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INVARIANT HYPERCOMPLEX STRUCTURES

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G-invariant Kähler structures (J_1, Ω) on the cotangent bundles $T^*(G/K)$ (symplectic manifolds with the canonical 2-form Ω) of Hermitian symmetric spaces with the standard antiholomorphic involution are considered. For arbitrary such a structure (J_1, Ω) a hypercomplex manifold $(T^*(G/K), \{J_1, J_2\})$ is constructed.

Key words: invariant hypercomplex structures.

1. A hypercomplex manifold $(X, \{J_1, J_2\})$ is a pair consisting of a $4n$-dimensional manifold X together with two anticommuting complex structures J_1, J_2. It then follows that X has a family of complex structures $J_\lambda = \lambda_1 J_1 + \lambda_2 J_2 + \lambda_3 J_3$, $J_3 = J_1 J_2$, parametrized by points $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ in the unit sphere $S^2 \subset \mathbb{R}^3$.

This article concerns the construction of hypercomplex structures on the cotangent bundle of Hermitian symmetric spaces. Non-compact homogeneous manifolds carrying such a structure were considered by Barberis and Miatello in [1], the case of compact homogeneous manifolds was considered by Joyce in [2].

Let $M = G/K$ be a Hermitian symmetric space and $\sigma : TM \to TM$ the involution which maps any tangent vector Y at $m \in M$ onto $-Y$ at m. Let Ω be the canonical symplectic structure on TM (the standard G-invariant metric $\mathrm{g_M}$ on M identifies the cotangent bundle T^*M and the tangent bundle TM). The main purpose of this note is to construct the following rich family of examples: let $\mathfrak{P}$ be a set of all G-invariant Kähler structures on some tube $T^sM = \{v \in TM : \mathbf{g}(v, v) < s^2\}$ (with Ω as the Kähler form) such that the mapping σ is an antiholomorphic involution. We prove here that for any $J_1 \in \mathfrak{P}$ there is a complex structure J_2 on T^sM for which $(T^sM, \{J_1, J_2\})$ is a hypercomplex manifold. The proof is simple because it is based on a Lie algebraic method of description of the elements $J_1 \in \mathfrak{P}$ [4] (usually the tensors J_1 are described in terms of geometric structures associated with the metric $\mathrm{g_M}$ on M [5,6]). Remark that the set $\mathfrak{P}$ is non-empty because it contains the adapted complex structure [3]; for all rank-one symmetric spaces this set $\mathfrak{P}$ is described in [4]. Moreover, the obtained set of hypercomplex structures on TM contains the hyper-Kähler structure constructed in [5,6].

2. Anticommuting structures. We recall some facts on Kähler and hypercomplex structures (see for example [4]). Let X be a (real) manifold with a symplectic form Ω and J be an almost complex structure on X. J is a complex structure if the complex subbundle F of $(0,1)$-vectors of J is an involutive subbundle of the complexified tangent bundle $T^{\mathbb{C}}X$. By definition, for any $x \in X$ we have $F(x) = \{Y + iJ_x(Y), Y \in T_xX\}$ $(J_x^2 = -1)$. We say that a complex structure J is a Kähler structure with the Käler form Ω if 1) $\Omega_x(J_x(Y_1), J_x(Y_2)) = \Omega_x(Y_1, Y_2)$ for any $Y_1, Y_2 \in T_xX$; 2) the quadratic form $B_x(Y_1, Y_2) = \Omega_x(J_xY_1, Y_2)$ is symmetric and positive-definite. Such a Kähler structure J will be denoted by (J, F, Ω).

A pair (J_1, J_2) formed by two anticommuting complex structures J_1, J_2 is a hypercomplex structure on X. Then $J_3 = J_1J_2$ is also a complex structure on X (for a proof see [7]).

3. G-invariant complex structures. Let $M = G/K$ be a symmetric space with a real reductive connected Lie group G and a compact connected subgroup K. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the groups G and K respectively,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m} \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \tag{1}$$

Suppose that there is a nondegenerate $\operatorname{Ad} G$-invariant bilinear form $\langle , \rangle$ on $\mathfrak{g}$ such that its restriction $\langle , \rangle|\mathfrak{m}$ is a positive definite form and $\mathfrak{k} \perp \mathfrak{m}$. This form defines the G-invariant Riemannian metric $\mathbf{g_M}$ on $M = G/K$. The metric $\mathbf{g_M}$ identifies the cotangent bundle T^*M and the tangent bundle TM and thus we can also talk about the canonical symplectic 2-form Ω on TM. This form Ω is G-invariant with respect to the natural action of G on TM.

Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\operatorname{Ad} K$-invariant (orthogonal) splitting of $\mathfrak{g}$, we can consider the trivial vector bundle $G \times \mathfrak{m}$ with the two Lie group actions (which commute) on it: the left G-action, $l_h : (g, w) \mapsto (hg, w)$ and the right K-action $r_k : (g, w) \mapsto (gk, \operatorname{Ad}_{k^{-1}} w)$. Let $\pi : G \times \mathfrak{m} \to G \times_K \mathfrak{m}$ be the natural projection. It is well known that $G \times_K \mathfrak{m}$ and TM are isomorphic. Using the corresponding G-equivariant diffeomorphism $\phi : G \times_K \mathfrak{m} \to TM$, $[(g, w)] \mapsto \frac{d}{dt}\Big|_0 g \exp(tw)K$ and the projection π define the G-equivariant submersion $\Pi : G \times \mathfrak{m} \to TM$, $\Pi = \phi \circ \pi$. Let ξ^l be the left-invariant vector field on the Lie group G defined by a vector $\xi \in \mathfrak{g}$. Since Ω is a symplectic form, the kernel $\mathcal{K} \subset T(G \times \mathfrak{m})$ of the 2-form $\tilde{\Omega} = \Pi^*\Omega$ is the kernel of Π_*, i.e. is generated by the global (left) G-invariant vector fields ζ^L, $\zeta \in \mathfrak{k}$ on $G \times \mathfrak{m}$, $\zeta^L(g, w) = (\zeta^l(g), [w, \zeta])$.

For given s, $0 < s \leqslant \infty$ consider the tube $T^sM \overset{\text{def}}{=} \{v \in TM \text{ of length } < s\}$. Put $W^s \overset{\text{def}}{=} \{w \in \mathfrak{m} : |w| < s\}$, where $|w| \overset{\text{def}}{=} \sqrt{\langle w, w\rangle}$. We will say that a smooth mapping $P : W^s \to GL(\mathfrak{m})$, $w \mapsto P_w$ is *K-equivariant* if

$$\operatorname{Ad}_k \circ P_w \circ \operatorname{Ad}_{k^{-1}} = P_{\operatorname{Ad}_k w} \quad \text{on } \mathfrak{m} \quad \text{for all} \quad w \in W^s, k \in K. \tag{2}$$

This mapping determines a complex (left) G-invariant subbundle $\mathcal{F}(P) \subset T^{\mathbb{C}}(G \times W^s)$ generated by nowhere vanishing on $G \times W^s$ (left) G-invariant vector fields ξ^L, $\xi \in \mathfrak{m}$ and $\zeta^L \in \Gamma\mathcal{K}$, $\zeta \in \mathfrak{k}$, where

$$\xi^L(g, w) = (\xi^l(g), iP_w(\xi)). \tag{3}$$

The subbundle $\mathcal{F}(P)$ is (right) K-invariant because the mapping P is K-equivariant. Therefore $F(P) \overset{\text{def}}{=} \Pi_*(\mathcal{F}(P))$ is a well-defined (smooth) complex subbundle of the complexified tangent bundle $T^{\mathbb{C}}(T^sM)$ $(\mathcal{K}^{\mathbb{C}} \subset \mathcal{F}(P))$.

Consider two (left) G-invariant and (right) K-invariant subbundles $\mathcal{T}_h$, $\mathcal{T}_v$ of the tangent bundle $T(G \times \mathfrak{m})$ given by

$$\mathcal{T}_h(g,w) = \{(\xi^l(g), 0),\ \xi \in \mathfrak{m}\}, \quad \mathcal{T}(g,w) = \{(0,u),\ u \in \mathfrak{m} = T_w\mathfrak{m}\}.$$

Put $\mathcal{T} = \mathcal{T}_h \oplus \mathcal{T}_v$. Since $T(G \times \mathfrak{m}) = \mathcal{K} \oplus \mathcal{T}$, the map $\Pi_*|\mathcal{T}_{(g,w)}$ is an isomorphism of the spaces $\mathcal{T}_{(g,w)}$ and $T_{\Pi(g,w)}(TM)$, in particular, by (right) K-invariance of $\mathcal{T}_h$ and $\mathcal{T}_v$ the images $\Pi_*(\mathcal{T}_h)$ and $\Pi_*(\mathcal{T}_v)$ are well-defined subbundles. But the natural projection $p : G \to G/K$ is a locally trivial fiber bundle so that for any $g \in G$ there is a (regular) submanifold $D \subset G$ such that the restriction $p : D \to G/K$ is an embedding. Since $\Pi = \phi \circ \pi$, the restriction $\Pi : D \times W^s \to TM$ is also an embedding. Denote by U_D the image $\Pi(D \times W^s)$. Now using the splitting

$$TU_D = \Pi_*(\mathcal{T}_h|D \times W^s) \oplus \Pi_*(\mathcal{T}_v|D \times W^s), \tag{4}$$

we obtain that the subbundle $F(P)|U_D$ is a subbundle of $(0,1)$-vectors of the almost complex tensor $J(P)|U_D : TU_D \to TU_D$, where $J_{\Pi(g,w)}(P) = \begin{pmatrix} 0 & -P_w^{-1} \\ P_w & 0 \end{pmatrix}$ $(J^2(P) = -1)$. The tensor field $J(P)$ on $T(T^sM)$ is smooth because $F(P)$ is a well-defined subbundle. $J(P)$ defines a complex structure if the subbundle $F(P)$ is involutive.

Fix base $\{W_b\}$ in $\mathfrak{m}$. Let $\{w_b\}$ be the coordinates in $\mathfrak{m}$ with respect to the basis $\{W_b\}$. For any vector-function $\tau : W^s \to \mathfrak{m}$, $\tau(w) = \sum_b \tau_b(w) W_b$ by $\overrightarrow{\tau}$ we denote the vector field $\overrightarrow{\tau} \overset{\text{def}}{=} \sum_b \tau_b \frac{\partial}{\partial w_b}$. Let $P(\xi)$, where $\xi \in \mathfrak{m}$, denote the vector-function $P(\xi) : w \mapsto P_w(\xi)$.

3.1. Proposition. [4] *Suppose that $M = G/K$ is a Riemannian symmetric space. Let $\mathcal{F}(P)$ be a complex subbundle of $T^{\mathbb{C}}(G \times W^s)$ defined by a K-equivariant mapping $P : W^s \to GL(\mathfrak{m})$. Then*
1) the subbundle $F(P) = \Pi_*(\mathcal{F}(P))$ is involutive on T^sM if and only if the Lie bracket identities $\left[\overrightarrow{P(\xi)}, \overrightarrow{P(\eta)}\right](w) = -[w, [\xi, \eta]]$ hold on W^s for all (fixed) $\xi, \eta \in \mathfrak{m}$;
2) the complex structure $J(P)$ such that $\sigma_*(F(P)) = \overline{F(P)}$ is a Kähler structure with the Kähler form Ω if and only if P_w is a symmetric positive-definite operator for each $w \in W^s$ (with respect to the bilinear form $\langle , \rangle$ on $\mathfrak{m}$).

For any G-invariant Kähler structure (J, F, Ω) on T^sM for which $\sigma_*(F) = \overline{F}$ there exists a unique K-equivariant mapping $P : W^s \to GL(\mathfrak{m})$ such that $J = J(P)$ and $F = F(P)$.

4. Examples: adapted complex structures. Let J^A be a (smooth) complex structure on some tube T^sM. The complex structure J^A on T^sM is called *adapted* [3,8] if for every geodesic γ in M a map $\hat{\gamma} : \mathbb{C} \to T(G/K)$, $(x + iy) \mapsto y\dot{\gamma}(x)$ is holomorphic on $\hat{\gamma}^{-1}(T^sM)$. Since the Riemannian manifold M is complete, an adapted complex structure on T^sM is unique (if it exists) [8]. Since the Riemannian manifold (M, g_M) is real-analytic and is also a symmetric space, on some tube T^sM there exists a real-analytic adapted structure J^A [8]. If the Lie group G is compact, by Corollary 21.1 of [9] (see also [5]) $F^A = F(P^A)$, where

$$P^A : W^s \to \mathrm{GL}(\mathfrak{m}),\ w \mapsto P^A_w,\ P^A_w = \left.\frac{\mathrm{ad}_w \cos \mathrm{ad}_w}{\sin \mathrm{ad}_w}\right|_{\mathfrak{m}}, \quad w \in \mathfrak{m}^s,\ s = \infty. \tag{5}$$

In this case the adapted structure F^A is defined on the whole tangent bundle TM and (J^A, F^A, Ω) is a Kähler structure.

If G is a noncompact semisimple Lie group and if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition of its Lie algebra $\mathfrak{g}$ then the (real) Lie algebra $\mathfrak{k} \oplus i\mathfrak{m} \subset \mathfrak{g}^{\mathbb{C}}$ is compact. Now it follows easily from Proposition 3.1 that formula (5) defines a Kähler structure $(J(P^A), F(P^A), \Omega)$ on some tube T^sM, $0 < s < \infty$. For this s eigenvalues of symmetric operators $\mathrm{ad}_w^2|\mathfrak{m}$, $w \in W^s$ are positive.

5. Hypercomplex structures on the cotangent bundles of Hermitian symmetric spaces. We continue with the previous notations but in this subsection it is assumed in addition that G/K is an irreducible Hermitian symmetric space (of the compact or noncompact type).

We will review a few facts about Hermitian symmetric spaces (see Ch.VIII, §§4–7,[10]). Since G/K is an irreducible Hermitian symmetric space, then $\mathfrak{g}$ is a simple Lie algebra and the center $\mathfrak{c}$ of $\mathfrak{k}$ is one-dimensional. There exists a unique (up to sign) element $z_0 \in \mathfrak{c} \subset \mathfrak{k}$ such that for the operator $I = \mathrm{ad}_{z_0}|\mathfrak{m}$ on $\mathfrak{m}$ we have $I^2 = -1$. Moreover, taking into account relations (1) and the Jacobi identity, we obtain that

$$[I\xi, I\eta] = [\xi, \eta], \qquad I[\xi, \zeta] = [I\xi, \zeta] \qquad \text{for all } \xi, \eta \in \mathfrak{m},\ \zeta \in \mathfrak{k}. \tag{6}$$

Since the Lie group K is connected, the group $\mathrm{Ad}(K)$ commutes elementwise with I (on $\mathfrak{m}$). This endomorphism determines an G-invariant complex structure on M [10].

Now fix some K-equivariant mapping $P : W^s \to GL(\mathfrak{m})$. The mapping PI, $(PI)_w \overset{\text{def}}{=} P_w I$ is also K-equivariant because the group $\mathrm{Ad}(K)$ commutes elementwise with I. As an application of the proposition above we will prove

5.1. Lemma. *If $J(P)$ is a complex structure then so is $J(PI)$.*

Proof. Suppose that $J(P)$ is a complex structure, i.e. $F(P)$ is an involutive subbundle. Since I is independent of w, by the definition of the Lie bracket and from relations (6) it follows that

$$\begin{aligned}
\left[\overrightarrow{(PI)(\xi)}, \overrightarrow{(PI)(\eta)}\right](w) &= \frac{d}{dt}\Big|_{t=0}\Big((PI)_{w+t(PI)_w(\xi)}(\eta) - (PI)_{w+t(PI)_w(\eta)}(\xi)\Big) \\
&= \frac{d}{dt}\Big|_{t=0}\Big(P_{w+tP_w(I\xi)}(I\eta) - P_{w+tP_w(I\eta)}(I\xi)\Big) \\
&= \left[\overrightarrow{P(I\xi)}, \overrightarrow{P(I\eta)}\right](w) \\
&= -[w, [I\xi, I\eta]] \\
&= -[w, [\xi, \eta]]
\end{aligned}$$

on W^s for any $\xi, \eta \in \mathfrak{m}$. Thus by assertion 1) of Proposition 3.1 the subbundle $F(PI)$ is involutive, i.e. $J(PI)$ is a complex structure on $T(T^sM)$.

Remark that locally on the open subset $U_D \subset T(T^sM)$ with respect to the splitting (4) the maps $J(P)$, $J(PI)$ and its product $J(P)J(PI)$ are represented by

$$J_{\Pi(g,w)}(P) = \begin{pmatrix} 0 & -P_w^{-1} \\ P_w & 0 \end{pmatrix}, \quad J_{\Pi(g,w)}(PI) = \begin{pmatrix} 0 & IP_w^{-1} \\ P_w I & 0 \end{pmatrix},$$

$$J_{\Pi(g,w)}(P)J_{\Pi(g,w)}(PI) = \begin{pmatrix} -I & 0 \\ 0 & P_w I P_w^{-1} \end{pmatrix}.$$

Therefore $J(P)$ and $J(PI)$ is a pair of anticommuting complex structures on T^sM. Thus we have proved

5.2. Theorem. *Let $(J(P), F(P), \Omega)$ be an arbitrary G-invariant Kähler structure on T^sM such that $\sigma_*(F(P)) = \overline{F(P)}$. Then $(T^sM, \{J_1 = J(P), J_2 = J(PI)\})$ is a hypercomplex manifold.*

5.3. *Remark.* Using the results of [5,6] we obtain that the constructed above hypercomplex structure $(T^sM, \{J_1 = J(P), J_2 = J(PI)\})$ is a hyper-Kähler structure if and only if $PI = IP$, i.e. if $P_wI = IP_w$ for any $w \in W^s$.

5.4. *Remark.* If G/K is a rank-one symmetric space isomorphic to $U(n+1)/(U(1)\times U(n))$, $n \geqslant 2$ then each G-invariant Kähler structure $J(P)$ on TM is determined by the following operator-function $P : \mathfrak{m} \to GL(\mathfrak{m})$, $w \mapsto P_w$ [4]

$$P_w(\xi) = \psi(r)\xi + \big(\psi(2r) - \psi(r)\big)r^{-2}\cdot\langle Iw, \xi\rangle Iw + \big(\lambda(r) - \psi(r)\big)r^{-2}\cdot\langle w, \xi\rangle w,$$

where $w \in \mathfrak{m}$, $r = \|w\| = \sqrt{-\frac{1}{2}\operatorname{Tr} w^2}$, $\lambda, \psi : [0, +\infty) \to \mathbb{R}$ are arbitrary positive functions satisfying the relations $\psi(r) = r\dfrac{\cosh\alpha(r)}{\sinh\alpha(r)}$, $\alpha'(r) = \dfrac{1}{\lambda(r)}$.

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ІНВАРІАНТНІ ГІПЕРКОМПЛЕКСНІ СТРУКТУРИ

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Розглянуто G-інваріантні келерові структури (J_1, Ω) на кодотичних розшаруваннях $T^*(G/K)$ (симплектичних многовидах з канонічною 2-формою Ω) ермітових симетричних просторів зі стандартною антиголоморфною інволюцією. Для довільної такої структури (J_1, Ω) побудовано гіперкомплексний многовид $(T^*(G/K), \{J_1, J_2\})$.

Ключові слова: інваріантні гіперкомплексні структури.

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