ВІСНИК ЛЬВІВ. УН-ТУ
Серія мех.-мат. 2003. Вип. 61. С. 180–183

VISNYK LVIV UNIV.
Ser. Mech-Math. 2003. Vol. 61. P. 180–183

УДК 512.552.12

ELEMENTARY ROW TRANSFORMATIONS OVER RINGS OF STABLE RANK $\leqslant 2$

Oleh ROMANIV

Ivan Franko National University of Lviv, 1 Universitetska Str. 79000 Lviv, Ukraine

It is proved that for a ring R of stable rank $\leqslant 2$ any row $(a, b, c) \in R^3$ with $aR + bR + cR = R$ can be reduced to $(1, 0, 0)$ by elementary transformations. Also it is shown that for a right Bezout ring R of stable rank $\leqslant 2$ any row $(a, b, c) \in R^3$ can be reduced to $(\alpha, \beta, 0)$, $\alpha, \beta \in R$, by means of elementary transformations.

Key words: stable rank, Bezout ring, elementary transformations.

In [1] B.V. Zabavsky has posed a problem of a complete description of the rings over which any matrix can be reduced to the diagonal form by elementary transformations. In this note we consider elementary transformations of rows over rings of stable rank 1 and 2.

Throughout this paper R will denote an associative ring with $1 \neq 0$.

Let us introduce the necessary definitions.

An *elementary matrix* with entries from a ring R is a square matrix of one of the following three types [2]: a diagonal matrix with invertible elements on the diagonal; a matrix differing from the identity matrix by a unique nonzero element outside of the main diagonal; permutation matrix, i.e., the identity matrix with its rows or columns permuted arbitrarily.

Denote by $GE_n(R)$ the group generated by elementary $(n \times n)$ matrices.

A ring R is called a *right Bezout ring* [3] if any finitely generated right ideal in R is principal.

A row $(a_1, a_2, \ldots, a_n)$ of elements of a ring R is a *right unimodular row* if there are elements $x_i \in R$, $1 \leqslant i \leqslant n$, with $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1$. A positive integer d is called a *the stable rank* [4] of R if for any unimodular row $(a_1, a_2, \ldots, a_n)$ with $n > d$, there are elements b_i, $1 \leqslant i \leqslant n-1$ such that the row $(a_1 + a_nb_1, \ldots, a_{n-1} + a_nb_{n-1})$ is again right unimodular.

Theorem 1. *Let R be a ring of stable rank 1. Then for any elements $a, b \in R$ with $aR + bR = R$ there is a matrix $M \in GE_2(R)$ such that*

$$(a, b)M = (1, 0).$$

Proof. Since R is a ring of stable rank 1, there are elements $t, w \in R$ such that

$$(a + bt)w = 1.$$

Then

$$(a,b)\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}\begin{pmatrix}1 & w(1-b)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -(a+bt) & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = (1,0).$$

The proof in complete.

Theorem 2. *Let R be a ring of stable rank 2. Then for any elements $a,b,c \in R$ with $aR+bR+cR=R$ there is a matrix $M \in GE_3(R)$ such that*

$$(a,b,c)M = (1,0,0).$$

Proof. Since R is a ring of a stable rank $\leqslant 2$, there exist $x,y,p,q \in R$ such that

$$(a+cx)p + (b+cy)q = 1.$$

Then

$$(a,b,c)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ x & y & 1\end{pmatrix} = (a+cx, b+cy, c)$$

and

$$(a+cx, b+cy, c)\begin{pmatrix}1 & 0 & p(1-c)\\ 0 & 1 & q(1-c)\\ 0 & 0 & 1\end{pmatrix} = (a+cx, b+cy, 1).$$

It is clear that the row $(a+cx, b+cy, 1)$ can be transformed into $(1,0,0)$ by elementary transformations.

The proof in complete.

Corollary 1. *Let R be a ring of stable rank 1. Then for any elements $a,b,c \in R$ with $aR+bR+cR=R$ there is a matrix $M \in GE_3(R)$ such that*

$$(a,b,c)M = (1,0,0).$$

Theorem 3. *Let R be a right Bezout ring of stable rank 1. Then for any elements $a,b \in R$ there is a matrix $M \in GE_2(R)$ such that*

$$(a,b)M = (\alpha,0), \quad \alpha \in R.$$

Proof. Since R is a right Bezout ring, the ideal $aR+bR$ is principal and thus equal to dR for some $d \in R$. Then $a = da_0$, $b = db_0$, $au+bv = d$, $a_0, b_0, u, v \in R$. Let $e_0 = 1 - a_0u - b_0v$. Then $de_0 = 0$ and $a_0R + b_0R = R$.

Since R is a ring of stable rank 1, there are elements $t, w \in R$ such that $(a_0+b_0t)w = 1$. Then

$$d(a_0+b_0t)w = (a+bt)w = d.$$

Thus

$$(a,b)\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}\begin{pmatrix}1 & w(1-b_0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -w\\ 0 & 1\end{pmatrix} = (a+bt, 0).$$

which finishes the proof.

Theorem 4. *Let R be a right Bezout ring of stable rank 2. Then for any elements $a, b, c \in R$ there is a matrix $M \in GE_3(R)$ such that*

$$(a, b, c)M = (\alpha, \beta, 0), \quad \alpha, \beta \in R.$$

Proof. Let $aR + bR + cR = dR$, $d \in R$. Then $a = da_0$, $b = db_0$, $c = dc_0$, $au + bv + cw = d$, $a_0, b_0, c_0, u, v, w \in R$. Let $e_0 = 1 - a_0u - b_0v - c_0w$. Then $de_0 = 0$ and $a_0R + b_0R + c_0R = R$.

Since R is a ring of stable rank 2, there exist $x, y, p, q \in R$ such that

$$(a_0 + c_0x)p + (b_0 + c_0y)q = 1.$$

Then

$$d(a_0 + c_0x)p + d(b_0 + c_0y)q =$$

$$= (a + cx)p + (b + cy)q = ap + bq + c(xp + yq) = d.$$

Thus

$$(a, b, c)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} = (a + cx, b + cy, c),$$

$$(a + cx, b + cy, c)\begin{pmatrix} 1 & 0 & p(1 - c_0) \\ 0 & 1 & q(1 - c_0) \\ 0 & 0 & 1 \end{pmatrix} = (a + cx, b + cy, d)$$

and

$$(a + cx, b + cy, d)\begin{pmatrix} 1 & 0 & -p \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{pmatrix} = (a + cx, b + cy, 0),$$

which finishes the proof.

Corollary 2. *Let R be a right Bezout ring of stable rank 1. Then for any elements $a, b, c \in R$ there is a matrix $M \in GE_3(R)$ such that*

$$(a, b, c)M = (\alpha, \beta, 0), \quad \alpha, \beta \in R.$$

---

1. *Zabavsky B. V.* Ring with elementary reduction of matrices // Ring Theory Conference. – Miskols, 1996 (July 15-20). – P. 14.
2. *Cohn P. M.* On the structure of the GL_2 of ring // I.H.E.S. Publ. Math. – 1966. – 30. – P. 365-413.
3. *Cohn P. M.* Free rings and their relations. – London, 1971.
4. *Warfield R. B.* Stable equivalence of matrices and resolutions // Comm. in Algebra. – 1978. – 6 (12). – P. 1811-1828.
5. *Menal P., Moncasi J.* On regular rings with stable range 2 // J. Pure Apll. Algebra. – 1982. – 24. – P. 25-40.

ЕЛЕМЕНТАРНІ ПЕРЕТВОРЕННЯ РЯДКІВ НАД КІЛЬЦЯМИ СТАБІЛЬНОГО РАНГУ $\leqslant 2$

О. Романів

Львівський національний університет імені Івана Франка, вул. Університетська, 1 79000 Львів, Україна

Доведено, що над кільцем R стабільного рангу $\leqslant 2$ довільний рядок $(a, b, c) \in R^3$ такий, що $aR + bR + cR = R$, елементарними перетвореннями зводиться до вигляду $(1, 0, 0)$. Показано, що над правим кільцем Безу R стабільного рангу $\leqslant 2$ довільний рядок $(a, b, c) \in R^3$ за допомогою елементарних перетворень зводиться до вигляду $(\alpha, \beta, 0)$, $\alpha, \beta \in R$.

Ключові слова: стабільний ранг, кільце Безу, елементарна редукція.

Стаття надійшла до редколегії 20.12.2001

Прийнята до друку 14.03.2003