for x', it now easily follows that $x = x'$.

CLAIM 3. The map h is a Z-embedding.

Since $h[K] = f[K]$ is a Z-set, it suffices to show that $h[Y]$ is a Z-set if $Y \subseteq Q \backslash K$ is compact. But this easily follows from the fact that the map $h': Q \longrightarrow \exp(\mathbb{I}^n)$ defined by

$$h'(x) = \bigcup_{y \in F(x)} \overline{O_\delta(y)} \cup [\mu(x)/4 \cdot \xi(x) + y]$$

maps Q into the complement of $h[Y]$, for every positive δ, and is δ-close to the identity. This completes the proof of the Lemma 1. □

Theorem 1. *If $n \geqslant 1$ and $\alpha \in (0, n)$, then the set $D_{>\alpha}(\mathbb{I}^n) = \{A \in \exp(\mathbb{I}^n) | \dim_H A > \alpha\}$ is strongly $\mathcal{F}_\sigma$-universal in* $\exp(\mathbb{I}^n)$.

Proof. Let $\varepsilon > 0$. Choose a sequence $A_1 \subseteq A_2 \subseteq \dots$ of compact subset in the Hilbert cube Q and let $A = \bigcup_{n=1}^\infty A_n$. Let $f: Q \longrightarrow \exp(\mathbb{I}^n)$ be a map that restricts to a Z-embedding on some compact subset K of Q. Let $\mu: Q \longrightarrow [0, 1]$, $H: \exp(\mathbb{I}^n) \times \mathbb{I} \longrightarrow \exp(\mathbb{I}^n)$, $F: Q \longrightarrow \exp(\mathbb{I}^n)$ be maps, as in the proof of Lemma 1.

For every $t \in [0, 1]$ let $\phi: \mathbb{I} \longrightarrow \exp(\mathbb{I}^n)$ be defined as follows: $\phi(t) = H_t(\mathbb{I}^n)$. Then, it is clear that $\phi(0) = \mathbb{I}^n$ and $\phi((0, 1]) \subseteq \exp_\omega(\mathbb{I}^n)$.

Let $\{B_i\}_{i=1}^\infty$ be a sequence of compact subsets of $\mathbb{I}^n$, as in the proof of Lemma 1, let $\beta_i: \mathbb{I}^n \longrightarrow B_i$ be a homeomorphism. For some $\lambda \in (0, 1]$ and $y \in \mathbb{I}^n$ define $(\beta_i)_y^\lambda = \lambda \beta_i + y + \lambda y_0$, where $y_0 = (1, 1, \dots, 1)$.

Let $h: Q \longrightarrow \exp(\mathbb{I}^n)$ be a map that satisfies the conditions of Lemma 1.

Now define $g: Q \longrightarrow \exp(\mathbb{I}^n)$ as follows

$$\begin{aligned} g(x) = h(x) \cup \bigcup_{y \in F(x)} \left[\bigcup_{i=1}^{\infty} (\beta_i)_y^{\mu(x)/4} (\phi(d(x, A_i))) \cup \{\mu(x)/2 \cdot y_0 + y\} \right] \\ \cup \{h(x) + \mu(x)/2 \cdot y_0\}. \end{aligned}$$

We claim that g is a required map, i.e., g is an approximation of f with the properties stated in the definition of strong $\mathcal{F}_\sigma$-universality.

CLAIM 1. The map g is well-defined, continuous, and satisfies $g|K = f|K$. Moreover, for every $x \in Q$, $d_H(f(x), g(x)) \leqslant \frac{11}{12} \min\{\varepsilon, d(f(x), f[K])\}$.

a) Let $x \in Q$. Then by Lemma 1, $h(x) \subseteq [0, 1-\mu(x)/2]^n$. For every $y \in F(x)$, the diameter of the set $\bigcup_{i=1}^{\infty} (\beta_i)_y^{\mu(x)/4}(\phi(d(x, A_i)))$ does not exceed $\mu(x)/4$, which implies that $g(x) \subseteq \mathbb{I}^n$.

b) If $\mu(x) > 0$, then $g(x)$ is compact and non-empty, being a finite union of compact non-empty sets. If $\mu(x) = 0$, then $g(x) = f(x)$ which is also compact and non-empty. Therefore for every $x \in Q$, $g(x) \in \exp(\mathbb{I}^n)$.

c) That g is continuous follows from the continuity of the involved maps.

d) Fix $x \in Q$. It is clear, by the proof of Lemma 1, that $d_H(f(x), g(x)) \leqslant 9/4 \cdot \mu(x) + \mu(x)/2 = 11 \cdot \mu(x)/4$, from which it follows that $d_H(f(x), g(x)) \leqslant 11/12 \cdot \min\{\varepsilon, d_H(f(x), f[K])\}$. So we are done because this inequality implies that $g|K = f|K$.

CLAIM 2. The map g is injective.

Injectivity of g follows from injectivity of h and construction of the map g.

CLAIM 3. We have $g^{-1}[D_{>\alpha}(\mathbb{I}^n)]\backslash K = A\backslash K$.

By analogy to the proof of Lemma 1, we first observe that from Claim 1 and the fact that f is an embedding it follows that

$$g[Q\backslash K] \cap g[K] = \emptyset. \qquad (*)$$

Choose $x \in Q\backslash K$. If $x \in A_k$ for certain k, then $d(x, A_k) = 0$. This implies that $\phi(d(x, A_k)) = \mathbb{I}^n$. In this case, we see that $g(x)$ contains the n-dimensional cube and this implies that $\dim_H(g(x)) \geqslant n$. Therefore, $g(x) \in D_{>\alpha}(\mathbb{I}^n)$.

If $x \notin A$, then $d(x, A_k) > 0$ for every $k \in \mathbb{N}$ and $\phi(d(x, A_k))$ is a finite set for all $k \in \mathbb{N}$. In this case, by construction, $g(x)$ is a countable set, being a countable union of finite sets. This implies that $\dim_H(g(x)) = 0$. Therefore, $g(x) \notin D_{>\alpha}(\mathbb{I}^n)$. Equality $(*)$ completes the proof of Claim 3.

CLAIM 4. The map g is a Z-embedding.

Follows from the same results for the map h.

This completes the proof of Theorem 1. □

Corollary. *In the assumptions of Theorem 1, the pair* $(\exp(\mathbb{I}^n), D_{>\alpha}(\mathbb{I}^n))$ *is homeomorphic to* $(Q, B(Q))$.

Proof follows from the standard results of the theory of absorbing sets in Q; see [4]. □

Theorem 2. *If* $n \geqslant 1$ *and* $\Gamma = \{\gamma_k\}_{k=1}^{\infty}$ *is a countable ordered set, where* $0 < \gamma_1 < \dots < \gamma_k < \dots < n$ *then the sequence* $\{D_{>\gamma_k}(\mathbb{I}^n)\}_{k=1}^{\infty}$ *is strongly* $\mathcal{F}_\sigma$*-universal in* $\exp(\mathbb{I}^n)$.

Proof. Let $\varepsilon > 0$. Choose a decreasing sequence of σ-compact subsets $\{\mathcal{A}_m\}_{m=1}^{\infty}$ in Q and a map $f: Q \longrightarrow \exp(\mathbb{I}^n)$ that restricts to a Z-embedding on some compact subset K of Q.

Write $\mathbb{N}$ as the disjoint union of infinitely many infinite sets, say, $N_1, N_2, \dots$. It is clear that for every $\gamma \in (0, n]$ there is a set $C \in \exp(\mathbb{I}^n)$ such that $\dim_H(C) = \gamma$. For $p \geqslant 1$ and $i \in N_p$, let $C_i \in \exp(\mathbb{I}^n)$ be a set such that $\dim_H(C_i) = \gamma_{p+1}$.

Since the set $\exp(\mathbb{I}^n)\setminus \exp_\omega(\mathbb{I}^n)$ is locally homotopy negligible in $\exp(\mathbb{I}^n)$(see [4]), we can find a continuous function $\phi_i\colon \mathbb{I} \longrightarrow \exp(\mathbb{I}^n)$ such that $\phi_i(0) = C_i$ and $\phi_i((0,1]) \subseteq \exp_\omega(\mathbb{I}^n)$.

Let $\mu\colon Q \longrightarrow [0,1]$, $F\colon Q \longrightarrow \exp(\mathbb{I}^n)$ be maps, as in the proof of Lemma 1.

For every $m \geqslant 1$ write $\mathcal{A}_m = \bigcup_{p=1}^{\infty} A_m^p$, where A_m^p are compact subsets of Q. Let $i(m,p)$ be the pth element of N_m.

Let $\{B_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of $\mathbb{I}^n$, as in the proof of Lemma 1, and $\beta_i\colon \mathbb{I}^n \longrightarrow B_i$ be a homeomorphism. For some $\lambda \in (0,1]$ and $y \in \mathbb{I}^n$ define $(\beta_i)_y^\lambda = \lambda\beta_i + y + \lambda y_0$, where $y_0 = (1,1,\ldots,1)$.

Let $h\colon Q \longrightarrow \exp(\mathbb{I}^n)$ be a map that satisfies the conditions of Lemma 1.

Now define $g\colon Q \longrightarrow \exp(\mathbb{I}^n)$ as follows:

$$g(x) = h(x) \cup \bigcup_{y\in F(x)} \left[\bigcup_{m=1}^{\infty} \bigcup_{p=1}^{\infty} (\beta_{i(m,p)})_y^{\mu(x)/4}(\phi_{i(m,p)}(d(x, A_m^p))) \cup \{\mu(x)/2 \cdot y_0 + y\} \right]$$
$$\cup \{h(x) + \mu(x)/2 \cdot y_0\}.$$

We claim that g is a required map, i.e., g is an approximation of f with the properties stated in the definition of strong $\mathcal{F}_\sigma$-universality.

CLAIM 1. The map g is well-defined, continuous and satisfies $g|K = f|K$. Moreover, for every $x \in Q$, $d_H(f(x), g(x)) \leqslant \frac{11}{12}\min\{\varepsilon, d(f(x), f[K])\}$.

a) Let $x \in Q$. Then by Lemma 1, $h(x) \subseteq [0, 1-\mu(x)/2]^n$. For every $y \in F(x)$, the diameter of the set $\bigcup_{m=1}^{\infty} \bigcup_{p=1}^{\infty} (\beta_{i(m,p)})_y^{\mu(x)/4}(\phi_{i(m,p)}(d(x, A_m^p)))$ does not exceed $\mu(x)/4$, which implies that $g(x) \subseteq \mathbb{I}^n$.

b) If $\mu(x) > 0$, then $g(x)$ is compact and non-empty, being a finite union of compact non-empty sets. If $\mu(x) = 0$, then $g(x) = f(x)$ which is also compact and non-empty. Therefore, for every $x \in Q$, $g(x) \in \exp(\mathbb{I}^n)$.

c) That g is continuous follows from the continuity of the involved maps.

d) Fix $x \in Q$. It is clear, by the proof of Lemma 1, that $d_H(f(x), g(x)) \leqslant 9/4 \cdot \mu(x) + \mu(x)/2 = 11 \cdot \mu(x)/4$, from which it follows that $d_H(f(x), g(x)) \leqslant 11/12 \cdot \min\{\varepsilon, d_H(f(x), f[K])\}$. So we are done, because this inequality implies that $g|K = f|K$.

CLAIM 2. The map g is injective.

Injectivity of g follows from injectivity of h and construction of the map g.

CLAIM 3. For every $k \in \mathbb{N}$ we have $g^{-1}[D_{>\gamma_k}(\mathbb{I}^n)]\setminus K = \mathcal{A}_k\setminus K$.

By analogy to the proof of Lemma 1, we first observe that from Claim 1 and the fact that f is an embedding it follows that

$$g[Q\setminus K] \cap g[K] = \emptyset. \qquad (*)$$

Choose $x \in Q\setminus K$. If $x \in \mathcal{A}_k\setminus\mathcal{A}_{k+1}$ for certain k, then $x \in A_k^p$ for some p. This implies that $d(x, A_k^p) = 0$ and $\phi_{i(k,p)}(d(x, A_k^p)) = C_{i(k,p)}$, where $\dim_H(C_{i(k,p)}) = \gamma_{k+1}$. Thus, $g(x)$ is a union of finitely many countable sets and countable union of the sets for which the Hausdorff dimension does not exceed γ_{k+1}. Therefore, $\dim_H(g(x)) = \gamma_{k+1}$. This implies that for $x \in \mathcal{A}_k \setminus \mathcal{A}_{k+1}$, $g(x) \in D_{>\gamma_k}(\mathbb{I}^n) \setminus D_{>\gamma_{k+1}}(\mathbb{I}^n)$.

If $x \notin \mathcal{A}_k$ for every $k \in \mathbb{N}$ then $g(x)$ is a countable set and therefore $\dim_H(g(x)) = 0$. Equality (*) completes the proof of Claim 3.

CLAIM 4. The map g is a Z-embedding.

Follows from the same results for the map h.

This completes the proof of Theorem 2. □

Corollary. *The pair* $(\exp(\mathbb{I}^n), D_{=n}(\mathbb{I}^n))$, *where* $D_{=n}(\mathbb{I}^n) = \{A \in \exp(\mathbb{I}^n) | \dim_H A = n\}$, *is homeomorphic to* $(Q^\omega, B(Q)^\omega)$.

Proof. Since $B(Q)^\omega$ is $\mathcal{F}_{\sigma\delta}$-absorbing set in Q^ω (see [4]) and we can write $D_{=n}(\mathbb{I}^n) = \bigcap_{i=1}^{\infty} D_{>n-1/i}(\mathbb{I}^n)$, this follows from Theorem 2. □

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ПОГЛИНАЮЧІ МНОЖИНИ ПОВ'ЯЗАНІ З ВИМІРОМ ГАУСДОРФА

Н. Мазуренко

Львівський національний університет імені Івана Франка,
вул. Університетська, 1 79000 Львів, Україна

Доведено, що гіперпростір компактних множин у n-вимірному кубі $\mathbb{I}^n$, вимір Гаусдорфа яких $> \alpha$, $0 < \alpha < n$, є $\mathcal{F}_\sigma$-поглинаючою множиною в гіперпросторі $\exp(\mathbb{I}^n)$. Крім того, для довільної послідовності (α_i), $0 < \alpha_1 < \alpha_2 < \ldots < n$, послідовність гіперпросторів компактних множин в $\mathbb{I}^n$, вимір Гаусдорфа яких $> \alpha_i$, є $\mathcal{F}_\sigma$-поглинаючою послідовністю в $\exp(\mathbb{I}^n)$.

Ключові слова: гіперпростір, вимір Гаусдорфа, Гільбертів куб, поглинаюча система.

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