ВІСНИК ЛЬВІВ. УН-ТУ
Серія мех.-мат.2003.Вип.61.С.156–166

VISNYK LVIV UNIV.
Ser.Mech-Math.2003.Vol.61.P.156–166

УДК 513.83

n-TRIVIAL KNOTS AND THE ALEXANDER POLYNOMIAL

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For each integer $n \geqslant 1$, we construct via the pure braid commutators $(2n-1)$-trivial knots with non-trivial Alexander polynomial. We formulate also a sufficient condition under which an $(n-1)$-trivial knot, $n > 2$, has trivial Alexander polynomial. As a particular case, for each $n > 1$, we describe some class of "geometric" $(n-1)$-trivial knots with trivial Alexander polynomial.

Key words: invariant of finite order, braid commutator, Seifert surface, Alexander polynomial, trivalent diagram, n-trivial knot.

1. Knots K and L are called V_n-equivalent (n-equivalent) if they cannot be distinguished by the Vassiliev invariants (additive Vassiliev invariants, respectively) of order $\leqslant n$, the invariants taking values in any abelian group. Goussarov [6] was the first who has characterized the relation of n-equivalence on knots in combinatorial terms. Later on, it turned out that the relations of V_n-equivalence and n-equivalence coincide on the knots in S^3 (see [7,15]). Habiro [7] has characterized n-equivalence of knots in terms of the so-called C_{n+1}-moves. Stanford [15] has given a description of n-equivalent knots in terms of the pure braid $(n+1)$-commutators. A knot in S^3 is called n-trivial if it is n-equivalent to the trivial one. In [9], Kalfagianni and Lin have introduced for each $n \geqslant 2$ the classes of "geometric" knots, among other the classes of n-hyperbolic, n-elliptic and n-parabolic knots, and showed that all they are n-trivial. Moreover, any n-hyperbolic and n-elliptic knot has the trivial Alexander polynomial [9]. The latter two classes do not exhaust however all n-trivial knots. Kalfagianni and Lin showed (Proposition 6.1 of [9]) that for each integer $n \geqslant 1$ there exists an n-trivial knot with the non-trivial Alexander polynomial. The proof of the proposition is based on Theorem 1 of [3] (which proves the Melvin-Morton-Rozansky conjecture) and is rather of existence character. In the present paper, for each integer $n \geqslant 1$ we indicate in an explicit form the $(2n-1)$-trivial knots having non-trivial Alexander polynomial. Our approach uses in essential way the characterization of n-equivalent knots in terms of the pure braid commutators (see [14] and [15]).

In [12], H. Murakami and T. Ohtsuki have described the filtration on the vector space S over the rationals $\mathbb{Q}$ spanned by Seifert matrices of knots,

$$\mathcal{S} \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \ldots,$$

and related this to the Goussarov-Vassiliev filtration of the vector space spanned by knots. They showed that a rational Vassiliev invariant of order n comes from the Alexander polynomial (i.e. can be expressed as a sum of products of the coefficients of the Alexander-Conway polynomial) if and only if it can be factored through the quotient space $\mathcal{S}/\mathcal{S}_{n+1}$. In this paper, using the above mentioned results of H. Murakami and T. Ohtsuki (see [12]) and the results of A. Kricker, B. Spence, and I. Aitchison [10] on the characterization of rational weight systems coming from the Alexander-Conway polynomial, we obtain a sufficient condition for an n-trivial knot, $n \geqslant 2$, to have the trivial Alexander polynomial. As a particular case, for each $n \geqslant 1$ we describe some class of "geometric" n-trivial knots with the trivial Alexander polynomial, where each such "geometric" n-trivial knot is obtained from the trivial one by inserting in it the "double" pure braid $(n+1)$-commutators (see [14] for details).

Now we define some needed notions and review briefly the results on the characterization of n-equivalent knots via pure braid commutators [15] and C_{n+1}-moves [7]. We shall also review the results of H. Murakami and T. Ohtsuki, and A. Kricker, B. Spence, and I. Aitchison on the characterization of the rational Vassiliev invariants and weight systems coming from the Alexander-Conway polynomial (see [12] and [10] for details).

Let $\mathcal{K}$ denote the free abelian group generated by the classes of equivalent oriented knots in S^3 and $\mathcal{K}_n$ the subgroup of $\mathcal{K}$ generated by all n-singular knots, $n \geqslant 1$. Here an n-singular knot we regard as an element of $\mathcal{K}$ so that each double point of this knot is replaced by a difference of the positive and negative crossings (see Fig. 1).

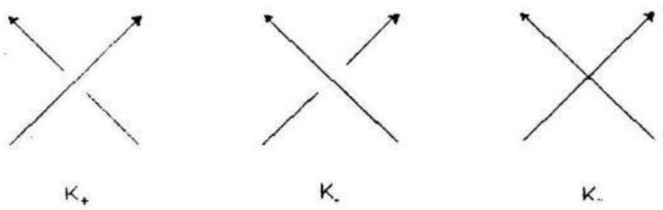

Fig. 1

Let

$$\mathcal{K} \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

be the Vassiliev-Goussarov filtration of $\mathcal{K}$. A Vassiliev invariant of type n, $n \geqslant 0$, taking values in an abelian group H is a map $v: \mathcal{K} \to H$ vanishing on the subgroup $\mathcal{K}_{n+1}$. The smallest number m such that v vanishes on $\mathcal{K}_{m+1}$ is called the order of v. The Vassiliev invariants are called also the invariants of finite type of knots (or links).

A trivalent diagram D of degree n is a connected graph with $2n$ vertices all of which are trivalent. There is a distinguished subgraph which is homeomorphic to a circle, called the external one. Each vertex of D which lies on the external circle is called external, otherwise it is internal. At each internal vertex a of a trivalent diagram one of two possible cyclic ordering of the edges around this vertex (the orientation at a) is chosen. The subgraph of D having the same vertex set as D and the edge set of which consists of all edges of D which do not lie on the external circle is called the internal graph of D. An orientation of the external circle is chosen and the other edges of D are taken to be non-oriented. All trivalent diagrams are considered up to an isomorphism of distinguished graphs that respects the above structures on them. Every trivalent diagram D is pictured in the plane in such a way that each its internal vertex has the counterclockwise orientation. If a trivalent diagram has no internal vertices it is called a chord diagram. Denote by $\mathcal{D}$ and $\mathcal{D}_n$ the collections of all trivalent diagrams and trivalent diagrams of degree n, respectively.

Define $\mathcal{A}_n$ and $\mathcal{A}$ to be the quotient groups, $\mathcal{A}_n = \mathbb{Z}\mathcal{D}_n/\{\text{all STU relations}\}$ and $\mathcal{A} = \mathbb{Z}\mathcal{D}/\{\text{all STU relations}\}$, where STU is the homogeneous relation on $\mathbb{Z}\mathcal{D}$ indicated in Fig. 2.

STU :

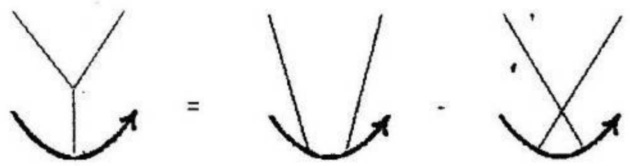

Fig. 2

Note that the graded abelian group $\mathcal{A}$ is naturally isomorphic to the graded abelian group $\mathcal{A}^c$, the quotient of the group freely generated by chords diagrams via 4T-relations (see [2]). A trivalent diagram D is called a tree diagram, if its internal graph is a tree. A trivalent diagram T is called a one-branch tree diagram of degree n, if its internal graph is isomorphic to the standard n-tree. There is a natural one-to-one correspondence between the permutations of the symmetric group S_n and the one-branch tree diagrams of degree n. For a given permutation $\sigma \in S_n$, denote by T_σ the one-branch trivalent diagram of degree n corresponding to σ (see [11]). Note that $\mathcal{A} \otimes \mathbb{Q}$ has an algebra structure with respect to the connected sum of external circles of trivalent diagrams (the product of trivalent diagrams) [2]. The co-product ∇ on $\mathcal{A} \otimes \mathbb{Q}$ is defined in a natural way (see [2]). With respect to these operations, $\mathcal{A} \otimes \mathbb{Q}$ is a commutative and co-commutative Hopf algebra and is generated by the primitive elements of $\mathcal{A} \otimes \mathbb{Q}$ [2]. The primitive subspace $\mathcal{P}$ of $\mathcal{A} \otimes \mathbb{Q}$ is generated (as a graded vector space) by the primitive trivalent diagrams, i.e. the trivalent diagrams with the connected internal graph. It is known (see [5]) that the primitive space $\mathcal{P}_d, d > 1$, is generated by the trivalent diagrams of the two types. The first type of generators consists of the primitive trivalent diagrams whose internal graph has the negative Euler characteristic (see [8]). The second type (only for even $d = 2n$) consists of the so-called "wheel" ω_{2n} (see Fig. 3). Therefore, for odd $d > 1$ the primitive space $\mathcal{P}_d$ is generated by the primitive trivalent diagrams of the first type. The space $\mathcal{P}_1$ is one-dimensional and is generated by a chord diagram D_1 with a single chord. It follows that as an algebra, $\mathcal{A} \otimes \mathbb{Q}$ is generated by D_1, the primitive trivalent diagrams with negative Euler characteristic and the "wheels" $\omega_{2n}, n \geqslant 1$.

An (unframed) $\mathbb{Q}$-valued weight system of degree n is a map $w: \mathcal{A}_n \to \mathbb{Q}$ which vanishes on each trivalent (chord) diagram of degree n having an isolated chord. A split diagram is a diagram which can be decomposed into a product of diagrams of lower order. By Kontsevich's integral, each $\mathbb{Q}$-valued weight system of degree n can be integrated (in a non-unique way) to a $\mathbb{Q}$-valued Vassiliev invariant of order n [2]. A rational-valued Vasiliev invariant v of order n is called canonical if it is determined uniquely by its weight system $w(v)$, of the same degree n [3]. Under the Alexander-Conway polynomial we shall mean a canonical Vassiliev power series $\tilde{C}$ satisfying the following two axioms $\mathbb{A}1$ and $\mathbb{A}2$:

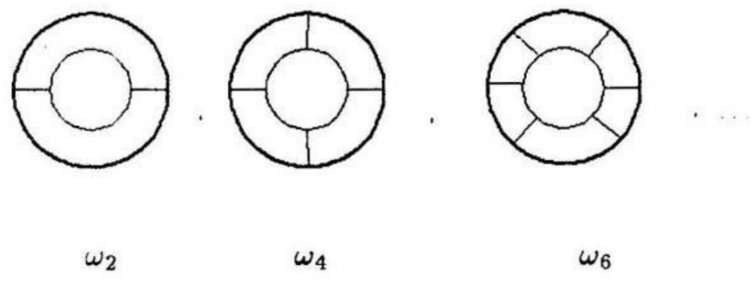

Fig. 3

$\mathbb{A}1$ (the skein-relation). $\tilde{C}(K_+) - \tilde{C}(K_-) = (e^{h/2} - e^{-h/2})\tilde{C}(K_0)$,
for any link diagrams K_+, K_- and K_0 which are the same outside some small disc in the plane where they look as positive crossing, negative crossing and smoothing, respectively;

$\mathbb{A}2$ (the initial data). $\tilde{C}$(c-component unlink)$= 1$ if $c = 1$ and 0, otherwise.

Therefore the Alexander-Conway polynomial is a renormalized and reparametrized version of both the Alexander and Conway polynomials. Bar-Natan and Garoufalidis described the Conway weight systems $w: \mathcal{A}^c \to \mathbb{Z}$ in terms of intersection graphs of chord diagrams (Theorem 3 of [3]) and the universal immanants of such intersection graphs (Theorem 5 of [3]). Chmutov [5] has calculated the Alexander-Conway weight systems on the generators of the space $\mathcal{P}_d, d > 1$. In particular, he showed that every (framed or unframed) Alexander-Conway weight system of degree $n > 1$ vanishes on primitive trivalent diagrams with the internal graph of negative Euler characteristic (see also [10]). Basing on the results of the paper [10], Kricker (Lemma 2.11 of [8]) has described the algebra of (framed) Alexander-Conway weight systems (see also Lemma 2.1 of [12]). We formulate its unframed version as follows.

1.1. Lemma. *If an (unframed) $\mathbb{Q}$-valued weight system w vanishes on every trivalent diagram, the internal graph of which has a component with the negative Euler characteristic, then it can be represented as a sum of products of weight systems coming from the coefficients of the Alexander-Conway polynomial.*

Let B_k be the braid group on k strands and P_k its subgroup of pure braids. For $0 \leqslant i < j \leqslant k-1$ let $p_{i,j} \in P_k$ be the braid that links the ith and jth strands behind the others (see Fig. 4). It is known [4] that the collection of braids $\{p_{i,j}\}_{0 \leqslant i < j \leqslant k-1}$ represents the standard generators of the group P_k.

By a tangle diagram we shall mean a knot diagram K with a single S^1-boundary which intersects each of the strands in the diagram transversely. To put it in another words, K is the closure of an oriented tangle T with $\operatorname{dom} T = \operatorname{codom} T$ which is positioned in $\mathbb{R}^2$ outside of a disc D, the latter being bounded by S^1. As in [16], for a fixed k by a tangle map $T: P_k \to$ {knot types} with $\operatorname{dom} T = \operatorname{codom} T = k$, we shall mean a canonical way of putting a pure braid $p \in P_k$ into a tangle diagram to get an oriented knot $T(p)$ (see also [14] for details).

Let $LCS_n(P_k)$ denote the nth subgroup of the lower central series of the group P_k. For each $\sigma \in S_n$ denote by $p_\sigma \in LCS_n(P_{n+1})$, the pure braid n-commutator of the following form $p_\sigma = [\ldots[p_{0,\sigma(n)}, p_{0,\sigma(n-1)}], \ldots], p_{0,\sigma(1)}]$, and let $p_n \in LCS_n(P_{n+1})$ be the pure braid n-commutator $p_n = [p_{n-1,n}, [p_{n-2,n-1}, \ldots, [p_{1,2}, p_{0,1}] \ldots]$. In [14], it is shown that each one-branch simple C_n-move on a knot, defined by Habiro [7], where $n \geqslant 2$, is equivalent to the insertion (in the non-oriented setting) in this knot via some tangle map of the pure braid n-commutator p_n.

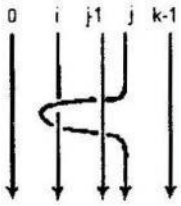

Fig. 4

The operation on oriented knots, inverse to the insertion, is called the deletion (of the pure braids) in knots. As discussed in [14], for each $n \geqslant 2$ the insertions of the coloured pure braid n-commutator p_n in a knot via the tangle maps can be considered as a topological realization of one-branch tree diagrams of degree n. Similarly, the closure $\widehat{p_\sigma}$ of the braid p_σ via the permutation $(1, 2, \ldots, n)$, where $\sigma \in S_n$, gives a topological realization of the one-branch tree diagram T_σ.

Let $\mathcal{K}'$ denote the vector space over $\mathbb{Q}$ spanned by all knot types in S^3 and let

$$\mathcal{K}' \supset \mathcal{K}'_1 \supset \mathcal{K}'_2 \supset \mathcal{K}'_3 \supset \ldots$$

be the Vassiliev-Goussarov filtration of $\mathcal{K}'$. A rational Vassiliev invariant of type n is a map $\mathcal{K}'/\mathcal{K}'_{n+1} \to \mathbb{Q}$.

Let $\mathcal{M}$ be the set of integer matrices of even size such that $M - M^t$ is unimodular. Denote by $[M]$ the S-equivalence class of matrices in $\mathcal{M}$ which contains M. Let S be the vector space over $\mathbb{Q}$ spanned by the S-equivalence classes of matrices in $\mathcal{M}$. H. Murakami and T. Ohtsuki [12] have defined a filtration of $\mathcal{S}$ in the following way. For a matrix $M \in \mathcal{M}$ of the size $m \times m$ and the integers $i_1, i_2, \ldots, i_n$, where $i_j \leqslant n, 1 \leqslant j \leqslant m$, define the alternating sum

$$\sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n = 0,1} (-1)^{\varepsilon_1 + \cdots + \varepsilon_n} [M + \varepsilon_1 E_{i_1 i_1} + \cdots + \varepsilon_n E_{i_n i_n}] \in S,$$

where E_{ii} is the matrix of the size $m \times m$ with (i,i)-entry 1 and the others 0. There is a natural linear map $s: \mathcal{K}' \to \mathcal{S}$ which takes a knot to the S-equivalence class of a Seifert matrix for this knot. The map s respects the filtrations of both the vector spaces $\mathcal{K}'$ and $\mathcal{S}$ and so, induces a map $\mathcal{K}'/\mathcal{K}'_{n+1} \to \mathcal{S}/\mathcal{S}_{n+1}$, denoted also by s. We shall say that a Vassiliev invariant $v: \mathcal{K}'/\mathcal{K}'_{n+1} \to \mathbb{Q}$ comes from Seifert matrices if v can be factored through the map s. H.Murakami and T.Ohtsuki [12] showed that a rational

Vassiliev invariant $v: \mathcal{K}'/\mathcal{K}'_{n+1} \to \mathbb{Q}$ comes from the Alexander-Conway polynomial if and only if it comes from Seifert matrices. As a consequence, any rational Vassiliev invariant of knots coming from Seifert matrices is equal to a linear sum of products of coefficients of Alexander-Conway polynomial.

Now let us recall some needed notions and facts from Habiro's clasper theory [7]. Let K be a knot in S^3. A clasper G for K is a framed uni-trivalent graph embedded in S^3 so that all its univalent vertices (and only they) are positioned on K and all possible intersections between the edges of G and K are transversal. We use the blackboard framing for description of claspers G. The degree $\deg(G)$ of the clasper G is the half of the number of its vertices. Any pair (K, G), where K is a knot and G is a clasper on it, defines a surgery of S^3 and S^3 surgered will be always a three sphere. Denote by K_G a knot obtained from the knot K by surgery of S^3 defined by the pair (K, G). Let $\mathcal{G}_n$ be the vector space over $\mathbb{Q}$ spanned by all the pairs (K, G) with $\deg(G) = n$. Habiro [7] has defined for each $n \geqslant 1$ a natural surjective map $e: \mathcal{G}_n \to \mathcal{K}'_n$. Let $\gamma: \mathcal{G}_n \to \mathcal{A}_n \otimes \mathbb{Q}$ be the map forgetting the embedding, $\varphi: \mathcal{A}_n \otimes \mathbb{Q} \to \mathcal{K}'_n/\mathcal{K}'_{n+1}$ the map which replaces chords by double points and let $p: \mathcal{K}'_n \to \mathcal{K}'_n/\mathcal{K}'_{n+1}$ be the canonical projection. Because of Kontsevich's integral over $\mathbb{Q}$, the map φ is an isomorphism [2]. Habiro actually showed [7] that the equality $p \cdot e = \varphi \cdot \gamma$ holds. As a consequence, the claspers on knots can be considered as topological realization of trivalent diagrams of the same degree.

2. C_n-moves and the Alexander-Conway polynomial.

2.1. Proposition. *Let a pair (K, G), where K is a trivial knot in S^3 and G is a clasper on K of degree $2n$, be a topological realization of the wheel $\omega_{2n}, n \geqslant 1$, and let K_G be the knot obtained by surgery of S^3 defined by the pair (K, G). Then K_G is $(2n-1)$-trivial knot with the non-trivial Alexander polynomial.*

Proof. The proof of the proposition follows from Habiro's clasper theory and the characterization of weight systems coming from the Alexander-Conway polynomial. Indeed, the knot K_G is obtained from the trivial knot K by C_{2n}-move defined by the pair (K, G). By Theorem 6.18 of [7], K_G is $(2n-1)$-trivial knot. Let $w: \mathcal{A}_n \otimes \mathbb{Q} \to \mathbb{Q}$ be the rational weight system of degree $2n$ defined as follows: $w(\omega_{2n}) = 1$ and $w(D) = 0$ for any trivalent diagram of degree $2n$, the internal graph of which has the negative Euler characteristic. By Kontsevich's integral over $\mathbb{Q}$, w can be integrated to a canonical $\mathbb{Q}$-valued Vassiliev invariant of order $2n$ [2]. By the definition of the topological realization of trivalent diagram, we have then $v(K_G) - v(K) = \pm v(\omega_{2n}) = \pm w(\omega_{2n}) = \pm 1$. Therefore, by Lemma 1.1, v is a non-trivial Vassiliev invariant of order $2n$ coming from the coefficients of the Alexander polynomial. It follows that the Alexander polynomial of K_G is non-trivial.

Now, for each $n \geqslant 1$ we indicate explicitly the $(2n-1)$-trivial knots with the non-trivial Alexander polynomial. For this, consider two the following pure braid $2n$-commutators: p_{σ_1} and p_{σ_2}, where $\sigma_1 = (1)(2)(3)\ldots(2n)$ and $\sigma_2 = (1, 2, 3, \ldots, 2n)$. Let $\hat{q}$ denote the closure of the braid $q \in LCS_{2n}(P_{2n+1})$ with the strands $u_0, u_1, u_2, \ldots,$
u_{2n} via the permutation $(0, 1, 2, \ldots, 2n)$ and let K be a trivial knot. Set $p = p_{\sigma_1} \cdot p_{\sigma_2}^{-1}$. Then the knot $K_{2n} = \hat{p}$ is the desired knot. Indeed, the knots K and K_{2n} are $LCS_{2n}(P_{2n+1})$-equivalent. Then, by Theorem 0.2 of [15], they are $(2n-1)$-equivalent.

It follows that K_{2n} is $(2n-1)$-trivial. On the other hand, by Lemma 1.11 of [11], for each Vassiliev invariant v of order $2n$ we have $v(K_{2n}) = v(K) \pm (v(T_{\sigma_1}) - v(T_{\sigma_2})) = \pm(v(T_{\sigma_1}) - v(T_{\sigma_2}))$. Note that in $\mathcal{A}_{2n} \otimes \mathbb{Q}$ we have $T_{\sigma_1} - T_{\sigma_2} = \omega_{2n}$, so $K_{2n} - K$ is a topological realization of the wheel ω_{2n}. Then the same reasoning as in the proof of the Proposition 2.1 shows that K_{2n} has the non-trivial Alexander polynomial.

2.2. Proposition. *Suppose that knots K and L in S^3 are related by a sequence of C_n-moves $M_i, n \geqslant 3$, and possibly isotopies, $K = K_0 \to K_1 \to \cdots \to K_l = L$, with the following properties. Each move $M_i, 1 \geqslant i \geqslant l$, is determined by the pair (K_{i-1}, G_{i-1}), where G_{i-1} is a clasper on the knot K_{i-1} such that the internal graph of the trivalent diagram $\gamma(G_{i-1})$ is connected and has a negative Euler characteristic. Then K and L are $(n-1)$-equivalent and have the same Alexander polynomial.*

Proof. The fact that K and L are $(n-1)$-equivalent follows directly from Theorem 6.18 of [7]. Therefore, we have only to show that the knots K and L share the same Alexander polynomial. The proof of the last assertion is by induction on the number l. Suppose that K and K_i, where $i \leqslant l-1$, have the same Alexander polynomial, $A_K(t) = A_{K_i}(t)$. We now proceed as in the proof of Theorem 1.2 of [12]. Let H_i be the internal graph of the trivalent diagram $D_i = \gamma(K_i, G_i)$. By the assumption, H_i is a connected graph having the negative Euler characteristic. It follows that there exists an internal vertex u of D_i which is not connected to any external vertex of D_i. It follows from the proof of Theorem 1.2 of [12], that the knots K_{i+1} and K_i have S-equivalent Seifert matrices. Since the Alexander-Conway polynomial of a knot is determined by the S-class of its Seifert matrix, the knots K_{i+1} and K_i have the same Alexander polynomial, $A_{K_{i+1}}(t) = A_{K_i}(t)$. Therefore, $A_K(t) = A_{K_{i+1}}(t)$. The induction step completes the proof. Note that the diagram $D = \sum_{i=1}^{l} D_i$, regarded in $\mathcal{A}_n \otimes \mathbb{Q}$, is an integral linear combination of trivalent diagrams, the internal graphs of which have the negative Euler characteristic, i.e. the generators of $\mathcal{A}_n \otimes \mathbb{Q}$ of the first type. Then for each Vassiliev invariant v of order $\leqslant n$ we have $v(L) - v(K) = \sum_{i=1}^{l} v(D_i) = v(\sum_{i=1}^{l} D_i)$.

2.1. Corollary. *Under the assumptions of Proposition 2.2, if the knot K is $(n-1)$-trivial and has the trivial Alexander polynomial, then L is also $(n-1)$-trivial and has the trivial Alexander polynomial.*

2.1 Remark. The restriction $n \geqslant 3$ in Proposition 2.2 is essential. Indeed, there are no trivalent diagram of degree 1 and 2, the internal graph of which has the negative Euler characteristic. On the other hand, it is well known that any two knots K and L in S^3 are related by a sequence of simple one-branch C_2-moves (C_2-move is also called the Δ-unknotting operation, see [13]). On the level of the vector space $\mathcal{A}_n \otimes \mathbb{Q}$, each Δ-operation on knots contributes the value $\pm(1/2)\omega_2$ to the total sum $\sum_{i=1}^{l} D_i$ of trivalent diagrams D_i. It follows that if for a Vassiliev invariant v_2 of order 2 there holds $v_2(K) - v_2(L) = 0$, then l must be even.

2.2. Remark. Recently Traczyk [17] has proved that for any integer $n \geqslant 3$ the Alexander (Conway) polynomial of oriented links is not changed by the rotation operation of Anstee, Przytycki and Rolfsen [1] of order n. Rotants (the pairs of links obtained via rotation operation) are known to share the same Homply polynomial for $n \leqslant 4$ and the same Kauffman polynomial for $n \leqslant 3$ [17]. In this context, it would be interesting to know whether the n-rotants of knots are m-equivalent for appropriate m (depending on n), and if this is the case, whether one can pass from a knot of the pair of rotants to another one of this pair via the particular C_{m+1}-moves just indicated in Proposition 2.2.

3. Band equivalence of knots. In the study of geometric properties of knot invariants of finite order, Kalfagianni and Lin [9] have introduced for each $n \geqslant 2$ several classes of knots, called n-hyperbolic, n-elliptic and n-parabolic. All these knots are characterized by the property that they bound in S^3 the regular Seifert surfaces having certain geometric properties and called n-hyperbolic, n-elliptic and n-parabolic surfaces, respectively. Thus Vassiliev invariants can be thought of the obstructions for knots to bound the regular Seifert surface of the corresponding type. One of the main result of Kalfagianni and Lin in [9] is that all n-hyperbolic, n-elliptic and n-parabolic knots are n-trivial. Kalfagianni and Lin proved also [9] that for each $n \geqslant 2$ all n-hyperbolic and n-elliptic knots have trivial Alexander polynomial, so they do not exhaust entirely the class of n-trivial knots. For example, there exists a 2-parabolic knot with the non-trivial Alexander polynomial. It is unknown likely if n-hyperbolic, n-elliptic and n-parabolic knots exhaust all the class of n-trivial knots. In the present paper, we consider Seifert surfaces for knots (not necessarily regular) in S^3 represented in the disc-band form and some specific moves on them, the band-analogues of insertions in knots of pure braids commutators.

Let K be a knot in $\mathbb{R}^3 \simeq \mathbb{R}^2 \times \mathbb{R}$ and S a Seifert surface for K given in the disc-band form in the projection to the plane $F = \mathbb{R}^2 \times \{0\}$. Suppose that in some disc $D^2 = I \times I \subset F$ the projection of S looks like the geometric trivial braid 1_m with each strand s_i, $i = 1, \ldots, m$, being replaced by a thin band b_i (see Fig. 5,a). Each band b_i, $i = 1, \ldots, m$, is bounded by two strands u_{2i-1} and u_{2i} (with opposite orientations). All the strands $u_j, 1 \leqslant j \leqslant 2m$, taken together with the appropriate orientations, give a diagram of the trivial braid $1_{2m} \in B_{2m}$ positioned in a disc $D^2 \subset \mathbb{R}^3$. Let p be a geometric pure braid representing an element of the group $LCS_n(P_m)$, where $m \geqslant 3$. We can also thicken the strands s'_i, $i = 1, \ldots, m$ of the braid p, replacing s'_i with a thin band b'_i, respecting all under-crossings and over-crossings of the strands s'_i of p. Now we can replace the part of the projection of Seifert surface S contained in the disc D^2 and consisting of m separate bands $b_i, i = 1, \ldots, m$ (the "thickened" braid 1_{2m}) with the "thickened" braid p consisting of m band-strands $b'_i, i = 1, \ldots, m$. To this operation on Seifert surfaces represented in the disc-band form there corresponds the operation of insertion of the "doubles" of pure braids in a knot diagram K [14]. The orientation of the boundary components u'_{2i-1} and u'_{2i} of b'_i is inherited from the orientation of the surface S (see Fig. 5,b). Denote by K_p and S_p, respectively, the surgered knot and the Seifert surface bounded it in $\mathbb{R}^3$. We shall say that S_p is obtained from S by inserting the thickened pure braid p or the "double" of p in it. The inverse move on the Seifert surfaces, represented in the disc-band form, and on the knots bounded by them, consists in replacement the "double" of the pure braid $p \in LCS_n(P_m), m \geqslant 3$, with the "double" of the trivial one with the same number of strands. Both the moves on Seifert surfaces for knots are called n-elementary moves with m bands, where $m \geqslant 3$. Any two knots K and L are called $LCS_n(P_m)$-band-equivalent, where $m \geqslant 3$, if there is a sequence of knots $K = K_1, K_2, \ldots, K_{l-1}, K_l = L$ and pairs of Seifert surfaces

$S_1, S'_1, S_2, S'_2, \dots, S_{l-1}, S'_{l-1}, S_l$, where S_i and S'_i bound the knot K_i, such that each Seifert surface S_{i+1} is obtained from S'_i by an n-elementary move with m bands or an isotopy. It follows from Proposition 2.3 of [14] that any two $LCS_n(B_m)$-band-equivalent knots are $LCS_n(P_m)$-equivalent. The converse implication does not hold, of cause.

3.1. Proposition. *If the knots K and L are $LCS_n(P_m)$-band-equivalent for some $n \geqslant 2$ and $m \geqslant 3$, then they share the same Alexander polynomial.*

Proof. Suppose that K and L are $LCS_n(P_m)$-band-equivalent for some $n \geqslant 2$ and $m \geqslant 3$. The Alexander polynomial of any knot K' in S^3 is determined by S-equivalence class of Seifert matrices for K'. Let S be any Seifert matrix for K', represented in the disc-band form. It is easy to see that an n-elementary move with m strands on S does not affect its Seifert matrix M. On the other hand, passing from any Seifert surface of K' to an other one, for a knot of the same knot type as K', does not also change the S-equivalence class of M. It follows that K and L have the S-equivalent Seifert matrices, completing the proof.

3.1. Corollary. *If a knot K is $LCS_n(P_m)$-band-equivalent to a trivial one for some $m \geqslant 3$, where $n \geqslant 2$, then K is $(n-1)$-trivial and has the trivial Alexander polynomial.*

Therefore, all the knots which are $LCS_n(P_m)$-band-equivalent to a trivial one for some $m \geqslant 3$, form a class of "geometric" $(n-1)$-trivial knots with trivial Alexander polynomial. Suppose that two knots K and L are related via an n-elementary band move. By Proposition 2.3 of [14], $L-K$, considered modulo $\mathcal{K}'_{n+1}$, can be represented as some integral linear combination of n-singular knots $\sum_i \lambda_i K_i$. Since the map $\varphi: \mathcal{A}_n \otimes \mathbb{Q} \to \mathcal{K}'_n/\mathcal{K}'_{n+1}$ is an isomorphism of vector spaces (see [2]), the diagram $D = \varphi^{-1}(\sum_i \lambda_i K_i)$ in $\mathcal{A}_n \otimes \mathbb{Q}$ is determined uniquely. Then it is not difficult to check directly that in $\mathcal{A}_n \otimes \mathbb{Q}$ the diagram D is a sum of trivalent diagrams, the internal graphs of which have the negative Euler characteristic.

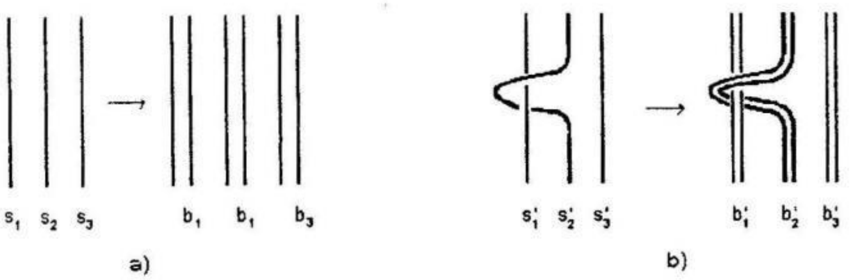

Fig. 5

Question. How does the class of "geometric" n-trivial knots described by Corollary 3.1 relate to the classes of n-hyperbolic and n-elliptic knots?

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n-ТРИВІАЛЬНІ ВУЗЛИ ТА ПОЛІНОМ АЛЕКСАНДЕРА

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Для кожного натурального $n \geqslant 1$, використовуючи комутатори групи чистих кос P_{2n}, побудовано $(2n-1)$-тривіальні вузли з нетривіальним поліномом Александера. Сформульовано достатню умову тривіальності полінома Александера $(n-1)$-тривіального вузла при $n > 2$. Описано клас "геометричних" $(n-1)$-тривіальних вузлів, $n > 1$ з тривіальним поліномом Александера.

Ключові слова: інваріант скінченного типу, комутатор кос, поверхня Зайферта, поліном Александера, тривалентна діаграма, n-тривіальний вузол.

Стаття надійшла до редколегії 15.03.2002

Прийнята до друку 14.03.2003