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ON TOTALLY BOUNDED SEMIGROUPS OF CONTINUOUS MAPPINGS

Igor PROTASOV

Kyiv Taras Shevchenko National University,
64 Volodymyrska Str. 01033 Kyiv, Ukraine

A semigroup S with the identity e endowed with a topology is called *left (right) totally bounded* if, for every neighborhood U of e, there exists a finite subset F such that $S = FU$ $(S = UF)$. For a topological space X, denote by $C(X)$ the semigroup of all continuous selfmappings of X with the topology of pointwise convergence. We give some sufficient conditions on X under which $C(X)$ is either left or right totally bounded.

Key words: totally bounded semigroup, distal group, homogeneous space.

For a topological space X, let $C(X)$ and $H(X)$ be the semigroup of all continuous selfmappings of X and a group of all homeomorphisms of X with the topology of pointwise convergence (i.e. the topology inherited from the Tychonov product X^X). Every subgroup of $H(X)$ is called a group of homeomorphisms of X. It is well known [1] that $C(X)$ is a semitopological semigroup (i.e. all shifts $x \mapsto sx$, $x \mapsto xs$, $s \in C(X)$ are continuous).

A semigroup S with the identity e endowed with a topology is called *left (right) totally bounded* if, for every neighborhood U of e, there exists a finite subset F such that $S = FU$ $(S = UF)$.

We give some sufficient conditions on X under which $C(X)$ is left or right totally bounded. In particular, we show that $C(X)$ is left and right totally bounded for a Cantor cube X of any weight. Under the Cantor cube of weight α we understand the product $\{0,1\}^\alpha$ of α copies of the discrete space $\{0,1\}$. On the other hand, $H(X)$ is neither left nor right totally bounded for every Cantor cube X of infinite weight.

Theorem 1. *Let X, Y be compact spaces such that X admits a base of the topology consisting of clopen subsets homeomorphic to Y. Then $C(X)$ is left totally bounded.*

Proof. Choose any distinct elements $a_1, a_2, \dots, a_n \in X$ and neighborhoods $A_1, A_2, \dots, A_n$ of $a_1, a_2, \dots, a_n$. Put

$$S = \{s \in C(X) : s(a_i) \in A_i \text{ for } i \leqslant n\}.$$

It suffices to find a finite subset $K \subseteq C(X)$ such that $C(X) = KS$.

For every element $x = (x_1, x_2, \dots, x_n) \in X^n$, choose clopen neighborhoods $V_1, V_2, \dots, V_n$ of $x_1, x_2, \dots, x_n$, homeomorphic to Y. We may suppose that X is infinite, so pick pair-wise disjoint clopen subsets $V_1', V_2', \dots, V_n'$ homeomorphic to Y such that

$$V_i' \subseteq V_i \text{ and } \{a_1, a_2, \dots, a_n\} \cap V_i' = \emptyset$$

for every $i \leqslant n$. Then choose pair-wise disjoint clopen neighborhoods $U_1, U_2, \dots, U_n$ of $a_1, a_2, \dots, a_n$ homeomorphic to Y such that $U_1 \subseteq A_1$, $U_2 \subseteq A_2, \dots, U_n \subseteq A_n$ and

$$(U_1 \cup U_2 \cup \dots \cup U_n) \cap (V_1' \cup V_2' \cup \dots \cup V_n') = \emptyset.$$

Fix any homeomorphisms $h_i : U_i \to V_i$, $t_i : V_i' \to U_i$, $i \leqslant n$, and define the mapping $g_x \in C(X)$ letting

$$g_x(a) = \begin{cases} h_i(a), & \text{if } a \in U_i, \\ t_i(a), & \text{if } a \in V_i', \\ a, & \text{otherwise.} \end{cases}$$

Put $V_x = V_1 \times V_2 \times \dots \times V_n$, consider the clopen cover $\{V_x : x \in X^n\}$ of X^n and choose its finite subcover $\{V_x : x \in F\}$. Put $K = \{g_x : x \in F\}$ and show that $C(X) = KS$.

Let f be an arbitrary element of $C(X)$. Choose $x = (x_1, x_2, \dots, x_n) \in F$ with $(f(a_1), f(a_2), \dots, f(a_n)) \in V_x$. Show that $f = g_x s$ for some element $s \in S$.

For every $i \leqslant n$ choose a clopen neighborhood W_i of a_i such that $f(W_i) \subseteq V_i$ and $W_i \subseteq U_i$. First define the mapping s on $W_1 \cup W_2 \cup \dots \cup W_n$. If $a \in W_i$, put $s(a) = h_i^{-1} f(a)$. Then $f(a) = g_x(s(a))$ and $s(a_i) \in U_i \subseteq A_i$.

To extend the mapping s onto X consider three cases in which $a \notin W_1 \cup W_2 \cup \dots \cup W_n$.

Case 1. $f(a) \notin (U_1 \cup U_2 \cup \dots \cup U_n) \cup (V_1' \cup V_2' \cup \dots \cup V_n')$. Put $s(a) = f(a)$ and note that $f(a) = g_x(s(a))$.

Case 2. $f(a) \in U_1 \cup U_2 \cup \dots \cup U_n$. If $f(a) \in U_i$, put $s(a) = t_i^{-1} f(a)$. Then $f(a) = g_x(s(a))$.

Case 3. $f(a) \in V_1' \cup V_2' \cup \dots \cup V_n'$. If $f(a) \in V_i'$, put $s(a) = h_i^{-1} f(a)$. Then $f(a) = g_x(s(a))$.

By the construction $f = g_x s$ and $s \in S$. □

Question 1. *Is the semigroup $C(X)$ left totally bounded for every zero-dimensional compact homogeneous space?*

A topological space X is called *homogeneous* if, for any points $x_1, x_2 \in X$, there exists a homeomorphism h of X with $h(x_1) = x_2$.

Theorem 2. *Let X be a topological space such that every point of X has a base of clopen neighborhoods homeomorphic to X. Then $C(X)$ is right totally bounded.*

Proof. Take any distinct points $x_1, x_2, \dots, x_n \in X$, choose disjoint open neighborhoods $U_1, U_2, \dots, U_n$ of $x_1, x_2, \dots, x_n$ and put

$$S = \{s \in C(X) : s(x_i) \in U_i \text{ for } i \leqslant n\}.$$

Take a clopen subset $V \subset U_1$ homeomorphic to Y with $x_1 \notin V$. Fix any homeomorphism $f : X \to V$ and show that $C(X) = Sf$. Take any mapping $h \in C(X)$ and define a continuous mapping $s' : V \to X$ such that $h = s'f$. Since X is zero-dimensional, s' can be extended to a mapping $s \in C(X) \cap S$. Hence, $h = sf$. □

Question 2. *Is the semigroup $C(X)$ right totally bounded for every zero-dimensional homogeneous space X?*

Theorem 3. *A semigroup $C(X)$ is left and right totally bounded for the Cantor cube X of any weight.*

Proof. Apply Theorems 1, 2. □

Let X be a topological space and let H be a group of homeomorphisms of X. The pair of distinct points $x_1, x_2 \in X$ is called *H-proximal*, if there exists a point $x \in X$ such that, for every neighborhood V of x there is $h \in H$ with $h(x_1) \in V$, $h(x_2) \in V$. If there are no H-proximal points in X, then H is called *distal*.

Theorem 4. *Let X be a topological space, H be a group of homeomorphisms of X which acts transitively on X. If H is left totally bounded then H is distal.*

Proof. Assume the converse. Since H acts transitively on X, there exist two distinct points $x_1, x_2 \in X$ such that, for every nonempty open subset U of X, there is a homeomorphism $h \in H$ with $h(x_1) \in U$, $h(x_2) \in U$. Choose disjoint open neighborhoods U_1, U_2 of x_1, x_2 and put

$$S = \{s \in H : s(x_1) \in U_1, s(x_2) \in U_2\}.$$

By assumption, there exists a finite subset $F = \{f_1, f_2, \dots, f_n\}$ of H such that $H = FS$. Put $V_1 = f(U_1)$. If $V_1 \cap f_2(U_1) \neq \emptyset$ put $V_2 = V_1 \cap f_2(U_1)$, otherwise, $V_2 = V_1$. If $V_2 \cap f_3(U_1) \neq \emptyset$ put $V_3 = V_2 \cap f_3(U_1)$, otherwise, $V_3 = V_2$. After n steps put $V = V_n$. By the construction, the subset V has the following property

$$(*) \quad \text{if } V \cap f_k(U_1) \neq \emptyset \text{ then } V \subseteq f_k(U_1),\ k \in \{1, 2, \dots, n\}.$$

Since H is not distal, there exists $h \in H$ such that $h(x_1) \in V$, $h(x_2) \in V$. Choose $f \in F$ and $s \in S$ with $h = fs$. Taking into account that $h(x_1) = f(s(x_1))$ and $h(x_1) \in V$, $s(x_1) \in U_1$ we conclude that $V \cap f(U_1) \neq \emptyset$. By the condition $(*)$, $V \subseteq f(U_1)$. Since $h(x_2) = f(s(x_2))$ and $h(x_2) \in V$, $s(x_2) \in U_2$ we get $V \cap f(U_2) \neq \emptyset$, a contradiction with $f(U_1) \cap f(U_2) = \emptyset$ and $V \subseteq f(U_1)$. □

Question 3. *Let X be a compact space and let H be a distal group of homeomorphisms of X which acts transitively on X. Is H left totally bounded?*

Theorem 5. *Let X be an infinite topological space and let H be a group of homeomorphisms which acts n-transitively on X for every natural number n. Then H is not right totally bounded.*

Proof. Fix any point $x \in X$ and choose a neighborhood U of x such that the subset $X \setminus U$ is infinite. Put $S = \{s \in H : s(x) \in U\}$ and suppose that there exists a finite subset F of H such that $H = SF$. Let $F^{-1}(x) = \{y_1, y_2, \dots, y_k\}$. Take any distinct point $z_1, z_2, \dots, z_k \in X \setminus U$. Choose $h \in H$ such that

$$h(y_1) = z_1, h(y_2) = z_2, \dots, h(y_k) = z_k.$$

Since $H = SF$, there exists $f \in F$ such that $h = sf$ and thus $hf^{-1} \in S$. Since $f^{-1}(x) \in \{y_1, y_2, \dots, y_k\}$ we get a contradiction: $hf^{-1} \notin S$. □

Theorem 6. *The group $H(X)$ of all homeomorphisms of a Cantor cube X of infinite weight is neither left nor right totally bounded.*

Proof. Apply Theorems 4, 5. □

Theorem 7. *Let X be an infinite discrete space. Then $C(X)$ is right totally bounded, $H(X)$ is neither left nor right totally bounded.*

Proof. Take any element $x \in X$ and put $S = \{s \in C(X) : s(x) = x\}$. Take any finite subset $F \subset C(X)$. Since $FS(x) = F(x)$ and the subset $F(x)$ is finite, we get $C(X) \neq FS$ so $C(X)$ is not left totally bounded. Put $S' = S \cap H$. The same argument shows that S' is a subgroup of infinite index in $H(X)$ so $H(X)$ is neither left nor right totally bounded.

To show that the semigroup $C(X)$ is right totally bounded, take any distinct elements $x_1, x_2, \dots, x_n \in X$ and put $S = \{s \in C(X) : s(x_i) = x_i \text{ for } i \leqslant n\}$. Fix any bijection $f : X \to X \setminus \{x_1, x_2, \dots, x_n\}$. Clearly, $C(X) = Sf$. □

Theorem 8. *Let X be an infinite discrete space, βX be the Stone-Čech compactification of X. Then $C(\beta X)$ is right but not left totally bounded, $H(\beta X)$ is neither left nor right totally bounded.*

Proof. We prove only that $C(\beta X)$ is right totally bounded. Identify βX with the set of all ultrafilters on X. Given any subset $A \subseteq X$, put $\overline{A} = \{p \in \beta X : A \in p\}$. Then the family $\{\overline{A} : A \in p\}$ is a base of neighborhoods of p. Take any distinct element $p_1, p_2, \dots, p_n \in \beta X$ and pick pairwise disjoint subsets $P_1 \in p_1, P_2 \in p_2, \dots, P_n \in p_n$ such that $|X \setminus (P_1 \cup P_2 \cup \dots \cup P_n)| = |X|$. Put $S = \{s \in C(\beta X) : s(p_i) \in \overline{P_i} \text{ for } i \leqslant n\}$.

Fix any bijection $f : X \to X \setminus (P_1 \cup P_2 \cup \dots \cup P_n)$ and denote by $\overline{f}$ the extension of f to βX. Then, $C(\beta X) = S\overline{f}$. □

Added in Proofs. Recently, T. Banakh and O. Hryniv answered Questions 1 and 2 in negative: they proved that the semigroup $C(X)$ of the paratopological first-countable zero-dimensional homogeneous compactum X constructed by E. van Douwen [2] is neither left nor right totally bounded; moreover, the homeomorhism group $H(X)$ of X is neither left nor left totally bounded in $C(X)$.

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ПРО ЦІЛКОМ ОБМЕЖЕНІ НАПІВГРУПИ НЕПЕРЕРВНИХ ВІДОБРАЖЕНЬ

І. Протасов

Київський національний університет імені Тараса Шевченка,
вул. Володимирська, 64 01033 Київ, Україна

Напівгрупа S з одиницею e, наділена топологією, називається *цілком обмеженою (зліва) справа*, якщо для кожного околу U одиниці e існує така скінченна підмножина F, що $S = FU$ $(S = UF)$. Через $C(X)$ позначимо напівгрупу всіх неперервних відображень топологічного простору X в себе, наділену топологією поточкової збіжності. Знайдено достатні умови на топологічний простір X, при яких напівгрупа $C(X)$ цілком обмежена зліва чи справа.

Ключові слова: цілком обмежена напівгрупа, дистальна група, однорідний простір.

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