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ON 2-TORSION OF BRAUER GROUPS OF HYPERELLIPTIC CURVES OVER PSEUDOLOCAL FIELDS

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It is proved an analogue of the Yanchevskii and Margolin's theorem about the 2-torsion of the Brauer group of hyperelliptic curve over a complete discretely valued field with pseudofinite residue field.

Key words: hyperelliptic curve, Brauer group, local field, pseudofinite field.

By a pseudolocal field K we mean a complete with respect to a discrete valution field with pseudofinite [1] residue fields k. Recall that an infinite field is called *pseudofinite* if it is perfect, pseudoalgebraically closed and possesses exactly one extension of each degree. If K_s is a separable closure of a field K, let $G_K = Gal(K_s/K)$ be its Galois group. Then $H^i(K, M)$ denotes the Galois cohomology group of G_K-module M. C_n and C/nC stand for the kernel and cokernel of multiplication by n in an abelian group C. For an abelian variety A defined over K by $A(K)$ (respectively $K(A)$) we denote the group of K-rational points of A (respectively the function field on A).

Consider the homomorphism $\mu : K(A)^* \to Div(A)$, which sends a function from $K(A)^*$ to its divisor. The map μ induces the corresponding homomorphism in Galois cohomology

$$\mu_* : H^2(G, K(A)^*) \to H^2(G, Div(A)).$$

The kernel of μ_* called the Brauer group of A, is denoted by BrA. It is known [1], that the group BrA consists of the classes of similar central simple K -algebras, which are unramified at all the valutions v of K.

Recall that two central simple K-algebras C_1, C_2 are *similar* if there exist two natural numbers m, n such that $C_1 \otimes M_n(K)$ and $C_2 \otimes M_m(K)$ are isomorphic. Recall also that a generalized quaternion algebra $(\frac{a,b}{K})$ over a field K is a K-algebra, generated by 1, x, y, $z \in K^*$, where $x^2 = a$, $y^2 = b$, $z^2 = -ab$, and $xy = -yx$, $xz = -zx$, $yz = -zx$. $[\frac{a,b}{K}]$ is the element of BrK with the representative $(\frac{a,b}{K})$.

The 2-torsion part of the Brauer group of an elliptic or hyperelliptic curve over a local field was described by V.I.Yanchevskii and G.L.Margolin [2]. In their description every element of $(BrA)_2$ is represented by a quaternion algebra over $K(A)$. It turns out that the analogous results remain true for hyperelliptic curves over a pseudolocal field.

Let K be a pseudolocal field. Denote by O_K the ring of integer of K. Let π be an uniformizing element of K, α be a unit of K which is not a square, n be a prime number, $n \neq chark$ and $|A|$ denote the order of finite set A.

Let A be a hyperelliptic curve over a preudolocal field K, that is a curve defined by the equation $y^2 = f(x)$, $f(x) = \beta f_0(x)$, where $f_0(x) \in O_K$ is a monic polynomial, $m = deg f(x)$, $\beta \in \{1, \alpha, \pi\}$.

Theorem 1. *Let A be either an elliptic or a hyperelliptic curve with good reduction defined over a pseudolocal field K with a pseudofinite residue field $\bar{K}$, $char K \neq 2$, $m \not\equiv 0 (mod 4)$. Then $(BrA)_2$ consists of the folowing pairwise distinct elements:*

$$\left[\left(\frac{\pi, 1}{K(A)}\right)\right], \left[\left(\frac{\pi, \alpha}{K(A)}\right)\right], \left[\left(\frac{\pi, g(x)}{K(A)}\right)\right] \left[\left(\frac{\pi, \alpha g(x)}{K(A)}\right)\right],$$

where

(i) if m is odd then $g(x)$ runs over all monic divisors of $f(x)$ of degree less than $m/2$.

(ii) if m is even then $g(x)$ runs over all monic divisors of $f(x)$ of even degree less than $m/2$.

To prove this result we need some preliminary statements which are of interest by its own right.

First, we will need the following three lemmas from [2].

Lemma 1. *Let $g(x)$ be a divisor of $f(x)$ and let either m be odd or $deg\, g(x)$ even. Then the quaternion algebra $\left(\frac{B, g(x)}{K(A)}\right)$ is unramified over $K(A)$ for any $B \in K^*$.*

Lemma 2. *Let β be either 1 or α, $g = Bg_0$, B either 1 or α and $g_0 \in O_K[x]$ be a monic divisor of $f(x)$, $f_0 = g_0 \hat{g_0}$. If $\bar{g}_0 \notin \bar{K}[x]^2$, $\bar{\hat{g_0}} \notin \bar{K}[x]^2$, where $\bar{K}$ is residue field of K, $\bar{g}_0$ and $\bar{\hat{g_0}}$ are polinomial g_0 and $\hat{g_0}$, regarded over the residue field $\bar{K}$. Then $\left(\frac{\pi, g(x)}{K(A)}\right) \not\cong 1$.*

Lemma 3. *Let K be a general local field and A has good reduction. If $\left(\frac{\pi, \alpha}{K}\right)$ is the quaternion division algebra over K, then $\left(\frac{\pi, \alpha}{K(A)}\right)$ is a division algebra.*

Note, that the statements and the proofs of these three results remain true for any complete discretely valued field.

Lemma 4. *Let K be a pseudolocal field, A be an abelian variety defined over K. Suppose that A has good reduction. Then $|A(K)/nA(K)| = |A(K)_n|$ for any n, $(n, char \bar{K}) = 1$.*

For local fields this was proved by V. I. Yanchevskii and G. L. Margolin [2]. The case of elliptic curves over pseudolocal fields was considered in [5]. The case of abelian variety with good reduction was investigated by V. I. Andrijchuk [3].

For completeness sake we sketch briefly the corresponding arguments. Any principal homogeneous space for A over K has a K-rational point, so $H^1(k, A) = 0$. Thus the Kummer exact sequence corresponding to multiplication by n yields the isomorphism of the groups $A(k)/nA(k)$ and $H^1(K, A_n)$. Besides, since the absolute Galois group of k is isomorphic to $\hat{Z}$, we obtain $|H^1(k, A_n)| = |H^0(k, A_n)|$ (see [6] for more details). Thus we have $|A(k)/nA(k)| = |A(k)_n|$. To finish the proof it is sufficient to use the reduction exact sequence and the snake lemma together with the fact that the kernel of the reduction map is uniquelly divisible by n.

Lemma 5. *Let A be an abelian variety with good reduction, defined over a pseudolocal field K, $\hat{A}$ be its dual variety. Then for any n, $(n, char k) = 1$ the Tate-Shafarevich pairing induces a nondegenerate pairing $A(K)/nA(K) \otimes H^1(K, A)_n \longrightarrow \mathbb{Z}/n\mathbb{Z}$. If A is an elliptic curve with bad reduction, then the last pairing is nondegenerate in the case of general local field.*

For an elliptic curve this was proved in [4]. According to [3] this fact remains true for an abelian variety of any dimention with good reduction.

To prove that the pairing $A(K)/nA(K) \otimes H^1(K, \hat{A}_n) \longrightarrow \mathbb{Z}/n\mathbb{Z}$ is non-degenerate, we consider the commutative diagram

$$\begin{array}{ccc} H^1(K, A_n) \times H^1(K, \hat{A}_n) & \xrightarrow{W} & \mathbb{Q}/\mathbb{Z} \\ \uparrow i_n \qquad\qquad \downarrow j_n & & \| \\ A(K)/nA(K) \times H^1(K, \hat{A})_n & \xrightarrow{W} & \mathbb{Q}/\mathbb{Z}, \end{array}$$

where i_n and j_n are the homomorphisms from the Kummer exact sequences for A, W is induced by the Weil pairing, and T is induced by the Tate-Shafarevich pairing. The homomorphis i_n is injective and it is known that the pairing W is non-degenerate. Since i_n is a monomorphism and j_n is an epimorphism, it follows that T is nondegenerate on the left. To prove that it is a duality, it suffices to prove that $|A(K)/nA(K)| = |H^1(K, \hat{A}_n)|$. But this follows from the equalities $|A(K)_n| = |\hat{A}(K)_n|$, $|H^1(K, A_n)| = |H^0(K, A_n)| \times |H^2(K, A_n)|$ and $|H^1(K, A_n)| = |H^1(K, \hat{A}_n)|$, which hold for any complete discretely valued field with quasifinite residue fields (the first of them holds for any field).

Proof of Theorem 1. We denote by $PicA$ (respectively $Pic^0 A$) the Picard group (respectively its subgroup of divisor classes of degree zero). As in [2], we begin by considering the following exact sequence

$$0 \to PicA \to H^0(K, Pic\bar{A}) \to BrK \to BrA \to H^1(K, Pic\bar{A}) \to H^3(K, K_s^*). \quad (1)$$

In this sequence $H^3(K, K_s^*) = 0$, since the cohomological dimension of K is 2, as it is follows from [6, Prop.12, p.105]. Since A has a K-rational point, the index of A is equal 1. Thus the homomorphism $Pic^0 A \to H^0(K, Pic^0 A)$ is surjective, and we obtain the exact sequence

$$0 \longrightarrow BrK \longrightarrow BrA \longrightarrow H^1(K, PicA) \longrightarrow 0. \quad (2)$$

Using $BrK \cong \mathbb{Q}/\mathbb{Z}$ for any general local field K and passing to n-torsion in the exact sequence (1) we obtain the following equality:

$$|(BrA)_n| = n|H^1(K, Pic\bar{A})|. \quad (3)$$

Since the period of A divides the index of A, it is 1, so passing to cohomology in the exact sequence

$$0 \longrightarrow Pic^0\bar{A} \longrightarrow Pic\bar{A} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

we obtain the isomorphism $H^1(K, Pic^0\bar{A}) \to H^1(K, Pic\bar{A})$ which induces the isomorphism $H^1(K, Pic^0\bar{A})_n \to H^1(K, Pic\bar{A})_n$

Now, Lemma 5 implies, that $|H^1(K, Pic^0\bar{A})| = |A(K)/nA(K)|$, and by the equality (3), $|(BrA)_n| = n|H^1(K, Pik^0\bar{A})| = n|Pic^0A/nPic^0A| = n|(Pic^0A)_n|$. Here the last equality follows from Lemma 4.

The order of $(BrA)_2$ is $2|Pic^0(A)_2|$. But, by [2, Corol. 4, p.19], there is bijective correspondence between elements of $(Pic^0\bar{A})_2$ and monic divisors of $f(x)$ defined over K of degree less than $m/2$ in the case of odd m and monic divisors of $f(x)$ defined over K of even degree less than $m/2$ in the case of even m. Thus, to finish the proof, it remains to show that all algebras from the statement of Theorem 1 are nontrivial, unramified and pairwise not isomorphic. But this follows from Lemmas 1, 2, 3 and from the fact that the tensor products of two such algebras is similar to an algebra of the same form.

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2-КРУЧЕННЯ ГРУП БРАУЕРА ГІПЕРЕЛІПТИЧНИХ КРИВИХ НАД ПСЕВДОЛОКАЛЬНИМИ ПОЛЯМИ

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Доведено аналог теореми В. І. Янчевського та Г. Л. Марголіна про 2-кручення групи Брауера гіпереліптичної кривої над повним дискретно нормованим полем з псевдоскінченним полем лишків.

Ключові слова: гіпереліптична крива, група Брауера, локальне поле, псевдоскінченне поле.

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