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ON H-CLOSED PARATOPOLOGICAL GROUPS

Oleksandr RAVSKY

Ivan Franko National University of Lviv, 1 Universitetska Str. 79000 Lviv, Ukraine

A Hausdorff paratopological group is H-closed if it is closed in every Hausdorff paratopological group containing it as a paratopological subgroup. We give a criterion of the H-closedness of an abelian topological group for some classes of abelian paratopological groups are obtained simple criteria of the H-closedness.

Key words: paratopological group, minimal topological group, absolutely closed topological group.

All topological spaces considered in this paper are Hausdorff, if the opposite is not stated. We shall use the following notations. Let A be a subset of a group and n be an integer. Put $A^n = \{a_1 a_2 \cdots a_n : a_i \in A\}$ and $nA = \{a^n : a \in A\}$. For a group topology τ the closure of a set A is denoted by $\overline{A}^{\tau}$ and $\mathcal{B}_\tau$ stands for a neighborhood base of the unit.

A topological space X is *C-closed* in a class C of topological spaces provided X is closed in any space $Y \in C$ containing X as a subspace. It is well known that when C is the class of Tychonoff spaces, then the C-closedness coincides with the compactness. For the class of Hausdorff spaces the following conditions for a space X are equivalent [1, 3.12.5]:

1) The space X is H-closed;

2) If $\mathcal{V}$ is a centered family of open subsets of X then $\bigcap\{\overline{V} : V \in \mathcal{V}\} \neq \varnothing$;

3) Every ultrafilter in the family of all open subsets of X is convergent;

4) Every cover $\mathcal{U}$ of the space X contains a finite subfamily $\mathcal{V}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$.

The group G with a topology τ is called a *paratopological group* if the multiplication on the group G is continuous. If the inversion on the group G is continuous, then (G, τ) is a *topological group*. A group (G, τ) is paratopological if and only if the following conditions (known as Pontrjagin conditions) are satisfied for a neighborhood base $\mathcal{B}$ at unit e of G [4,5]:

1. $\bigcap\{UU^{-1} : U \in \mathcal{B}\} = \{e\}$;
2. $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B}) : W \subset U \cap V$;
3. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^2 \subset U$;
4. $(\forall U \in \mathcal{B})(\forall u \in U)(\exists V \in \mathcal{B}) : uV \subset U$;
5. $(\forall U \in \mathcal{B})(\forall g \in G)(\exists V \in \mathcal{B}) : g^{-1}Vg \subset U$.

The paratopological group G is a topological group if and only if

6. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^{-1} \subset U$.

A topological group is *absolutely closed* if it is closed in every Hausdorff topological group containing it as a topological subgroup. A topological group G is closed in the class of topological groups if and only if it is *Rajkov-complete*, that is complete with respect to the upper uniformity which is defined as the least upper bound $\mathcal{L} \vee \mathcal{R}$ of the left and the right uniformities on G. Recall that the sets $\{(x, y) : x^{-1}y \in U\}$, where U runs over a base at unit of G, constitute a base of entourages for the left uniformity $\mathcal{L}$ on G. In the case of the right uniformity $\mathcal{R}$, the condition $x^{-1}y \in U$ is replaced by $yx^{-1} \in U$. The *Rajkov completion* $\hat{G}$ of a topological group G is the completion of G with respect to the upper uniformity $\mathcal{L} \vee \mathcal{R}$. For every topological group G the space $\hat{G}$ has a natural structure of a topological group. The group $\hat{G}$ can be defined as a unique (up to an isomorphism) Rajkov complete group containing G as a dense subgroup.

A paratopological group is *H-closed* if it is closed in every Hausdorff paratopological group containing it as a subgroup. In the present section we shall consider H-closed paratopological groups.

Question. Let G be a regular paratopological group which is closed in every regular paratopological group containing it as a subgroup. Is G H-closed?

1. Lemma. *Let (G, τ) be a paratopological group. If there exists a paratopology σ on the group $G \times \mathbb{Z}$ such that $\sigma|G \subset \tau$ and $e \in \overline{(G,1)}^{\sigma}$ then (G, τ) is not H-closed.*

Proof. We shall build the paratopology ρ on the group $G \times \mathbb{Z}$ such that $\rho|G = \tau$ and $\overline{G}^{\rho} \neq G$. Determine a base of unit $\mathcal{B}_\rho$ as follows. Let $S = \{(x, n) : x \in G, n > 0\}$. For every neighborhoods $U_1 \in \tau$, $U_2 \in \sigma$ such that $U_1 \subset U_2$ put $(U_1, U_2) = U_1 \cup (U_2 \cap S)$. Put $\mathcal{B}_\rho = \{(U_1, U_2) : U_1 \in \mathcal{B}_\tau, U_2 \in \mathcal{B}_\sigma\}$. Verify that $\mathcal{B}_\rho$ satisfies the Pontrjagin conditions.

1. It is satisfied since $(U_1, U_2) \subset U_2$.

2. It is satisfied since $(U_1 \cap V_1, U_2 \cap V_2) \subset (U_1, U_2) \cap (V_1, V_2)$.

3. Select $V_2 \in \mathcal{B}_\sigma$ and $V_1 \in \mathcal{B}_\tau$ such that $V_2^2 \subset U_2$, $V_1^2 \subset U_1$ and $V_1 \subset V_2$. Let $y_1, y_2 \in (V_1, V_2)$. The following cases are possible

A. $y_1, y_2 \in V_1$. Then $y_1y_2 \in V_1^2 \in (U_1, U_2)$.

B. $y_1 \in V_1, y_2 \in V_2 \cap S$. Then $y_1y_2 \in V_2^2 \in U_2$. Since $y_1 \in G$ and $y_2 \in S$, we get $y_1y_2 \in S$ and hence $y_1y_2 \in U_2 \cap S$.

C. $y_1 \in V_2 \cap S, y_2 \in V_1$ is similar to the case B.

D. $y_1, y_2 \in V_2 \cap S$. Since S is a semigroup, $y_1y_2 \in U_2 \cap S$.

4. Let $y \in (U_1, U_2)$. There exist $V_2 \in \mathcal{B}_\sigma$ and $V_1 \in \mathcal{B}_\tau$ such that $yV_2 \subset U_2$ and $V_1 \subset V_2$. The following cases are possible:

A. $y \in U_1$. We may suppose that $yV_1 \subset U_1$. Since $y \in G$, $y(V_2 \cap S) \subset U_2 \cap S$.

B. $y \in U_2 \cap S$. Since $V_1 \subset G$ then $yV_1 \in U_2 \cap S$. Since S is a semigroup and $y \in S$ then $y(V_2 \cap S) \subset U_2 \cap S$. Therefore $y(V_1, V_2) \subset (U_1, U_2)$.

5. Let $(g, n) \in G \times \mathbb{Z}$. There exist $V_2 \in \mathcal{B}_\sigma$ and $V_1 \in \mathcal{B}_\tau$ such that $V_1 \subset V_2$, $g^{-1}V_1g \subset U_1$ and $g^{-1}V_2g \subset U_2$. Then $(g, n)^{-1}(V_1, V_2)(g, n) = g^{-1}(V_1, V_2)g = g^{-1}(V_1 \cup (V_2 \cap S)g \subset U_1 \cup (U_2 \cap S) = (U_1, U_2)$.

Therefore (H, ρ) is a paratopological group. Taking into account that $(U_1, U_2) \cap G = U_1$ we get $\rho|G = \tau$.

Since $e \in \overline{(G,1)}^{\sigma}$, for every $U_2 \in \mathcal{B}_\sigma$ there exists $g \in G$ such that $(g, 1) \in U_2$. Then $g \in (e, -1)(U_2 \cap S)$ and therefore $(e, -1) \in \overline{G}^{\rho}$. $\square$

A group topology τ_1 on the group G is called *complementable* if there exist a nondiscrete group topology τ_2 on G and neighborhoods $U_i \in \tau_i$ such that $U_1 \cap U_2 = \{e\}$. In this case we say that τ_2 is a *complement* to τ_1. Proposition 1.4 from [1] implies that in this case the topology $\tau_1 \wedge \tau_2$ is Hausdorff.

A *Banach measure* is a real function μ defined on the family of all subsets of a group G and satisfies the following conditions:

a) $\mu(G) = 1$;

b) if $A, B \subset G$ and $A \cap B = \varnothing$ then $\mu(A \cup B) = \mu(A) + \mu(B)$;

c) $\mu(gA) = \mu(A)$ for every element $g \in G$ and for every subset $A \subset G$.

2. Lemma. [3, p.37]. *Let G be an abelian group and let μ be a Banach measure on G. Let τ be a group topology on G. Suppose that the set nG is U-unbounded for some natural number n and for some neighborhood U of zero in (G,τ). Then $\mu(\{x \in G : nx \in gW\}) = 0$ for every element $g \in G$ and for every neighborhood W of zero satisfying $WW^{-1} \subset U$.*

Let U be a neighborhood of zero in a topological group (G, τ). We say that a subset $A \subset G$ is U-unbounded if $A \not\subset KU$ for every finite subset $K \subset G$.

Given any elements $a_0, a_1, \ldots, a_n$ of an abelian group G put

$$Y(a_0, a_1, \ldots, a_n) = \{a_0^{x_0} a_1^{x_1} \cdots a_n^{x_n} : 0 \leqslant x_i \leqslant i+1, i \leqslant n, \sum x_i^2 > 0\},$$

$$X(a_0, a_1, \ldots, a_n) = \{a_0^{x_0} a_1^{x_1} \cdots a_n^{x_n} : -(i+1) \leqslant x_i \leqslant i+1, i \leqslant n\}.$$

Then $X(a_0, a_1, \ldots, a_n) = Y(a_0, a_1, \ldots, a_n) Y(a_0, a_1, \ldots, a_n)^{-1}$.

3. Lemma. *Let (G, τ) be an abelian paratopological group of infinite exponent. If there exists a neighborhood $U \in \mathcal{B}_\tau$ such that the group nG is UU^{-1}-unbounded for every natural number n, then the paratopological group (G, τ) is not H-closed.*

Proof. Define a seminorm $|\cdot|$ on the group G such that $|xy| \leqslant |x| + |y|$ for all $x, y \in G$. Suppose that there exists a non periodic element $x_0 \in G$. Determine a map $\phi_0 : \langle x_0 \rangle \to \mathbb{Z}$ putting $\phi_0(x_0^n) = n$. Since $\mathbb{Q}$ is a divisible group, the map ϕ_0 can be extended to a homomorphism $\phi : G \to \mathbb{Q}$. Put $|x| = |\phi(x)|$ for every element $x \in G$. If G is periodic, then put $|e| = 0$ and $|x| = [\ln ord(x)] + 1$, where $ord(x)$ denotes the order of the element x.

Fix a neighborhood $V \in \mathcal{B}_\tau$ such that $V^2 \subset U$ and put $W = VV^{-1}$. We shall construct a sequence $\{a_n\}$ such that

a) $|a_n| > n$;

b) $W \cap X(a_0, a_1, \ldots, a_n) = \{e\}$;

c) $Y(a_0, a_1, \ldots, a_n) \not\ni e$;

d) if $-n \leqslant k \leqslant n, k \neq 0$ then $a_n^k \notin 2X(a_0, a_1, \ldots, a_{n-1})$.

Take any element $a_0 \notin W$. Suppose that the elements $a_0, \ldots, a_n$ have been chosen to satisfy conditions (a) and (b). Put

$$B_n = \{x \in G : (\forall g \in X(a_0, a_1, \ldots, a_{n-1}))(\forall k \in \mathbb{Z}\backslash\{0\} : -e^{n+1} \leqslant k \leqslant e^{n+1}) : kx \notin gW\}$$

If the group G is periodic, then $|x| > n$ for every element $x \in B_n$. Lemma 2 implies that $\mu(B_n) = 1$. If the group G is not periodic, then the construction of the seminorm $|\cdot|$ implies that $\mu(\{x \in G : |x| \leqslant n\}) = \mu(\phi^{-1}[-n;n]) = 0$. In both cases there exists an element $a_n \in B_n$ such that $|a_n| > n$. Then $W \cap X(a_0, a_1, \dots, a_n) = \varnothing$. Considering a subsequence and applying condition (a) we can satisfy conditions (c) and (d) also.

Define a base $\mathcal{B}_{\tau\{a_n\}}$ at the unit of group topology $\tau\{a_n\}$ on the group $G \times \mathbb{Z}$ as follows. Put $A_n^+ = \{(e,0)\} \cap \{(a_k, 1) : k \geqslant n\}$. For every increasing sequence $\{n_k\}$ put $A[n_k] = \bigcup_{l \in \mathbb{N}} A_{n_1}^+ \cdots A_{n_l}^+$. Put $\mathcal{B}_{\tau\{a_n\}} = \{A[n_k]\}$. We claim that $(G \times Z, \tau\{a_n\})$ is a zero dimensional paratopological group.

Put $F = \bigcup_{n \in \omega} X(a_0, a_1, \dots, a_n)$. Let $A[n_k] \in \mathcal{B}_{\tau\{a_n\}}, (x, n_x) \notin A[n_k]$. If $x \notin F$, then $(x, n_x)A[n] \cap A[n_k] = \varnothing$. Let $x \in X(a_0, a_1, \dots, a_m)$. Put $m_k = m + k$. Suppose that $(x, n_x)A[m_k] \cap A[n_k] \neq \varnothing$. Select the minimal k such that $(x, n_x)(A_{m_1}^+ \cdots A_{m_k}^+) \cap A[n_k] \neq \varnothing$. Let

$$(*) \qquad (x, n_x)(a_{l_1}, 1) \cdots (a_{l_k}, 1) = (a_{l'_1}, 1) \cdots (a_{l'_{k'}}, 1)$$

and for all i, i' holds $m_i \leqslant l_i \leqslant l_{i+1}$, $n'_i \leqslant l'_{i'} \leqslant l'_{i'+1}$. Remark that a member a_q occurs in each part of the equality (**) no more than q times. If $l_k > l'_{k'}$ then if we move all members which are not equal to $(a_{l_k}, 1)$ from the left side of the equality (*) to the right one, we obtain contradiction to condition (d). The case $l_k < l'_{k'}$ is considered similarly. Therefore $l_k = l'_{k'}$, a contradiction with the choice of k as a minimal number satisfying the equality (*). It is showed similarly that if $x \neq e$ and $m_k = m + k + 1$, then $(x, n_x) \notin A[m_k]$. If $x = e$ and $n_x \neq 0$, then the condition (c) implies that $A[n] \not\ni (x, n_x)$. Hence Pontrjagin condition 1 for $\mathcal{B}_{\tau\{a_n\}}$ is satisfied. Since $A[n_{2k}]^2 \subset A[n_k]$, Pontrjagin condition 3 is satisfied. All the other Pontrjagin conditions are obvious.

Condition (b) implies that $A[n]A[n]^{-1} \cap VV^{-1} = \{(e,0)\}$. Therefore the topology $\tau\{a_n\}_g$ is a complement to the topology $(\tau \times \{0\})_g$, where $\tau \times \{0\}$ is the product topology on the group $(G, \tau) \times \mathbb{Z}$. Therefore the topology $\sigma = \tau\{a_n\}(\tau \times \{0\})$ is Hausdorff. Since $(e, 0) \in \overline{(G,1)}^{\tau\{a_n\}} \subset \overline{(G,1)}^{\sigma}$ we can apply Lemma 1 to show that (G, τ) is not H-closed. □

We shall need the following lemma.

4. Lemma. *Let G be a paratopological group and H be a normal subgroup of G. If H and G/H are topological groups then so is the group G.*

Proof. Let U be an arbitrary neighborhood of the unit. There exist neighborhoods V, W of the unit such that $V \subset U$, $(V^{-1})^2 \cap H \subset U$ and $W \subset V$, $W^{-1} \subset VH$. If $x \in W^{-1}$, then there exist elements $v \in V, h \in H$ such that $x = vh$. Then $h = v^{-1}x \in V^{-1}W^{-1} \cap H \subset U$. Therefore $x \in VU \subset U^2$. Hence G is a topological group. □

The following criterion was suggested by T. Banakh.

5. Theorem. *An abelian topological group (G, τ) is H-closed if and only if (G, τ) is Rajkov complete and for every group topology $\sigma \subset \tau$ on G the quotient group $\hat{G}/G$ is periodic, where $\hat{G}$ is the Rajkov completion of the group (G, σ).*

Proof. Suppose that there exists a group topology $\sigma \subset \tau$ on G such that the quotient group $\hat{G}/G$ is not periodic, where $\hat{G}$ is the Rajkov completion of the group

(G,σ). Select a non periodic element $x \in \hat{G}$ such that $\langle x\rangle \cap G = \{e\}$. Then $G \times \langle x\rangle$ is naturally isomorphic to the group $G \times \mathbb{Z}$ and Lemma 1 implies that the group (G,τ) is not H-closed.

Let a paratopological group (H,τ') contains (G,τ) as non closed subgroup. Since G is abelian, $\overline{G}$ is an abelian semigroup. Choose an arbitrary element $x \in \overline{G}\backslash G$. Then the group hull $F = \langle G, x\rangle$ with the topology $\tau'|F$ is an abelian paratopological group. Then the group G is dense in $(F, {\tau'}_g)$. Since the Rajkov completion $\hat{F}$ of the topological group $(F, \tau'|F_g)$ is periodic, there exists a natural number n such that $x^n \in G$. Therefore $F^n \subset G$. Lemma 4 implies that F is a topological group and therefore G is closed in $(F, {\tau'}_g)$, a contradiction. □

6. Corollary. *A Rajkov completion of a isomorphic condensation of H-closed abelian topological group is H-closed.*

7. Proposition. *Let G be a Rajkov complete topological group, H be H-closed paratopological subgroup of the group G. If a group G/H has finite exponent then G is an H-closed paratopological group.*

Proof. Select a number n such that $g^n \in H$ for every element $g \in G$. Let $F \supset G$ be a paratopological group. Since H is closed in F then for every element $g \in \overline{G}$ we get $g^n \in H$. Denote the continuous maps $\phi : \overline{G} \to \overline{G}$ as $\phi(g) = g^{n-1}$ and $\psi : \overline{G} \to H$ as $\psi(g) = (g^n)^{-1}$. Then for every element $g \in \overline{G}$ we get $g^{-1} = \phi(g)\psi(g)$ and hence the inversion on the group $\overline{G}$ is continuous. Since $\overline{G}$ is a topological group and G is Rajkov complete, $\overline{G} = G$. □

8. Proposition. *Let G be a paratopological group and K be a compact normal subgroup of the group G. If the group G/K is H-closed then the group G is H-closed.*

Proof. Suppose that there exists a paratopological group F containing the group G such that $\overline{G} \neq G$. Since K is compact then F/K is a Hausdorff paratopological group by Proposition 1.13 from [4]. Let $\pi : F \to F/K$ be the quotient homomorphism map. Then $\overline{G/K} \supset \pi(\overline{\pi^{-1}(G/K)}) \supset \pi(\overline{G}) \neq \pi(G) = G/K$. This implies that the group G/K is not H-closed, a contradiction. □

Let G be a topological group, N be a closed normal subgroup of the group G. If N and G/N are Rajkov complete, then so is the group G [5]. This suggests the following

9. Question. Let G be a paratopological group, N be a closed normal subgroup of the group G and the groups N and G/N are H-closed. Is the group G H-closed?

Let (G,τ) be a paratopological group. Then there exists the finest group topology τ_g coarser than τ (see [2]), which is called *the group reflection* of the topology τ.

10. Proposition. *Let (G,τ) be an abelian paratopological group. If (G,τ_g) is H-closed then (G,τ) is H-closed. If (G,τ) is H-closed and (G,τ_g) is Rajkov complete then (G,τ_g) is H-closed.*

Proof. Suppose that the group (G,τ_g) is H-closed and (G,τ) is not. Suppose a paratopological group $(H,\hat{\tau})$ contains (G,τ) as a non closed subgroup. Without loss of generality we may suppose that there exists an element $x \in H\backslash G$ such that $H = \langle G, x\rangle$ and the group H is abelian. Let $\hat{\tau}_g$ be the group reflection of the topology $\hat{\tau}$. Since $\hat{\tau}_g|G \subset \tau_g$, Theorem 5 implies that the group H/G is periodic. Without loss of generality we may suppose that $x^p \in G$ for some prime p.

Denote by $\mathcal{B}_{\hat{\tau}}$ the family of neighborhoods at unit in the topology τ. Let $U \in \mathcal{B}_{\hat{\tau}}$. If $U \cap xG = \varnothing$ then there exists a neighborhood V of unit such that $V^p \subset U$ and thus $V \subset G$ and G is open in $(H, \hat{\tau})$. Therefore a set $\mathcal{F} = \{x^{-1}(xG \cap U) : U \in \mathcal{B}_\tau\}$ is a filter. Let $U \in \mathcal{B}_{\hat{\tau}}$. There exists $V \in \mathcal{B}_{\hat{\tau}}$ such that $V^p \subset U$. Then $(xG \cap V)^p \subset U$. Let $xg \in (xG \cap V)$. Then $x^{-1}(xG \cap V) \subset x^{-1}((xg)^{1-p}(xG \cap V)^p) \cap G \subset x^{-p}g^{1-p}(U \cap G)$ and hence $\mathcal{F}$ is a Cauchy filter in the group (G, τ_g). Let $h \in G$ be a limit of the filter $\mathcal{F}$ on the group (G, τ_g). But then for every neighborhood of the unit U in the topology $\hat{\tau}_g$ we get $U \cap xhU \supset U \cap xh(U \cap G) \neq \varnothing$ and therefore $(H, \hat{\tau}_g)$ is not Hausdorff, a contradiction.

Let (G, τ_g) is Rajkov complete and (G, τ_g) is not H-closed. Then Theorem 5 implies that there exists a group topology $\sigma \subset \tau$ on G such that the quotient group $\hat{G}/G$ of the Rajkov completion $\hat{G}$ of the group (G, σ) is not periodic. Then Lemma 1 implies that a group (G, τ) is not H-closed. □

11. Lemma. *Let topological group (H, σ_H) be a closed subgroup of an abelian topological group (G, τ) and $\sigma_H \subset \tau|H$. Then there exists a group topology $\sigma \subset \tau$ on the group G such that $\sigma|H = \sigma_H$.*

Proof. Let $\mathcal{B}_\tau$ and $\mathcal{B}_{\sigma H}$ be bases of unit of (G, τ) and (H, σ_H) respectively.

Put $\mathcal{B}_\sigma = \{U_1U_2 : U_1 \in \mathcal{B}_\tau, U_2 \in \mathcal{B}_{\sigma H}\}$. Verify that the family $\mathcal{B}_\sigma$ satisfies the Pontrjagin conditions.

2. It is satisfied since $(U_1 \cap V_1)(U_2 \cap V_2) \subset U_1U_2 \cap V_1V_2$.

3. Select $V_2 \in \mathcal{B}_{\sigma H}$ and $V_1 \in \mathcal{B}_\tau$ such that $V_2^2 \subset U_2$, $V_1^2 \subset U_1$. Then $(V_1V_2)^2 \subset U_1U_2$.

4. Let $y \in U_1U_2$. Then there exist points $y_1 \in U_1$ and $y_2 \in U_2$ such that $y = y_1y_2$. Therefore there exist a neighborhoods $V_1 \in \mathcal{B}_\tau$ and $V_2 \in \mathcal{B}_{\sigma H}$ such that $y_iV_i \subset U_i$. Then $yV_1V_2 \subset U_1U_2$.

5. It is satisfied since G is abelian.

6. $(U_1^{-1}U_2^{-1})^{-1} \subset U_1U_2$.

1. Since all others Pontrjagin conditions are satisfied, it suffices to show that $\bigcap \mathcal{B}_\sigma = \{e\}$. Let $x \in G$ and $x \neq e$. If $x \in H$ then there exists $U_2 \in \mathcal{B}_{\sigma H}$ such that $U_2^2 \not\ni x$ and $U_1 \in \mathcal{B}_\sigma$ such that $U_1 \cap H \subset U_2$. Then $U_1U_2 \cap \{x\} = U_1U_2 \cap \{x\} \cap H \subset U_2^2 \cap \{x\} = \varnothing$. If $x \notin H$ then $(G \backslash xH)H \not\ni x$.

Therefore (G, σ) is a topological group. Since $U_1U_2 \cap H = (U_1 \cap H)U_2$, we conclude $\sigma|H = \sigma_H$. □

12. Proposition. *A closed subgroup of an H-closed abelian topological group is H-closed.*

Proof. Let H be a closed subgroup of an H-closed abelian group (G, τ). Then G and H are Rajkov complete. Let $\sigma_H \subset \tau|H$ be a group topology on the group H. Lemma 11 implies that there exists a group topology σ on the group G such that $\sigma|H = \sigma_H$. Let $(\hat{G}, \hat{\sigma})$ be the Rajkov completion of the group (G, σ). Then a closure $\overline{H}^{\hat{\sigma}}$ of the group H in the group $(\hat{G}, \hat{\sigma})$ is a Rajkov completion of the group (H, σ_H). Let $x \in \overline{H}^{\hat{\sigma}}$. Theorem 5 implies that there exists $n > 0$ such that $x^n \in G$. Since $\overline{H}^{\hat{\sigma}} \cap G = H$ then $x^n \in H$. Therefore Theorem 5 implies that H is H-closed. □

13. Proposition. *Let G be a H-closed abelian topological group. Then $K = \bigcap_{n\in\mathbb{N}} \overline{nG}$ is compact and for each neighborhood U of zero in G there exists a natural n with $\overline{nG} \subset KU$.*

Proof. Let Φ be the filter on G generated by base $\{\overline{nG} : n \in \mathbb{N}\}$, and Ψ be an arbitrary ultrafilter on G with $\Psi \supset \Phi$. Let U be a closed neighborhood of the unit in G. Lemma 2 implies that there exists a number n such that the set $\overline{nG}$ is U-bounded. Since $\overline{nG} \in \Phi$ and Ψ is an ultrafilter, there exists $g \in G$ with $gU \in \Psi$. Hence Ψ is a Cauchy filter on G. By the completeness of G, Ψ is convergent. Therefore each ultrafilter Ψ on G with $\Psi \supset \Phi$ converges. In particular each ultrafilter on K is convergent, and since K is closed, K is compact.

To show that there exists a number n with $\overline{nG} \subset KU$, it suffices to prove that $KU \in \Phi$. Assume that $KU \notin \Phi$. Then there exists an ultrafilter $\Psi \supset \Phi$ with $G \setminus KU \in \Psi$. As we have proved, Ψ is convergent. Clearly $\lim \Psi \in K$. Therefore $KU \in \Psi$ which is a contradiction. Hence $KU \in \Phi$, and this completes the proof. □

14. Corollary. *A divisible abelian H-closed topological group is compact.* □

15. Proposition. *Every H-closed abelian topological group is a union of compact groups.*

Proof. Let G be such a group. It suffice to show that every element $x \in G$ is contained in a compact subgroup. Let X be the smallest closed subgroup of G containing the element x. Then $X = \bigcup_{k=0}^{n}(kx + \overline{nX})$ for every natural n. Let U be an arbitrary neighborhood of the zero. By Lemma 15 there exists a natural number n such that nG is U-bounded. Then X is also U-bounded. Hence X is a precompact group. Since X is Rajkov complete then X is compact. □

16. Conjecture. An abelian topological group G is H-closed if and only if G is Rajkov complete and nG is precompact for some natural n.

17. Proposition. *The Conjecture 16 is true provided the group (G, τ) satisfies the following two conditions:*

1) There exists a σ-compact subgroup L of G such that G/L is periodic.

2) There exists a group topology $\tau' \subset \tau$ such that the Rajkov completion $\hat{G}$ of the group (G, τ') is Baire.

Proof. Let G be such a group and $L = \bigcup_{k\in\mathbb{N}} L_k$ be a union of compact subsets L_k. Put $G(n,k) = \{x \in \hat{G} : nx \in L_k\}$ for every natural n and k. Then every set $G(n,k)$ is closed. By Theorem 5 $\hat{G} = \bigcup_{n,k\in\mathbb{N}} G(n,k)$. Since $\hat{G}$ is Baire, there exist natural numbers n and k such that $\operatorname{int} G(n,k) \neq \varnothing$. Then $F = G(n,k) - G(n,k)$ is a neighborhood of the zero. By Corollary 6 the group $\hat{G}$ is H-closed. Put $K = \bigcap_{n\in\mathbb{N}} \overline{n\hat{G}}$. By Proposition 13 there exists a natural m such that $m\hat{G} \subset F + K$. Then $mnG \subset mn\hat{G} \subset L_k - L_k + K$ and hence the group mnG is precompact. □

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ПРО H-ЗАМКНЕНІ ПАРАТОПОЛОГІЧНІ ГРУПИ

О. Равський

Львівський національний університет імені Івана Франка, вул. Університетська, 1 79000 Львів, Україна

Гаусдорфова паратопологічна група називається H-замкненою, якщо вона замкнена у довільній гаусдорфовій паратопологічній групі, що її містить. Отримано критерій H-замкненості абелевої топологічної групи і для деяких класів абелевих паратопологічних груп одержано прості критерії H-замкненості.

Ключові слова: паратопологічна група, мінімальна топологічна група, абсолютно замкнена топологічна група.

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