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SOME PROPERTIES OF MINORS OF INVERTIBLE MATRICES

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The invariants of matrices which reduced a matrix over a commutative elementary divisor domain to canonical diagonal form is investigated. Some properties of minors of invertible matrices are considered.

Key words: commutative elementary divisor domain, transformable matrix, invertible matrix, minors, invariants of matrices.

Let R be a commutative elementary divisor domain [1]. Let A be a nonsingular $n\times n$ matrix with the canonical diagonal form $\Phi = \operatorname{diag}(\varphi_1, \dots, \varphi_n)$. Therefore, there exist invertible matrices P and Q such that $PAQ = \Phi$. The matrices P and Q are called the left and the right transformable matrices of the matrix A, respectively. By $\mathbf{P}_A$ we denote the set of left transformable matrices of the matrix A. It follows from the results of the papers [2, 3] that $\mathbf{P}_A = \mathbf{G}_\Phi P$, where $\mathbf{G}_\Phi$ is the group of invertible matrices of the form

$$\left\| \begin{matrix} h_{11} & h_{12} & \dots & h_{1n-1} & h_{1n} \\ \frac{\varphi_2}{\varphi_1}h_{21} & h_{22} & \dots & h_{2n-1} & h_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\varphi_n}{\varphi_1}h_{n1} & \frac{\varphi_n}{\varphi_2}h_{22} & \dots & \frac{\varphi_n}{\varphi_{n-1}}h_{nn-1} & h_{nn} \end{matrix} \right\|. \qquad (1)$$

It means that the set $\mathbf{P}_A$ is a left conjugate class of the complete linear group $GL_n(R)$ with respect to the subgroup $\mathbf{G}_\Phi$. In the papers [2, 3, 4, 5] it is proved that the set $\mathbf{P}_A$ plays the main role in the description of the divisors of matrices. The papers [2, 3, 6, 7] are dedicated to investigations of invariants and description of properties of the set $\mathbf{P}_A$. Our paper is connected to this topic. We also study properties of minor determinants of invertible matrices.

Let U be an $m \times n$ matrix over R, $m \leqslant n$. The matrix U is called *primitive* if the greatest common divisor (g.c.d.) of the minors of order m of the matrix U is equal to 1. We denote by $U_{i_1,\dots,i_k}$ the matrix consisting of $i_1, \dots, i_k$ columns of the matrix U, where $k \leqslant m$, $1 \leqslant i_1 < \dots < i_k \leqslant n$.

Lemma. *The g.c.d. of the minors of order m of the matrix U, which contain the matrix $U_{i_1,\dots,i_k}$ is equal to the g.c.d. of the minors of order k of the matrix $U_{i_1,\dots,i_k}$.*

Proof. If $k = m$ the lemma clearly holds. Suppose that $k < m$. A suitable permutation of the columns of U gives $\| S \quad U_{i_1,\dots,i_k} \| = UL$, where L is a suitable permutation matrix. There exists an invertible matrix K such that

$$KU_{i_1,\dots,i_k} = \left\| \begin{array}{ccc} u_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & u_k \\ \hline & \mathbf{0} & \end{array} \right\| = \left\| \begin{array}{c} U'_{i_1,\dots,i_k} \\ \mathbf{0} \end{array} \right\|,$$

where $u_1 \cdots u_k$ is g.c.d. of the minors of order k of the matrix $U_{i_1,\dots,i_k}$. Then

$$KUL = K \| S \quad U'_{i_1,\dots,i_k} \| = \left\| \begin{array}{cc} * & U'_{i_1,\dots,i_k} \\ T & \mathbf{0} \end{array} \right\|.$$

Every minor of order m of the matrix KUL which contains the matrix $\left\| \begin{array}{c} U'_{i_1,\dots,i_k} \\ \mathbf{0} \end{array} \right\|$ has the form

$$\det \left\| \begin{array}{cc} * & U'_{i_1,\dots,i_k} \\ T_{m-k} & \mathbf{0} \end{array} \right\|,$$

where T_{m-k} is an $(m-k) \times (m-k)$ submatrix of the matrix T. Therefore, g.c.d. of such minors is equal to $u_1 \dots u_k \tau$, where τ is the g.c.d. of the minors of order $m-1$ of the matrix T. Since the matrix KUL is primitive, it follows that the matrix T is primitive. Thus $\tau = 1$. Since any minor of the order m of the matrix U differs from respective minor of the matrix KUL by a unit multiplier of the ring R, the proof of our statement is complete. □

Let $U \in \mathbf{P}_A$. We denote by U^m the matrix consisting of the last m rows of the matrix U, $1 \leqslant m < n$, by $U^m_{i_1,\dots,i_k}$ the matrix consisting of $i_1, \dots, i_k$ columns of the matrix U^m, $1 \leqslant k \leqslant n$, $1 \leqslant i_1 < \dots < i_k \leqslant n$, and by $\delta^m_{i_1,\dots,i_k}$ the g.c.d. of the minors of order $\min\{m, k\}$ of the matrix $U^m_{i_1,\dots,i_k}$.

Theorem. *The elements $\left(\delta^m_{i_1,\dots,i_k}, \dfrac{\varphi_{n-m+1}}{\varphi_{n-m}} \right)$ are invariant with respect to the choice of transformable matrices from $\mathbf{P}_A$, for all indices $i_1, \dots, i_k$, $1 \leqslant i_1 < \dots < i_k \leqslant n$, $k = 1, \dots, n$, $m = 1, \dots, n-1$.*

Proof. Let $V \in \mathbf{P}_A$. By $\Delta^m_{i_1,\dots,i_k}$ we denote the g.c.d. of the minors of order $\min\{m, k\}$ of the matrix $V^m_{i_1,\dots,i_k}$. At first we consider the case $k = m$. Since $V = HU$, where H is an invertible matrix of form (1), it follows that

$$V^m_{i_1,\dots,i_m} = \left\| \begin{array}{cccccc} \frac{\varphi_s}{\varphi_1} h_{s\,1} & \dots & \frac{\varphi_s}{\varphi_{s-1}} h_{s\,s-1} & h_{s\,s} & \dots & h_{s\,n-1} & h_{s\,n} \\ \frac{\varphi_{s+1}}{\varphi_1} h_{s+1\,1} & \dots & \frac{\varphi_{s+1}}{\varphi_{s-1}} h_{s+1\,s-1} & \frac{\varphi_{s+1}}{\varphi_s} h_{s+1\,s} & \dots & h_{s+1\,n-1} & h_{s+1\,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\varphi_n}{\varphi_1} h_{n\,1} & \dots & \frac{\varphi_n}{\varphi_{s-1}} h_{n\,s-1} & \frac{\varphi_n}{\varphi_s} h_{n\,s} & \dots & \frac{\varphi_n}{\varphi_{n-1}} h_{n\,n-1} & h_{n\,n} \end{array} \right\| \times$$

$$\times \begin{Vmatrix} u_{1\,i_1} & \dots & u_{1\,i_m} \\ \dots & \dots & \dots \\ u_{n\,i_1} & \dots & u_{n\,i_m} \end{Vmatrix},$$

where $s = n - m + 1$. Since $\frac{\varphi_{s+i}}{\varphi_j}$ are divisible by $\frac{\varphi_s}{\varphi_{s-1}}$ for any $i = 1, \dots, n-s$, $j = 1, \dots, s-1$, $s + i > j$, all minors of order m which are built on last m rows of the matrix H, except the minor

$$\begin{vmatrix} h_{s\,s} & \dots & h_{s\,n-1} & h_{s\,n} \\ \frac{\varphi_{s+1}}{\varphi_s} h_{s+1\,s} & \dots & h_{s+1\,n-1} & h_{s+1\,n} \\ \dots & \dots & \dots & \dots \\ \frac{\varphi_n}{\varphi_s} h_{n\,s} & \dots & \frac{\varphi_n}{\varphi_{n-1}} h_{n\,n-1} & h_{n\,n} \end{vmatrix} = |H_s|$$

are divisible by $\frac{\varphi_s}{\varphi_{s-1}}$. Since H is an invertible matrix, we have $\left(|H_s|, \frac{\varphi_s}{\varphi_{s-1}}\right) = 1$. By the Cauchy-Binet formula and from what has been said it follows that

$$|V^m_{i_1,\dots,i_m}| = \frac{\varphi_s}{\varphi_{s-1}} d + |H_s| \left|U^m_{i_1,\dots,i_m}\right|,$$

where

$$|U^m_{i_1,\dots,i_m}| = \begin{vmatrix} u_{1\,i_1} & \dots & u_{1\,i_m} \\ \dots & \dots & \dots \\ u_{n\,i_1} & \dots & u_{n\,i_m} \end{vmatrix}.$$

Then

$$\left(\frac{\varphi_s}{\varphi_{s-1}}, |V^m_{i_1,\dots,i_m}|\right) = \left(\frac{\varphi_s}{\varphi_{s-1}}, |H_s| \left|U^m_{i_1,\dots,i_m}\right|\right) = \left(\frac{\varphi_s}{\varphi_{s-1}}, \left|U^m_{i_1,\dots,i_m}\right|\right).$$

We remark that $\delta^m_{i_1,\dots,i_m} = |U^m_{i_1,\dots,i_m}|$ and $\Delta^m_{i_1,\dots,i_m} = |V^m_{i_1,\dots,i_m}|$, hense the result holds for $k = m$.

Let $m < k \leqslant n$. We choose a minor μ of order m of the matrix $U^m_{i_1,\dots,i_k}$. The corresponding minor of the matrix $V^m_{i_1,\dots,i_k}$ is denoted by ν. The first case implies that

$$\left(\mu, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\nu, \frac{\varphi_s}{\varphi_{s-1}}\right).$$

Since $\delta^m_{i_1,\dots,i_k}$ is g.c.d. of the minors of order m of the matrix $U^m_{i_1,\dots,i_k}$, we have

$$\left(\delta^m_{i_1,\dots,i_k}, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\Delta^m_{i_1,\dots,i_k}, \frac{\varphi_s}{\varphi_{s-1}}\right).$$

Finally we consider the case $1 \leqslant k < m$. Let $\mu_1, \dots, \mu_t$ be minors of order m of the matrix U^m which contains the matrix $U^m_{i_1,\dots,i_k}$. By $\nu_1, \dots, \nu_t$ we denote the respective minors of the matrix V^m. By Lemma we get

$$(\mu_1, \dots, \mu_t) = \delta^m_{i_1,\dots,i_k}, \ (\nu_1, \dots, \nu_t) = \Delta^m_{i_1,\dots,i_k}.$$

Thus,

$$\left(\delta^m_{i_1,\dots,i_k}, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\mu_1, \dots, \mu_t, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\left(\mu_1, \frac{\varphi_s}{\varphi_{s-1}}\right), \dots, \left(\mu_t, \frac{\varphi_s}{\varphi_{s-1}}\right)\right) =$$

$$= \left(\left(\nu_1, \frac{\varphi_s}{\varphi_{s-1}}\right), \ldots, \left(\nu_t, \frac{\varphi_s}{\varphi_{s-1}}\right)\right) = \left(\nu_1, \ldots, \nu_t, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\Delta^m_{i_1,\ldots,i_k}, \frac{\varphi_s}{\varphi_{s-1}}\right).$$

The proof of the Theorem is complete. □

Proposition 1. *Let A be an $m \times s$ matrix with the canonical diagonal form* $\operatorname{diag}(\underbrace{1,\ldots,1}_{k}, \alpha_{k+1}, \ldots, \alpha_s)$, *where $\alpha_{k+1} \notin U(R)$, $m \geqslant s$. Then the matrix A is a submatrix of some invertible $n \times n$ matrix if and only if $n - m \geqslant s - k$.*

Proof. **Necessity.** Let A be a submatrix of an invertible $n \times n$ matrix U. We may assume, without loss of generality, that

$$U = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix},$$

where B, C, D are matrices of respective sizes. Let P and Q be transformable matrices of A, i.e.

$$PAQ = \begin{Vmatrix} \begin{matrix} E_k & 0 \\ 0 & S \end{matrix} \\ \hline 0 \end{Vmatrix} = \Phi,$$

where E_k is the identity $k \times k$ matrix, $S = \operatorname{diag}(\alpha_{k+1}, \ldots, \alpha_s)$. Then

$$\begin{Vmatrix} P & 0 \\ 0 & E_{n-m} \end{Vmatrix} U \begin{Vmatrix} Q & 0 \\ 0 & E_{n-s} \end{Vmatrix} = \begin{Vmatrix} \Phi & PB \\ CQ & D \end{Vmatrix}.$$

Put F be an $(m-k) \times n$ matrix consisting of $k+1, k+2, \ldots, m$ columns of the matrix $\begin{Vmatrix} \Phi & PB \\ CQ & D \end{Vmatrix}$. Therefore

$$F = \begin{Vmatrix} 0 \\ S \\ 0 \\ C_k \end{Vmatrix},$$

where C_k is a submatrix of C. Since $\alpha_{k+1} | \alpha_j$ for $j = k+1, \ldots, s$, we see that α_{k+1} divides all minors of order $s-k$ of the matrix F, which contain at least one of first m rows of F. The matrix F is primitive, therefore there exists a minor of order $s-k$ which does not contain the first m rows of F. It means that the $(n-m) \times (s-k)$ matrix C_k contains at least $s-k$ rows, i.e. $n - m \geqslant s - k$.

Sufficiency. Let

$$U = \begin{Vmatrix} P^{-1} & 0 \\ 0 & E_{s-k} \end{Vmatrix} \begin{Vmatrix} E_k & 0 & 0 \\ 0 & S & E_{m-k} \\ 0 & 0 & \\ 0 & E_{s-k} & 0 \end{Vmatrix} \begin{Vmatrix} Q^{-1} & 0 \\ 0 & E_{m-k} \end{Vmatrix}.$$

The matrix U is an example of a desired invertible matrix of order $n = m + s - k$. □

Corollary. *Let U be an invertible $n \times n$ matrix, V be a square submatrix of U and* $\operatorname{diag}(\underbrace{1,\ldots,1}_{k}, \alpha_{k+1}, \ldots, \alpha_{n-t})$ *be the canonical diagonal form of V. Then $k \leqslant t$.* □

Proposition 2. *Let be $U = \|u_{ij}\|_1^n$ and $U^{-1} = V = \|v_{ij}\|_1^n$. Then g.c.d. of the minors of maximal order of matrices*

$$U' = \begin{Vmatrix} u_{i_1k_1} & u_{i_1k_2} & \dots & u_{i_1k_p} \\ u_{i_2k_1} & u_{i_2k_2} & \dots & u_{i_2k_p} \\ \dots & \dots & \dots & \dots \\ u_{i_sk_1} & u_{i_sk_2} & \dots & u_{i_sk_p} \end{Vmatrix} \text{ and } V' = \begin{Vmatrix} v_{k'_1i'_1} & v_{k'_1i'_2} & \dots & v_{k'_1i'_{n-s}} \\ v_{k'_2i'_1} & v_{k'_2i'_2} & \dots & v_{k'_2i'_{n-s}} \\ \dots & \dots & \dots & \dots \\ v_{k'_{n-p}i'_1} & v_{k'_{n-p}i'_2} & \dots & v_{k'_{n-p}i'_{n-s}} \end{Vmatrix},$$

coincides, where $i_1 < i_2 < \ldots < i_s$ together with $i'_1 < i'_2 < \ldots < i'_{n-s}$ and $k_1 < k_2 < \ldots < k_p$ together with $k'_1 < k'_2 < \ldots < k'_{n-p}$ form the complete system of indices $1, 2, \ldots, n$.

Proof. Suppose that $s \geqslant p$. It is well known that the minors $U\begin{pmatrix} i_1 \ldots i_p \\ k_1 \ldots k_p \end{pmatrix}$ and $V\begin{pmatrix} k'_1 \ldots k'_{n-p} \\ i'_1 \ldots i'_{n-p} \end{pmatrix}$ differ from each other by a unit multiplier of R (see: [6, Part 1, §4]). All the minors $V\begin{pmatrix} k'_1 \ldots k'_{n-p} \\ i'_1 \ldots i'_{n-p} \end{pmatrix}$ can be characterized as all minors that contain the submatrix V'. Then Lemma implies our statement. The case $s < p$ is similar. □

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ДЕЯКІ ВЛАСТИВОСТІ МІНОРІВ ОБОРОТНИХ МАТРИЦЬ

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Досліджено інваріанти тих матриць, які зводять матрицю над комутативною областю елементарних дільників до її канонічної діагональної форми. Зазначено деякі властивості мінорів оборотних матриць.

Ключові слова: комутативна область елементарних дільників, перетворювальні матриці, оборотні матриці, мінори, інваріанти матриць.

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