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STANDARD FORM OF PAIRS OF MATRICES WITH RESPECT TO GENERALIZED EQUIVALENCE

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Pairs (A_1, A_2) and (B_1, B_2) of matrices over an adequate ring R are called *generalized equivalent pairs* if $A_1 = UB_1V_1$, $A_2 = UB_2V_2$ for some invertible matrices U, V_1, V_2 over R. A standard form to which a pair of matrices can be reduced by means of generalized equivalent transformations is established. Conditions under which pairs of matrices will be generalized equivalent are founded. Classes of pairs of matrices which have the unique standard form are given.

Key words: pairs of matrices, generalized equivalence, canonical diagonal form, standard form.

Let R be an adequate ring, i.e. R be domain of integrity in which every finitely generated ideal is principal and for every $a, b \in R$ with $a \neq 0$, a can be represented as $a = cd$ where $(c, b) = 1$ and $(d_i, b) \neq 1$ for any non-unit factor d_i of d [1]. Further let $M(n, k, R)$ and $M(n, R)$ be the sets of $n \times k$ and $n \times n$ matrices over R respectively; d_m^A be the greatest common divisor of minors of the order m of the matrix $A \in M(n, k, R)$; D^A be the canonical diagonal form (the Smith normal form) of the matrix A, i.e.

$$D^A = UAV = \text{diag}(\varphi_1, \varphi_2, \ldots, \varphi_r, 0, \ldots, 0), \quad \varphi_r \neq 0, \quad \varphi_1 \mid \varphi_2 \mid \cdots \mid \varphi_r$$

for certain invertible matrices $U \in GL(n, R)$ and $V \in GL(k, R)$. Pairs of matrices (A_1, B_1) and (A_2, B_2), where $A_1, A_2 \in M(n, k_1, R)$ and $B_1, B_2 \in M(n, k_2, R)$ are called *generalized equivalent pairs* if $A_2 = UA_1V_1$ and $B_2 = UB_1V_2$ for certain matrices $U \in GL(n, R)$ and $U_i \in GL(k, R)$, $i = 1, 2$.

The reducibility of finite sets and pairs of matrices over polynomial and other rings by the same transformations to the triangular forms and their applications is considered in [2-7]. V. Dlab and C. M. Ringel [8] have established the canonical form of the pairs of complex matrices (A_1, A_2) with respect to the transformation $(A_1, A_2)(Q, P_1, P_2) = (QA_1P_1^{-1}, QA_2P_2^{-1})$, where Q is a complex invertible matrix, P_1 and P_2 are real invertible matrices.

The problem of the classification up to generalized equivalence of pairs of matrices over the rings as and the problem of the classification up to equivalence of matrices and of pairs of matrices, is wild [9]. Therefore such classification of pairs of matrices is possible only in some cases.

In this paper some form to which a pair of matrices can be reduced by means of generalized equivalent transformations is established. Conditions for generalized equivalence of pairs of matrices are given.

Lemma 1. *Let $B \in M(n,k,R)$ and $D^B = \operatorname{diag}(\psi_1,\dots,\psi_r,0\dots,0)$. Then there exist an upper unitriangular matrix $U \in GL(n,R)$ and an invertible matrix $V \in GL(k,R)$ such that*

$$UBV = T^B = \left\| \begin{matrix} \psi_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ b_{21}\psi_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{r-1,1}\psi_1 & \dots & \psi_{r-1} & 0 & 0 & \dots & 0 \\ b_{r1}\psi_1 & \dots & b_{r,r-1}\psi_{r-1} & b_{rr}\psi_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n1}\psi_1 & \dots & b_{n,r-1}\psi_{r-1} & b_{nr}\psi_r & 0 & \dots & 0 \end{matrix} \right\|,$$

where $(b_{rr},\dots,b_{nr}) = 1$.

Proof. By Lemma 1 [7] there exists a row matrix $u = \| 1 \quad u_2 \quad \dots \quad u_n \|$ such that $uB = \| c_1 \quad c_2 \quad \dots \quad c_k \|$, where $(c_1,c_2,\dots,c_k) = d_1^B = \psi_1$. Then for the matrix

$$U_1 = \left\| \begin{matrix} 1 & u_2 & \dots & u_n \\ 0 & & I_{n-1} & \end{matrix} \right\|,$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix, and for some matrix $V_1 \in GL(k,R)$ we get

$$U_1BV_1 = \left\| \begin{matrix} \psi_1 & 0 \ \dots \ 0 \\ b_{21}\psi_1 & \\ \dots & B_{n-1,k-1} \\ b_{n1}\psi_1 & \end{matrix} \right\| = B_1, \quad B_{n-1,k-1} \in M(n-1,k-1,R).$$

We now carry out similar reasoning on the matrix $B_{n-1,k-1}$ etc. In the end we obtain the matrix B_{r-1}, such that at the lower right corner of B_{r-1} there is a matrix $B_{n-(r-1),k-(r-1)}$ and $\operatorname{rank} B_{n-(r-1),k-(r-1)} = 1$.

Then for some matrix $V_{r-1} \in GL(k-(r-1),R)$ we have

$$B_{n-(r-1),k-(r-1)}V_{r-1} = \left\| \begin{matrix} b_{rr}\psi_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ b_{nr}\psi_r & 0 & \dots & 0 \end{matrix} \right\|.$$

The lemma is proved.

Corollary 1. *Let $B \in M(n,k,R)$, $\operatorname{rank} B = n$. Then there exist an upper unitriangular matrix $U \in GL(n,R)$ and an invertible matrix $V \in GL(k,R)$ such that $UBV = T^B = TD^B$, where T is a lower unitriangular matrix in $GL(n,R)$.*

Further by $U(R)$ we denote the group of units of the ring R, by R_δ a complete set of residues modulo δ, and by R'_δ the maximal subset of R_δ such that $ua \not\equiv b \pmod{\delta}$ for any $a,b \in R'_\delta$ and every $u \in U(R)$.

Theorem 1. *Let $A \in M(n,k_1,R)$, $B \in M(n,k_2,R)$, $n \leqslant k_1$, $n \leqslant k_2$, and $D^A = \operatorname{diag}(\varphi_1,\dots,\varphi_r,0,\dots,0)$, $D^B = \operatorname{diag}(\psi_1,\dots,\psi_s,0,\dots,0)$, $r \leqslant s \leqslant n$. Then a pair of matrices (A,B) is generalized equivalent to the pair (D^A,T^B), where*

(i) if $s=r=n$, then

$$T^B = \begin{Vmatrix} \psi_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ t_{21}\psi_1 & \psi_2 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_{n1}\psi_1 & t_{n2}\psi_2 & \dots & \psi_n & 0 & \dots & 0 \end{Vmatrix}$$

and $t_{ij} \in R'_{\delta_{ij}}$, where $\delta_{ij} = \left(\frac{\varphi_i}{\varphi_j}, \frac{\psi_i}{\psi_j}\right)$, $i,j=1,\dots,n$, $i>j$;

(ii) if $r<s\leqslant n$, then

$$T^B = \begin{Vmatrix} \psi_1 & & & & & & & & & \\ t_{21}\psi_1 & \psi_2 & & & & & & & & \\ \dots & \dots & \ddots & & & & \mathbf{0} & & & \\ t_{r1}\psi_1 & t_{r2}\psi_2 & \cdots & \psi_r & & & & & & \\ t_{r+1,1}\psi_1 & t_{r+1,2}\psi_2 & \cdots & t_{r+1,r}\psi_r & \psi_{r+1} & & & & & \\ \dots & \dots & \dots & \dots & \dots & \ddots & & & & \\ t_{s1}\psi_1 & t_{s2}\psi_2 & \cdots & t_{sr}\psi_r & 0 & \cdots & \psi_s & & & \\ t_{s+1,1}\psi_1 & t_{s+1,2}\psi_2 & \cdots & t_{s+1,r}\psi_r & 0 & \cdots & 0 & 0 & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \\ t_{n1}\psi_1 & t_{n2}\psi_2 & \cdots & t_{nr}\psi_r & 0 & \cdots & 0 & 0 & \cdots & 0 \end{Vmatrix}$$

and $t_{ij} \in R'_{\delta_{ij}}$, $i=1,\dots,n$, $j=1,\dots,r$, $i>j$, where

$$\delta_{ij} = \begin{cases} \left(\frac{\varphi_i}{\varphi_j}, \frac{\psi_i}{\psi_j}\right), & \text{if } i,j=1,\dots,r,\ i>j, \\ \frac{\psi_i}{\psi_j}, & \text{if } i=r+1,\dots,s,\ j=1,\dots,r, \\ 0, & \text{if } i=s+1,\dots,s,\ j=1,\dots,r. \end{cases}$$

(iii) if $r=s<n$, then

$$T^B = \begin{Vmatrix} \psi_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ t_{21}\psi_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_{r-1,1}\psi_1 & \dots & \psi_{r-1} & 0 & 0 & \dots & 0 \\ t_{r1}\psi_1 & \dots & t_{r,r-1}\psi_{r-1} & t_{rr}\psi_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_{n1}\psi_1 & \dots & t_{n,r-1}\psi_{r-1} & t_{nr}\psi_r & 0 & \dots & 0 \end{Vmatrix},$$

$(t_{rr},\dots,t_{nr})=1$, and $t_{ij} \in R'_{\delta_{ij}}$, where $\delta_{ij} = \left(\frac{\varphi_i}{\varphi_j}, \frac{\psi_i}{\psi_j}\right)$, $i,j=1,\dots,r-1$, $i>j$.

Proof. The pair of matrices (A,B) is generalized equivalent to the pair (D^A,B_1), where $D^A = PAQ$, $B_1 = PB$ for some matrices $P \in GL(n,R)$ and $Q \in GL(k_1,R)$.

Further we shall reduce the matrix B_1 to a triangular form by means of admissible equivalent transformations (U, V), $U \in GL(n, R)$ and $V \in GL(k_2, R)$ such that $UD^A = D^A V_1$ for some matrix $V_1 \in GL(k_1, R)$. Then the matrix U has a form $U = \|u_{ij}\|_1^n$, where $u_{ij} = \frac{\varphi_i}{\varphi_j} u'_{ij}$ for all $i > j$, $i, j = 1, \dots, n$.

Let $\operatorname{rank} B_1 = \operatorname{rang} B = s > r$. Then by Lemma 2 there exist an upper unitriangular matrix $U_1 \in GL(n, R)$ and a matrix $V_1 \in GK(k_2, R)$ such that

$$U_1 B V_1 = \left\| \begin{matrix} \psi_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ b_{21}\psi_1 & \psi_2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{r1}\psi_1 & b_{r2}\psi_2 & \cdots & \psi_r & 0 & \cdots & 0 \\ b_{r+1,1}\psi_1 & b_{r+1,2}\psi_2 & \cdots & b_{r+1,r}\psi_r & b_{r+1,r+1} & \cdots & b_{r+1,k_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1}\psi_1 & b_{n2}\psi_2 & \cdots & b_{nr}\psi_r & b_{n,r+1} & \cdots & b_{nk_2} \end{matrix} \right\| = B_2.$$

Therefore for some matrices $P_{n-r} \in GL(n-r, R)$ and $Q_{k_2-r} \in GK(k_2 - r, R)$ we have $P_{n-r} B_{n-r,k_2-r} Q_{k_2-r} = \operatorname{diag}(\psi_{r+1}, \dots, \psi_s, 0, \dots, 0)$, where

$$B_{n-r,k_2-r} = \left\| \begin{matrix} b_{r+1,r+1} & \cdots & b_{r+1,k_2} \\ \cdots & \cdots & \cdots \\ b_{n,r+1} & \cdots & b_{nk_2} \end{matrix} \right\|.$$

Put $U_2 = I_r \oplus P_{n-r}$, $V_2 = I_r \oplus P_{k_2-r}$. Thus $U_2 B_2 V_2$ is a lower triangular matrix with the principal diagonal D^B.

Then similarly as in the proof of Theorem 1 [6] we reduce the elements b_{ij} of the matrix T_1^B modules δ_{ij} $i = 2, \dots, n$, $j = 1, \dots, r$, $j < i$. Thus we get a matrix $T_2^B = U_3 T_1^B V_3$ whose (i, j)-element is equal to $c_{ij}\psi_j$, where $c_{ij} \in R_{\delta_{ij}}$. If $c_{ij} \notin R'_{\delta_{ij}}$, then there exists $t_{ij} \in R'_{\delta_{ij}}$ such that $u_i c_{ij} \equiv t_{ij} (\operatorname{mod} \delta_{ij})$ for some $u_i \in U(R)$. Then similarly as in the previous case we obtain the matrix $T_3^B = U_4 T_2^B V_4$ whose (i, j)-element is equal to $t_{ij}\psi_j$, $i = 2, \dots, n$, $j = 1, \dots, r$, $j < i$, ets. Therefore we get the matrix T^B which is defined in Theorem 1. Since we made admissible equivalent transformations over matrices B_i and T_i^B, $i = 1, 2, \dots$ the pair of matrices (A, B) is generalized equivalent to the pair of matrices (D^A, T^B). In cases (i) and (iii) the proofs are similar. Therefore the proof of the theorem is complete.

Definition 1. The pair (D^A, T^B) which defined in Theorem 1 is called *the standard form* of the pair of matrices (A, B).

We remark that the standard form (D^A, T^B) of the pair of matrices (A, B) with respect to generalized equivalence is uniquely determined only in some cases.

Corollary 2. *Let $A \in M(n, k_1, R)$, $B \in M(n, k_2, R)$, $n \leqslant k_1$, $n \leqslant k_2$, and $(d_n^A, d_n^B) = 1$. Then the pair diagonal matrices (D^A, D^B) is the unique standard form of the pair of matrices (A, B).*

Further we shall establish some conditions for generalized equivalence of pairs of matrices. Since for any matrix $A \in M(n, k, R)$, $n < k$ there exists a matrix $V \in GL(k, R)$ such that $AV = \| A_1 \quad 0 \|$, where $A_1 \in M(n, R)$ it is sufficient to consider the generalized equivalence of pairs of square $n \times n$-matrices.

Any pair of matrices is generalized equivalent to the pair of matrices of standard form. Hence it is sufficient to consider the generalized equivalence of pairs of matrices of the standard form.

Theorem 2. *Let (D, T_1) and (D, T_2) be pairs of matrices of standard form, i.e. $D = \operatorname{diag}(\varphi_1, \ldots, \varphi_n)$, $\varphi_n \neq 0$, $\varphi_1 \mid \varphi_2 \mid \cdots \mid \varphi_n$ and T_1, T_2 are lower triangular matrices with the principal diagonal $\Psi = \operatorname{diag}(\psi_1, \ldots, \psi_s, 0, \ldots, 0)$, $\psi_1 \mid \psi_1 \mid \ldots \psi_s$. Then the pairs of matrices (D, T_1) and (D, T_2) are generalized equivalent if and only if the following condition holds:*

(i) the matrices $(\operatorname{adj} D)T_1$ and $(\operatorname{adj} D)T_2$ are equivalent, i.e.

$$S(\operatorname{adj} D)T_1 = (\operatorname{adj} D)T_2Q, \quad S, Q \in GL(n, R); \tag{1}$$

(ii) in the set $\mathcal{S} = \{S \mid S(\operatorname{adj} D)T_1 = (\operatorname{adj} D)T_2Q,\ S, Q \in GL(n, R)\}$ the exists a matrix

$$S = \| s_{ij} \|, \text{ such that } s_{ij} = \frac{\varphi_j}{\varphi_i} s'_{ij}, \ i, j = 1, \ldots n, \ j > i. \tag{2}$$

Proof. **Necessity.** Let be $UDV_1 = D$ and $UT_1V_2 = T_2$, where $U, V_1, V_2 \in GL(n, R)$. The matrix $V_1 = \| v_{ij} \|_1^n$ has the form (2). Then

$$(\operatorname{adj} D)T_2 = (\operatorname{adj} V_1)(\operatorname{adj} D)(\operatorname{adj} U)UT_1V_2 = (\operatorname{adj} V_1)(\operatorname{adj} D)T_1V_2v, \tag{3}$$

$v \in U(R)$. Since $\operatorname{adj} V_1 = v_1 V_1^{-1}$, $v_1 \in U(R)$, then the equality (3) implies

$$V_1^{-1}(\operatorname{adj} D)T_1 = (\operatorname{adj} D)T_2Q, \ Q \in GL(n, R).$$

Since the matrix V_1 has a form (2), therefore V_1^{-1} has the same form [10].

Sufficiently. Assume the conditions (1) and (2) hold. It is easily to see, that $S(\operatorname{adj} D) = (\operatorname{adj} D)U$, where

$$U = \| u_{ij} \|_1^n, \ u_{ij} = \frac{\varphi_i}{\varphi_j} u'_{ij}, \ i, j = 1, \ldots, n, \ i > j. \tag{4}$$

Then the equality (1) implies $(\operatorname{adj} D)UT_1 = (\operatorname{adj} D)T_2Q$ or $UT_1 = T_2Q$. Since the matrix U has the form (4), then $UD = DV$. Therefore the pairs of matrix (D, T_1) and (D, T_2) are generalized equivalent.

Corollary 3. *Let $A, B \in M(n, R)$ and A be a nonsingular matrix. Then the pair of matrices (A, B) is generalized equivalent to the pair of diagonal matrices (D^A, D^B) if and only if the matrices $(\operatorname{adj} A)B$ and $(\operatorname{adj} D^A)D^B$ are equivalent.*

Proof. The pair matrices (A, B) is generalized equivalent to the pair of matrices of standard form (D^A, T^B). The matrix $(\operatorname{adj} D^A)T^B$ is equivalent to the matrix $(\operatorname{adj} A)B$ and hence it is equivalent to $(\operatorname{adj} D^A)D^B$. Then there exist lower unitriangular matrices $S, Q \in GL(n, R$ such that $S(\operatorname{adj} D^A)T^B = (\operatorname{adj} D^A)D^BQ$. Therefore the statement (ii) of Theorem 2 holds for the matrix S. .

Let (D,T_1) and (D,T_2) are pairs of matrices of such form: $D = \operatorname{diag}(1,\dots,1,\varphi_n)$, $\varphi_n \neq 0$,

$$T_i = \begin{Vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \psi_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \psi_{n-1} & 0 \\ t_{i1} & t_{i2}\psi_2 & \dots & t_{i,n-1}\psi_{n-1} & \psi_n \end{Vmatrix}, \quad \psi_2 \mid \psi_3 \mid \cdots \mid \psi_n, \ \ \psi_n \neq 0, \qquad (5)$$

$i = 1, 2$. Without loss of generality we may assume that $\psi_1 = 1$. Further by d_i we denote the greatest common divisor of elements of the last row of the matrix T_i:

$$d_i = (t_{i1}, t_{i2}\psi_2, \dots, t_{i,n-1}\psi_{n-1}, \psi_n), \quad i = 1, 2.$$

Lemma 3. *If the pairs of matrices (D,T_1) and (D,T_2) are generalized equivalent, then $(d_1,\varphi_n) = (d_2,\varphi_n)$.*

The proof of lemma follows from Theorem 2.

Lemma 4. *Let (D,T_1) be a pair of matrices of the form (5) and $(d_1,\varphi_n) = 1$. Then the canonical diagonal form of the matrix $(\operatorname{adj} D)T_1$ is equal to the product of canonical diagonal forms of the matrices $\operatorname{adj} D$ and T_1, i.e.*

$$D^{(\operatorname{adj} D)T_1} = D^{\operatorname{adj} D} D^{T_1} = \operatorname{diag}(1, \varphi_n\psi_2, \dots, \varphi_n\psi_n).$$

Proof. By Cauchy-Binet formula the minors of product of matrices we get that $\varphi_n^{p-1}\psi_2 \dots \psi_p$ divides every minor of order p of the matrix $(\operatorname{adj} D)T_1$. Since $(d_1,\varphi_n) = 1$ we have $d_p^{(\operatorname{adj} D)T_1} = \varphi_n^{p-1}\psi_2 \dots \psi_p$. This implies the statement of the lemma.

Theorem 3. *Let (D,T_1) and (D,T_2) be pairs of matrices of form (5). If $(d_1,\varphi_n) = (d_2,\varphi_n) = 1$, then the pairs of matrices (D,T_1) and (D,T_2) are generalized equivalent.*

Proof. By Lemma 4 the canonical diagonal forms of the matrices $(\operatorname{adj} D)T_1$ and $(\operatorname{adj} D)T_2$ coincide, i.e. the following matrices are equivalent:

$$S(\operatorname{adj} D)T_1 = (\operatorname{adj} D)T_2Q, \quad S, Q \in GL(n,R), \quad S = \| s_{ij} \|_1^n. \qquad (6)$$

Thus we have $\varphi_n \mid s_{in}t_{1j}\psi_j$, for all $j = 1,\dots,n$, $i = 1,\dots,n-1$, where $\psi_1 = t_{1n} = 1$. Since $(d_1,\varphi_n) = 1$ we obtain that $\varphi_n \mid s_{in}$ for all $i = 1,\dots,n-1$, i.e. statement (ii) of Theorem 2 holds for the matrix S. Therefore the proof of the theorem is complete.

Corollary 5. *Let (D,T_1) be a pair of matrices of form (5). If $(d_1,\varphi_n) = 1$, then (D,T_1) is generalized equivalent to the pair $(D,T\Psi)$, where $\Psi = \operatorname{diag}(1,\psi_2,\dots,\psi_n)$,*

$$T = \begin{Vmatrix} & I_{n-1} & & & 0 \\ t & 0 & \dots & 0 & 1 \end{Vmatrix},$$

and

$$t = \begin{cases} 0, & \text{if the matrices } (\operatorname{adj} D)T_1 \text{ and } (\operatorname{adj} D)\Psi \text{ are equivalent;} \\ 1, & \text{otherwise.} \end{cases}$$

Put $R''_\delta = \{a \in R'_\delta \mid (a,\delta) \neq 1 \text{ for all } a \neq 1\}$.
Then Theorems 1 and 3 imply

Corollary 6. *Let $A, B \in M(n, R)$,*

$$D^A = \operatorname{diag}(1, \dots, 1, \varphi_n), \ \ D^B = \operatorname{diag}(1, \psi_2, \dots, \psi_n),$$

$(\varphi_n, \psi_n) = \delta_n$ and $(\psi_{n-1}, \delta_n) = 1$. Then the pair of matrices (A, B) is generalized equivalent to the pair (D^A, TD^B), where

$$T = \begin{Vmatrix} & I_{n-1} & & & \mathbf{0} \\ t_1 & t_2 & \dots & t_{n-1} & 1 \end{Vmatrix}, \ \ t_j \in R''_\delta, \ \ j = 1, \dots, n-1.$$

Theorem 4. *Let $A_i, B_i \in M(n, R)$, $D^{A_i} = D^A = \operatorname{diag}(1, \dots, 1, \varphi_n)$, $D^{B_i} = D^B = \operatorname{diag}(1, \psi_2, \dots, \psi_n)$, $i = 1, 2$. Let $(\varphi_n, \psi_n) = p$, $(\psi_{n-1}, p) = 1$ and p be a prime element of the ring R. Then*

(i) the pairs of matrices (A_1, B_1) and (A_2, B_2) are generalized equivalent if and only if the matrices $(\operatorname{adj} A_1)B_1$ and $(\operatorname{adj} A_2)B_2$ are equivalent;

(ii) the pair of matrices (A_1, B_1) is generalized equivalent to the pair (D^{A_1}, TD^{B_1}), where

$$T = \begin{Vmatrix} & I_{n-1} & & & \mathbf{0} \\ t & 0 & \dots & 0 & 1 \end{Vmatrix}$$

and

$$t = \begin{cases} 0, & \text{if the matrices } (\operatorname{adj} A_1)B_1 \text{ and } (\operatorname{adj} D^{A_1})D^{B_1} \text{ are equivalent;} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. The pair of matrices (A_i, B_i) $(i = 1, 2)$ is generalized equivalent to the pair $\left(D^{A_i}, T_i^{B_i}\right)$ of form (5) and $t_{ij} \in R_p$, $j = 1, \dots, n-1$. Then $(d_i, \varphi_n) = 1$ if there exists $t_{ij} \neq 0$, $j = 1, \dots, n-1$. Further we use Theorem 3, which completes the proof of the statement.

Example. Let $\mathcal{N} = \{(A, B) \mid A, B \in M(2, \mathbb{Z})$ such that $D^A = \operatorname{diag}(1, 25), D^B = \operatorname{diag}(1, 175)\}$. Then $\delta = (25, 175) = 25$, $R_\delta = \{0, 1, \dots, 24\}$, $R'_\delta = \{0, 1, \dots, 12\}$ and $R''_\delta = \{0, 1, 5, 10\}$. Then the set $\mathcal{N}$ is partitioned up to the generalized equivalence on four disjoint classes with representations

$$\left(\begin{Vmatrix} 1 & 0 \\ 0 & 25 \end{Vmatrix}, \begin{Vmatrix} 1 & 0 \\ t & 175 \end{Vmatrix} \right), \ \ t = 0, 1, 5, 10.$$

The direct verification shows that the pairs of matrices

$$\left(\begin{Vmatrix} 1 & 0 \\ 0 & 25 \end{Vmatrix}, \begin{Vmatrix} 1 & 0 \\ 5 & 175 \end{Vmatrix} \right) \text{ and } \left(\begin{Vmatrix} 1 & 0 \\ 0 & 25 \end{Vmatrix}, \begin{Vmatrix} 1 & 0 \\ 10 & 175 \end{Vmatrix} \right)$$

are not generalized equivalent.

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СТАНДАРТНА ФОРМА ПАР МАТРИЦЬ ВІДНОСНО УЗАГАЛЬНЕНОЇ ЕКВІВАЛЕНТНОСТІ

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Пари матриць (A_1, A_2) і (B_1, B_2) над адекватним кільцем R називаються узагальнено еквівалентними, якщо $A_1 = UB_1V_1, A_2 = UB_2V_2$ для деяких оборотних матриць U, V_1, V_2 над R. Щодо таких перетворень визначено стандартну форму пар матриць та зазначено умови їх загальної еквівалентності. Виділено класи пар матриць, для яких ця форма визначається однозначно.

Ключові слова: пари матриць, узагальнена еквівалентність, канонічно діагональна форма, стандартна форма.

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