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Серія механіко–математична

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ON THE BRAUER GROUP AND THE HASSE PRINCIPLE FOR PSEUDOGLOBAL FIELDS

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We prove the Tate critérion for the Hasse principle in finite extensions of an algebraic function field K with pseudofinite constant field. Also any central simple algebra of finite dimension over such a field is cyclic and its index and exponent coincide.

Key words: class field theory, algebraic function field, Brauer group, Hasse principle, finite dimensional central simple algebra.

The aim of this paper is to show that the basic properties of the Brauer group of a global field hold as well for the Brauer group of an algebraic function field with pseudofinite [1] constant field. We call such a field pseudoglobal field. We prove also that any central simple algebra of finite degree over a pseudoglobal field K is cyclic and its index and exponent coincide. Besides, we discuss the Hasse principle in finite extensions of a pseudoglobal field.

The basic properties of the Brauer group of a pseudoglobal field will follow as the simple corollaries from the fundamental sequence

$$0 \to \mathrm{Br}K \xrightarrow{\ i\ } \bigoplus_{v\in V_K} \mathrm{Br}K_v \xrightarrow{\ j\ } \mathbb{Q}/\mathbb{Z} \to 0, \qquad (1)$$

which is exact both for global and pseudoglobal fields [1,2]. Here V_K denotes the set of all the valuations of pseudoglobal field K (which are trivial on the constant field), $\mathrm{Br}K$ is the Brauer group of K, and $\mathrm{Br}K_v$ is the Brauer group of the corresponding completion of field K at the valuation $v \in V_K$.

The elements of $\mathrm{Br}K$ are the equivalence classes of central simple K-algebras A of finite dimension with respect to the following equivalence relation: two algebras A and B are equivalent if there exist two natural numbers $m, n \geqslant 1$ such that the algebras $A \otimes_K M_n(K)$ and $B \otimes_K M_m(K)$ are isomorphic. All matrix algebras over K are equivalent and form the zero element of Brauer group. The class of the opposite algebra A° (that is A° is the additive group A equipped with the new multiplication $*$ such that $a * b = ba$) is the inverse for the class of A. We shall denote the class of A in the Brauer group by $[A]$.

The field extension L of K is said to be a splitting field of algebra A, if the algebras $A \otimes_K L$ and $M_m(L)$ are isomorphic. Two equivalent algebras have the same splitting fields. The subset $\mathrm{Br}(L/K)$ of $\mathrm{Br}K$, consisting of all the elements of $\mathrm{Br}(K)$ which split in L, is a subgroup of $\mathrm{Br}K$.

In the exact sequence (1) the map i sends $[A] \in \mathrm{Br}K$ to $(..., [A \otimes_K K_v], ...) \in \oplus \mathrm{Br}K_v$ (notice that there are only finitely many of valuations v of K such that $[A \otimes_K K_v]$ is a nontrivial element of $\mathrm{Br}K_v$ (see e.g. [3], p. 441, or [4])). Any local algebra $A_v = A \otimes K_v$ is a simple central algebra over a general local field K_v, so it determines an element of the Brauer group $\mathrm{Br}\, K_v$. It is known [5] that for a general local field K_v there exists an isomorphism $\mathrm{inv}_v : \mathrm{Br}K_v \longrightarrow \mathbb{Q}/\mathbb{Z}$.

The image of the element $[A_v]$ under this isomorphism is said to be the invariant of A_v (or the local invariant of the algebra A at the valuation v). It is denoted by $\mathrm{inv}_v(A)$. The homomorphism j maps an element of the group $\oplus_{v \in V_K} \mathrm{Br}K_v$ into sum of all corresponding local invariants.

The following proposition is an analogue for pseudoglobal fields of the classical Albert-Brauer-Hasse-Noether theorem on central simple algebras over global fields.

Proposition 1. *A central simple K-algebra A splits over pseudoglobal field K if and only if it splits locally everywhere, that is all its local invariants vanish.*

Proof. If the algebra A splits locally everywhere, then all its local invariants vanish. The injectivity of the homomorphism i in the exact sequence (1) show that the only trivial element of the group $\mathrm{Br}K$ (that is the matrix algebras over K) may have all trivial local invariants.

Proposition 2. *Suppose that a central simple K-algebra A over a pseudoglobal field K splits locally at all the valuations of K except possibly the valuation v_0. Then it splits over K.*

Proof. The exact sequence (1) yields that the sum of all local invariants of A is zero, so the local invariant of A_{v_0} must be zero as well. Thus the algebra A splits locally everywhere, and by Proposition 1 it splits over K.

The following proposition is an analogue for pseudoglobal fields of the Hasse norm theorem for cyclic extensions of global fields.

Proposition 3. *Let L/K be a cyclic extension of a pseudoglobal field K. An element $a \in K$ is a norm from L if and only if a is a norm locally everywhere, that is $a \in N_{L^v/K_v} L^v$ for all $v \in V_K$, where L^v denotes the completion of L at an extension of valuation $v \in V_K$ to L.*

Proof. Consider the exact sequence $0 \to L^* \to J_L \to C_L \to 0$, where J_L and C_L are the idèle group and the idèle class groups of the field L respectively. It was proved in [2] that $H^1(Gal(L/K), C_L) = 0$, thus we have the short exact cohomological sequence

$$0 \to H^2(G, L^*) \to \Pi_{v \in V_K} H^2(G_v, L^{v*}), \tag{2}$$

where L^v is the completion of L at some extension of the valuation v, and G_v is the decomposition group of the valuation v.

Since the extension L/K is cyclic, we have

$$H^2(G, L^*) \simeq K^*/N_{L/K}L^*,\ H^2(G_v, L^{v*}) \simeq K_v^*/N_{L^v/K_v}L^{v*},$$

so the exact sequence (2) may be written as follows

$$0 \to K^*/N_{L/K}L^* \to \Pi_{v \in V_K} K_v^*/N_{L^v/K_v}L^{v*}.$$

This exact sequence yields the desired statement.

Remark. Using the arguments given in the hints to the exercise 4 in [4, pp. 465–469], one can prove that Proposition 3 implies the Minkowski-Hasse theorem on quadratic forms: a nonsingular quadratic form over a pseudoglobal field K is isotropic if and only if it is isotropic over all the completions K_v of K.

To formulate the next Proposition, let us denote by $\hat{i}_K$ the composition of homomorphism i from sequence (1) and the isomorphism $\mathrm{Br}K_v \simeq \mathbb{Q}/\mathbb{Z}$. Then $\hat{i}_K$ maps the class of algebra A over K into the collection of its local invariants, $\hat{i}_K(A) = (..., \hat{i}_{K_v}[A \otimes_K K_v], ...)$.

Proposition 4. *Let K be a pseudoglobal field.*

a) $\hat{i}_K$ defines an injective homomorphism $BrK \to \oplus_{v \in V_K} \mathbb{Q}/\mathbb{Z}$.

b) Two K-algebras A and B are equivalent if and only if $\hat{i}_K(A) = \hat{i}_K(B)$.

c) Two K-algebras A and B are isomorphic if and only if $\hat{i}_K(A) = \hat{i}_K(B)$ and $\deg A = \deg B$.

Proof. a) $\hat{i}_K(A)$ depends only on the class $[A]$ of A. The injectivity of $\hat{i}_K : \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$ follows from the injectivity of the homomorphism i in the exact sequence (1).

b) This follows immediately from a).

c) If the algebras A and B are isomorphic, it is obvious that $\hat{i}_K(A) = \hat{i}_K(B)$ and $\deg A = \deg B$.

Conversely, if $\hat{i}_K(A) = \hat{i}_K(B)$, then the algebras A and B are equivalent, so there exists a skew field D of finite dimension over K such that $A = M_m(D), B = M_n(D)$ for some natural numbers $m, n \geqslant 1$. Since $\deg A = \deg B$, we have $m = n$ and $A \simeq B$.

Proposition 5. *a) Let L/K be a finite Galois extension of a pseudoglobal field K, A be a central simple algebra of finite dimension over K, $v \in V_K$, $w \in V_L$, w is an extension of the valuation v to the field L, K_v and L_w be the corresponding completions of the fields K and L. The algebra A splits over L if and only if* $[L_w; K_v] \cdot \mathrm{inv}_v(A) = 0$.

b) The field L is isomorphic to a strongly maximal subfield of the algebra A if and only if $\deg A = [L : K]$, *and* $[L_w : K_v] \cdot \mathrm{inv}_v(A) = 0$ *for all valuations v of K and their extensions w to L.*

Proof. a) By Proposition 1 the field L splits A if and only if $\mathrm{inv}_w(A \otimes_K L_w) = 0$ for all valuations w of L. It is easy to check that the following diagram

$$\begin{array}{ccc} \mathrm{Br}K_v & \xrightarrow[\mathrm{inv}_v]{\simeq} & \mathbb{Q}/\mathbb{Z} \\ \big\downarrow{\scriptstyle res} & & \big\downarrow{\scriptstyle n} \\ \mathrm{Br}L_w & \xrightarrow[\mathrm{inv}_w]{\simeq} & \mathbb{Q}/\mathbb{Z} \end{array}$$

commutes, and we have $\mathrm{inv}_w(A \otimes_K L) = [L_w : K_v] \,\mathrm{inv}_v(A)$.

b) Let L be a maximal subfield of A, then it is known (see e.g. [6] or [3]) that A splits over L, and $\deg A = [L : K]$.

Thus, by the above arguments $[L_w : K_v] \,\mathrm{inv}_v A = 0$. Conversely, if for a subfield L of A the condinions $\deg A = [L : K]$ and $[L_w : K_v] \,\mathrm{inv}_v A = 0$ hold, then the first of them implies that L is the maximal subfield of A, and the second one implies that the algebra A splits over K by statement a).

Proposition 6. *Let $\alpha_1, ..., \alpha_s$ be a finite set of elements of the group $\mathbb{Q}/\mathbb{Z}$. Then there is a simple central algebra A over a pseudoglobal field K with the invariants $\alpha_1, ..., \alpha_s$ if and only if $\sum_{i=1}^{s} \alpha_i = 0$.*

Proof. This is an immediate corollary from the exactness of sequence (1) in the middle term $\oplus_{v \in V_K} \mathrm{Br} K_v$.

Proposition 7. *Let L/K be a finite abelian extension of a pseudoglobal field K. If an element $a \in K$ is a local norm at all the completions K_v, where $v \in V_K \setminus \{v_0\}$, then a is a local norm in the field K_{v_0}.*

Proof. Consider the local and the global norm residue symbols θ_v i $\tilde{\theta}_K$ determined in [2] . They are related by the equality $\tilde{\theta}_K = \Pi_{v \in V_K} \theta_v$. Using the product formula for pseudoglobal field, we get $\tilde{\theta}_K(a) = 1 = \Pi_{v \in V_K} \theta_v(a) = \theta_{v_0}(a)$. The last equality and Proposition 7 follow from the fact that $\theta_v(a) = 1$, for $a \in K_v$, if and only if $a \in N_{L_w/K_v} L_w^*$.

The following Proposition 8 is a counterpart for pseudoglobal fields of Theorem 10 from [7, Chapter 6].

Proposition 8. *Let L/K be a finite abelian extension of a pseudoglobal field K, and let $\sigma_v \in G_v$ be the set of elements of the decomposition groups G_v such that almost all of them are trivial. Suppose that $\Pi_{v \in V_K} \sigma_v = 1$. Then there is an element $a \in K$ such that $\theta_v(a) = \theta_{L_w/K_v}(a) = \sigma_v$.*

Proof. Let $\theta_{L_w/K_v} : K_v/NL_w \to Gal(L_w/K_v)$ be the local norm residue symbol [2]. By the local class field theory generalized to general local fields (see [5]), one can find an idèle $(a_v) \in J_K$ such that $\theta_{L_w/K_v}(a_v) = \sigma_v$. Since $\Pi_{v \in V_K} \sigma_v = 1$, we have $\Pi_{v \in V_K} \theta_v(a_v) = \tilde{\theta}_K((a_v)) = 1$. Then $(a_v) \in K^* N_{L/K} J_L, (a_v) = a(b_v)$, where $(b_v) \in N_{L/K} J_L$. Thus $\tilde{\theta}_K((b_v)) = 1$, and we have $\theta_v(a) = \sigma_v$.

The following Proposition asserts that the conditions for a valuation v of K to be unramified or to split completely in a given Galois extension L/K can be formulated in terms of subgroups of idèle class group of the field K exactly in the same manner as for the global fields (see [7, Chapter 8, Theorem 3]).

Proposition 9. *Let L/K be a finite abelian extension of a pseudoglobal field K. The valuation v of the field K is unramified in the field L if and only if $U_v \subset N_{L/K} C_L$. The valuation v of the field K splits completely in the field L if and only if $K_v \subset N_{L/K} C_L$.*

Proof. In Proposition 9 it is assumed that the completion K_v is embedded into the group C_K by using the composition $K_v \hookrightarrow J_K \longrightarrow C_K$. To prove Proposition 9 we follow the arguments which were used in the case of global field (see [7, Ch.8]). First, we show that $N_{L/K} C_L \bigcap K_v = N_{L_w/K_v} L_w$, where w is an extension of the valuation v to L. It is enough to prove the inclusion $K N_{L/K} J_L \bigcap K K_v \subset K N_{L_w/K_v} L_w$. Let $a \in K, a_v \in K_v$, and $a a_v = N_{L/K}((a_w))$, where $(a_w) \in J_L$. Thus a is a local norm at all the valuation, except possibly at v, but then it follows from Proposition 7 that a is a local norm everywhere, so a_v is a local norm in K_v, and the desired inclusion is proved.

Let U_v be the unit group of K_v. If the valuation v is unramified, then all elements of U_v are norms by [5], thus $U_v \subset N_{L/K} C_L$. If $U_v \subset N_{L/K} C_L$, we have

$U_v \subset (N_{L/K}C_L \bigcap K_v) = N_{L_w/K_v}L_w$. Using the local class field theory for general local fields [5], we get that the valuation v is unramified in L.

To finish the proof of Proposition 9, it is enough to consider in the above argument the group K_v instead of U_v and the property "to split completely" instead of "to be unramified".

Let X be an algebraic curve defined over the field k. The Brauer group $\mathrm{Br}(X)$ of the curve X is the kernel of the homomorphism $\mathrm{Br}K \longrightarrow \oplus_{v\in V_K}\mathrm{Br}K_v$, where K is the function field on X.

Proposition 10. *For a pseudoglobal field K with constant field k the following equivalent properties hold:*

a) the reciprocity law holds for K/k;

b) for any finite cyclic extension L/K the sequence

$$Br(L/K) \longrightarrow \bigoplus_{v\in V_K} Br(L_w/K_v) \rightarrow [L:K]^{-1}\mathbb{Z}/\mathbb{Z} \rightarrow 0$$

us exact;

c) for any finite cyclic extension L/K,

$$H^1(Gal(L/K), Br(Y)) = 0,$$

where $Br(Y)$ is the Brauer group of the nonsingular projective algebraic curve Y with function field L;

d) for any finite cyclic extension L/K the map

$$K^*/N_{L/K}L^* \longrightarrow \bigoplus_{v\in V_K} K_v^*/N_{L_w/K_v}L_w^*$$

is injective;

e) $H^1(G(k), Jac_X(k_s)) = 0$, where $G(k)$ is the absolute Galois group of k and $Jac_X(k_s)$ is the jacobian of the curve X regarded over a separable closure k_s of the field k;

f) $Br(X) = 0$.

Proof. For a pseudoglobal field K/k assertion a) was proved in [2]. Assertion d) follows from Proposition 3. Conversely, as was proved in [2] d) implies a). The equivalence of a), b) and c) follows from Proposition A.12 [8, p. 167], and the equivalence of d),e),f) follows from Proposition A.13 [8, p. 168].

Now we shall show that the existence of class formation for pseudoglobal field K yields the same corollaries about the 3-dimensional Galois cohomology groups of the field K (respectively of idèle group and idèle class group of K) as in the case of a global field. Besides, it turns out that for abelian extensions of a pseudoglobal field one can get the Tate criterion for the Hasse principle.

Let L/K be a finite Galois extension of a pseudoglobal field K and let $G = Gal(L/K)$ be its Galois. Let H be a subgroup of G. Since the idèle classes of K form the class formation, it follows from Tate's theorem [4, p. 181] that the multiplication by the fundamental class $u_{L/K} \in H^2(G, C_L)$ defines the isomorphisms

$$H^n(H,\mathbb{Z}) \rightarrow H^{n+2}(H, C_L) \qquad (3)$$

for all $n \in \mathbb{Z}$.

Let K' be the subfield of L corresponding to the subgroup H by Galois theory, $H = \mathrm{Gal}(L/K')$.

Proposition 11. *The diagrams*

$$\begin{array}{ccc} H^n(G,\mathbb{Z}) & \rightarrow & H^{n+2}(G,C_L) \\ \downarrow \mathrm{res} & & \downarrow \mathrm{res} \\ H^n(H,\mathbb{Z}) & \rightarrow & H^{n+2}(H,C_L) \end{array} \tag{4}$$

and

$$\begin{array}{ccc} H^n(G,\mathbb{Z}) & \rightarrow & H^{n+2}(G,C_L) \\ \uparrow \mathrm{cor} & & \uparrow \mathrm{cor} \\ H^n(H,\mathbb{Z}) & \rightarrow & H^{n+2}(H,C_L) \end{array} \tag{5}$$

commute.

Proof. The commutativity of (4) follows from the equalities

$$\mathrm{res}(a \cup u_{L/K}) = (\mathrm{res}\, a) \cup u_{L/K'} = \mathrm{res}\, a \cup \mathrm{res}\, u_{L/K},$$

and diagram (5) commutes according to

$$(\mathrm{cor}\, a) \cup u_{L/K} = \mathrm{cor}(a \cup u_{L/K'}) = \mathrm{cor}(a \cup \mathrm{res}\, u_{L/K}).$$

By using isomorhisms (3) one can prove, exactly in the same manner as in the case of global fields [4, c. 301], the Tate criterion for the Hasse principle for Galois extensions of pseudoglobal fields.

The kernel of the homomorphism

$$f_{0,L/K} : \hat{H}^0(G,L^*) \longrightarrow \hat{H}^0(G,J_L)$$

is called the obsruction for the Hasse principle for the Galois extension L/K with Galois group G. One says that the Hasse principle holds for L/K if $\mathrm{Ker}\, f_{0,L/K} = 0$.

Proposition 12. *Let L/K be a finite Galois extension of a pseudoglobal field K, $G = Gal(L/K)$. Then*

$$\mathrm{Ker}\, f_{0,L/K} \simeq \mathrm{Ker}(H^3(G,\mathbb{Z}) \rightarrow \Pi_{v \in V_K} H^3(G^v,\mathbb{Z})),$$

where G^v is a decomposition group G_w of an extension to L of the valuation v of K.

Proof. For the sake of completeness, we present the proof, despite it essentially coincides with that for the global field (see [4, p. 301]). Consider the exact sequence of G-modules

$$0 \rightarrow L^* \rightarrow J_L \rightarrow C_L \rightarrow 0, \tag{6}$$

and the corresponding sequence of Tate's Galois cohomology

$$\ldots \longrightarrow H^{n-1}(G,J_L) \xrightarrow{g_{n-1}} \hat{H}^{n-1}(G,C_L) \longrightarrow$$
$$\longrightarrow \hat{H}^n(G,L^*) \xrightarrow{f_n} H^n(G,J_L) \longrightarrow \ldots.$$

From the exactness of this last sequence we get

$$\mathrm{Ker}\, f_n \simeq \mathrm{Coker}\, g_{n-1}.$$

Now, by the local class field theory for general local fields [5],

$$H^{n-1}(G, J_L) \simeq \Pi_{v\in V_K} \hat{H}^{n-1}(G^v, L_v^*) \simeq \Pi_{v\in V_K} H^{n-3}(G^v, \mathbb{Z}),$$

and by (3),

$$\hat{H}^{n-1}(G, C_L) \simeq \hat{H}^{n-3}(G, \mathbb{Z}).$$

Using the fact that the groups $H^n(G,\mathbb{Z})$ and $H^{-n}(G,\mathbb{Z})$ are dual, one can write

$$\begin{aligned}\operatorname{Ker} f_n &\simeq \operatorname{Coker}(\amalg_{v\in V_K} \hat{H}^{n-3}(G^v, \mathbb{Z}) \xrightarrow{g'_{n-1}} \hat{H}^{n-3}(G, \mathbb{Z})) \\ &\simeq \operatorname{Ker}(H^{3-n}(G, \mathbb{Z}) \xrightarrow{h_{3-n}} \textstyle\prod_{v\in V_K} H^{3-n}(G^v, \mathbb{Z})),\end{aligned}$$

where $g'_{n-1}(\sum_v z_v) = \sum_v \operatorname{cor} z_v$, $h_{3-n}(z) = \prod_{v\in V_K} \operatorname{res} z$. Setting $n = 0$, we get

$$\operatorname{Ker} f_0 \simeq \operatorname{Ker}(H^3(G, \mathbb{Z}) \to \Pi_{v\in V_K} H^3(G^v, \mathbb{Z})),$$

as was to be proved.

Proposition 13. *Let L/K be a finite Galois extension of a pseudoglobal field K, $n = [L : K]$, g_v be the number of all distinct valuations w of L which are the extensions of a valuation v of K, d be the greatest common divisor of all g_v. Then, by identifiyng the group $H^2(G, L^*)$ with a subgroup of $H^2(G, J_L)$, the quotient group $H^2(G, J_L)/H^2(G, L^*)$ is a cyclic group of order $\frac{n}{d}$, and the image of the group $H^2(G, C_L)$ in $H^3(G, L^*)$ is a cyclic group of order d.*

Proof. We have $H^2(G, J_L) \simeq \bigoplus_{v\in V_K}(\frac{1}{n_v}\mathbb{Z}/\mathbb{Z})$, where $n_v = [L^v : K_v]$. On the other hand, $H^1(G, C_L) = 0$, and $H^2(G, C_L) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$, thus the exact Galois cohomology sequence corresponding to (6) can be written as follows

$$0 \to H^2(G, L^*) \to \bigoplus_{v\in V_K} (\frac{1}{n}\mathbb{Z}/\mathbb{Z}) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to H^3(G, L^*). \tag{7}$$

Consequently, the quotient group $H^2(G, J_L)/H^2(G, L^*)$ is isomorphic to a subgroup of $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, so it is cyclic. Let us find in this quotient group an element of maximal order.

Let $\{i_v\}_{v\in V_K}$ be the set of integers such that almost all of them are zero, and $\sum_v i_v g_v = d$. Since $n = n_v g_v$, we have $\sum_v \frac{i_v}{n_v} = \frac{g}{n}$. Hence, it follows that the element of the quotient group $H^2(G, J_L)/H^2(G, L^*)$ with representative $\left(\left(\frac{i_v}{n_v}\right)(\text{mod} 1)\right)$ has the order $\frac{n}{d}$.

Futher, if $\left(\left(\frac{j_v}{n_v}\right)(\text{mod} 1)\right)$ is the representative of another element $\bar{\alpha}$ of this quotient group, then one can find an integer m, such that $\sum j_v n_v = md$. Consequently, $\sum_v \frac{j_v}{n_v} = m\frac{d}{n}$, thus the order of $\bar{\alpha}$ divides $\frac{n}{d}$. Hence the order of considered quotient group is $\frac{n}{d}$.

Now, the exact sequence (7) shows that the image of $H^2(G, C_L)$ in the group $H^3(G, L^*)$ is a cyclic subgroup of order d generated by the image of fundamental class $u_{L/K} \in H^2(G, C_L)$.

Finally, we consider the central simple algebras of finite dimension over a pseudoglobal field. We shall show that any such algebra A is cyclic, its index and exponent coincide, and the reduced Whitehad group $SK_1(A)$ is trivial.

We shall use one result of Saltman [9] on existence of abelian extensions of valued fields, namely, a part of Theorem 5.10 from [9].

Theorem (Saltman [9]). *Let G be an abelian group, K be a field with real valued valuations $v_1, ..., v_m$. Let r be the highest power of 2 dividing the exponent G. Let K_i be the completions of K with respect to $v_i, 1 \leqslant i \leqslant m$. Denote by $\rho(r)$ the primitive 2^r-th root of 1.*

a) Suppose that K has nonzero characteristic or $K_i(\rho(r))/K_i$ is cyclic for all i. Then if L_i/K_i are G Galois extensions, there is a G Galois extension L/K such that $L \otimes_K K_i = L_i$.

This result will play the same role in the proof of the theorem below that the Grunwald-Wang theorem [7, Chap.10] plays in the proof of classical result which asserts that any finite-dimensional central simple algebra over a global field is cyclic.

Theorem 1. *Any central simple algebra A of finite dimension over a pseudoglobal field K is cyclic and* $\operatorname{ind} A = \exp A$.

Proof. The proof we give is a slight modification of the proof for the classical case of algebras over global fields [3, p. 441-443]. Let $v_1, ..., v_r$ be all the valuations of the pseudoglobal field K at which the algebra A has nontrivial local invariants. Set $n_i = \operatorname{ind} A_{v_i}$, where $A_{v_i} = A \otimes_K K_{v_i}$. Let m be the smallest common multiple of $n_1, ..., n_r$. By [3, Proposition 13.4] $n_i|n$, where $n = \deg A = [A : K]^{\frac{1}{2}}$ for all $i, 1 \leqslant i \leqslant r$. Thus $m|n$. Now we use the Saltman theorem instead of the Grunwald-Wang theorem. By Saltman's theorem there are the cyclic extensions L and M of field K, of degrees m and n respectively, such that L/K and L_i/K_{v_i} are cyclic extensions of degree m, and M/K, M_i/K_{v_i} are cyclic extensions of degree n. Notice that one can take L_i and M_i to be the unramified extensions of K_{v_i} of degrees m and n respectively. All the number n_i divide m and n. It is easy to show that for the algebras A_{v_i} over general local fields K_{v_i}, we have, just as in the case of algebras over local fields, that n_i are the orders of local invariants of algebras A_{v_i}. Therefore it follows from Proposition 5 a) that A splits over the fields L and M. Then it follows from Proposition 5 b) that the field M is isomorphic to a strongly maximal subfield of A, thus the algebra A is cyclic.

It remains to prove that $\operatorname{ind} A = \exp A$. Since $\exp A | \operatorname{ind} A$ for an algebra over any field [3], it is enough to prove that $\operatorname{ind} A \leqslant \exp A$. Since the field L splits A, by [3, Proposition 13, p. 301] $\operatorname{ind} A \leqslant m$. But for the exponent e of A we have $e \cdot i[A] = i([A]^e) = 0$, where i is the homomorphism from the exact sequence (1). It follows that $e \cdot \hat{i}_{K_v} = 0$, where $\hat{i}_{K_v}$ is the local invariant of A for $v \in V_K$. Therefore $n_i|e, 1 \leqslant i \leqslant r$, hence $m|e$. Finally, $\operatorname{ind} A \leqslant m \leqslant e = \exp A$, and this completes the proof.

Let A be a central simple algebra of finite dimension over a field K. Let L be a maximal subfield of D. It is known [6] that there is an isomorphism $\varphi : A \otimes_K L \simeq M_n(L)$, where $n = [L : K]$. The composition map $N_{\text{red}} : A \to K$

$$x \mapsto x \otimes 1 \longrightarrow \phi(x \otimes 1) \longrightarrow det(\phi(x \otimes 1)).$$

is called the reduced norm. It turns out that the reduced norm does not depend either on a choice of maximal subfield L, or on a choice of a homomorphism φ, and its image is contained in K.

We shall prove that the reduced norm homomorphism is surjective for the algebras over pseudoglobal fields. For this purpose we will need the following simple lemma.

Lemma 1. *Any pseudoglobal field is a C_2-field.*

Proof. Recall that a field K is called C_i-field, if any homogeneous polynomial of degree d on $n > d^i$ variables has a nontrivial zero in K^n. Every condition C_i, in particular C_1, can be formulated as a sentence of the first order logic. Therefore, as the pseudofinite fields are elementarily equivalent to ultraproducts of finite fields, the pseudofinite constant field of K is a C_1-field.

S. Lang [10] and M. Nagata [11] proved that the property of a field to be a C_i - field is preserved under algebraic extensions. Besides, if k is a C_i–field, and K is an extension of k of transcendence degree n, then K is a C_{i+n} - field. Hence, a pseudoglobal field is a C_2-field.

Corollary. *Any quadratic form on $\geqslant 5$ variables defined over a pseudoglobal field K, has a nontrivial zero over K.*

Proposition 14. *Let A be a central simple algebra of finite dimension over a pseudoglobal field K. Then*

1) The reduced norm homomorphism $N_{\text{red}} : A \longrightarrow K$ is surjective.

2) The reduced Whitehad group $SK_1 A = SL_1(A)/[A^*, A^*]$ of A is trivial. Here $SL_1(A) = \{a \in A | N_{\text{red}}(a) = 1\}$, $[A^*, A^*]$ is the commutant of multiplicative group A^* of algebra A.

Proof. 1) The algebra A is a matrix algebra over a skew field D. Clearly, it suffices to prove that $N_{\text{red}} : D \longrightarrow K$ is surjective. Let $[D : K] = n^2$. The map N_{red} is given by a homogeneous polynomial $\nu(\bar{x})$ of degree n on n^2 variables, and besides $\nu(\bar{x}) = 0$ if and only if $\bar{x} = \bar{0}$. We need to prove that the equation $\nu(\bar{x}) = a$ has a nontrivial solution over K for any $a \in K^*$. But, using that K is a C_2 - field by Lemma 1, this follows from the fact that the form $\nu(\bar{x}) = a x^n_{n^2+1}$ is of degree n and has $n^2 + 1$ variables.

2) The above arguments show that a pseudoglobal field is a C'_2 - field (a field K is called C'_2 - field if for any algebraic extension K'/K, and for any finite dimensional skew field D with center K', the reduced norm homomorphism $N_{\text{red}} : D \rightarrow K'$ is surjective). V. I. Yanchevskii [12] proved that, if K is a C'_2 – field, then $SK_1(A) = 0$ for any finite dimensional central simple K-algebra A. This completes the proof of Proposition 14.

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ПРО ГРУПУ БРАУЕРА ТА ПРИНЦИП ГАССЕ ДЛЯ ПСЕВДОГЛОБАЛЬНИХ ПОЛІВ

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Доведено критерій Тейта виконання принципу Гассе в скінченних розширеннях поля алгебричних функцій K із псевдоскінченним полем констант. Крім того, кожна скінченновимірна алгебра над таким полем є циклічною, а її індекс дорівнює експоненті.

Ключові слова: теорія полів класів, поле алгебричних функцій, група Брауера, принцип Гассе, скінченновимірні центральні прості алгебри.

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SOME OPEN PROBLEMS IN TOPOLOGICAL ALGEBRA

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In the paper we consider and comment some open problems in topological algebra posed by participants of the conference dedicated to the 20th anniversary of the Chair of Algebra and Topology of Lviv National University, that was held in September, 2001.
Key words: topological group, paratopological group, topological semigroup.

This is the list of open problems in topological algebra posed on the conference dedicated to the 20th anniversary of the Chair of Algebra and Topology of Lviv National University, that was held in September, 2001.

Problem 1 (Čhoban). *Is every topological group a quotient group of a zero-dimensional topological group of the same weight? When an almost metrizable topological group is a quotient of a zero-dimensional group of the same weight?*

A space X is zero-dimensional if $\dim X = 0$. A topological group is *almost metrizable* if contains a compact subset of countable character. Let us make some comments to this problem. If a topological group G is a quotient of an almost metrizable group H with $\operatorname{ind} H = 0$, then G contains a zero-dimensional compact subgroup of countable character in G.

If H is a zero-dimensional subgroup of countable character in a group G and $xh = hx$ for every $x \in G$ and $h \in H$, then G is a quotient group of some almost metrizable zero-dimensional group of the same weight. In particular the answer to the first part of Problem 1 is positive for metrizable groups (see [10], [8]) and for almost metrizable abelian groups. A. V. Arhangel'skii proved that every topological group is a quotient of a zero-dimensional σ-discrete group. For universal algebras this fact was proved in [9].

Problem 2 (Čhoban). *Under which conditions is the free universal algebra of an uncountable signature over a metrizable space X paracompact? In particular, is the free topological linear space $L(X)$ of X over the discrete field of real numbers paracompact?*

A semigroup S with the identity e endowed with a topology is called a *left* (resp. *right*) *bounded* if for every neighborhood U of e there is a finite subset F of S such that $S = FU$ (resp. $S = UF$); S is called *bounded* if S is both left and right bounded. Replacing the "finite subset F" by "countable subset F" we get the definition of a (*left, right*) *ω-bounded* semigroup.

For a topological space X let $S(X) \subset X^X$ be the semigroup of all continuous selfmappings of X endowed with the topology of pointwise convergence (i.e., the topology inherited from the Tychonov product X^X). A topological space X is called *homogeneous* if for any points $x_1, x_2 \in X$ there is a homemorphism h of X with $h(x_1) = x_2$.

Problem 3 (Protasov). *Is the semigroup $S(X)$ left bounded for every zero-dimensional compact homogeneous space?*

The answer is affirmative provided X has a base of the topology consisting of pairwise homeomorphic clopen subsets, see [22].

Problem 4 (Protasov). *Is the semigroup $S(X)$ right bounded for any zero-dimensional homogeneous space X?*

The answer is affirmative provided X has a base of the topology consisting of clopen subsets homeomorphic to X, see [22].

Let X be a topological space. A subgroup H of $S(X)$ is called *distal* if for any distinct points $x_1, x_2 \in X$ and any point $x \in X$ there is a neighborhood U of x such that $\{h(x_1), h(x_2)\} \not\subset U$ for all $h \in H$. It is proven in [22, Theorem 4] that a left bounded subgroup $H \subset S(X)$ is distal provided H acts transitively on X.

Problem 5 (Protasov). *Let X be a compact space and H be a distal subgroup of $S(X)$ acting transitively on X. Is H left bounded?*

Under a *left-topological group* we understand a pair (G, τ) consisting of a group G and a topology τ invariant with respect to the left shifts $l_g : x \mapsto gx$. If, in addition τ is invariant with respect to the right shifts, then (G, τ) is called a *semitopological group*. A semitopological group G with continuous inverse mapping $x \mapsto x^{-1}$ is called a *quasitopological group*. If the group operation of G is continuous with respect to the topology τ, then (G, τ) is a *paratopological group*. If, additionally, the operation of taking the inverse is continuous, then (G, τ) is a topological group.

It is well known that σ-compact topological groups have countable cellularity [29] while compact topological groups support a strictly positive probability measure (i.e., a Borel probability measure μ such that $\mu(U) > 0$ for any nonempty open subset U of the group). Recently T. Banakh and O. Ravsky [4] (see also [6]) proved that any bounded paratopological group G has countable cellularity (moreover, each cardinal of uncountable cofinality is a precaliber of G). On the other hand, according to [23] for every infinite cardinal τ there is a left bounded left topological group of cellularity τ.

Problem 6 (Protasov). *Let G be a bounded semitopological (quasitopological) group. Has G countable cellularity?*

The answer to this problem is positive for bounded first countable left topological groups. This follows from an easy observation that a first countable left topological group G has countable cellularity provided G is *left ω-narrow* in the sense that for each neighborhood $U \subset G$ of the unit and each uncountable subset $F \subset G$ there are two distinct elements $x, y \in F$ with $xU \cap yU \neq \emptyset$. It is easy to see that a left topological group is left ω-narrow provided it is right ω-bounded. Therefore each right ω-bounded first countable left topological group has countable cellularity.

Problem 6 is related to another

Problem 7 (Protasov). *Does every zero-dimensional compact homogeneous space admit a structure of a left topological group?*

According to [25], each homogeneous topological (T_1-)space X is the quotient space G/H of the homeomorphism group G of X endowed with a suitable left invariant topology by a (closed discrete) subgroup H of G. For countable spaces there is a much stronger result proved recently by E. Zelenyuk: for each countable group G and each regular homogeneous countable topological space X there is a left topological group H homeomorphic to X and algebraically isomorphic to G. Thus each countable regular space is homeomorphic to a semitopological group. This Zelenyuk's result is specific for countable spaces and cannot be generalized onto zero-dimensional homogeneous spaces: there are examples of zero-dimensional homogeneous spaces homeomorphic to no semitopological group. Many such examples can be constructed with help of a recent result [18] implying that *each Tychonov almost Čech-complete semitopological group is a Čech-complete topological group.* A Tychonov space X is defined to be *almost Čech-complete* if X contains a dense Čech-complete subspace. Examples of compact left topological groups which are not topological groups (see [19]) show that the mentioned result of [18] cannot be generalized onto left topological groups. Nonetheless we do not know the answer to

Problem 8 (Banakh). *Is every almost Čech complete left topological group Čech complete?*

Let us note that each almost Čech complete left topological group G is *non-expandable* in the sense that G coincides with any Tychonov left topological group $\tilde{G}$ containing G as a dense subgroup. The non-expandable left topological groups can be thought as "complete" in some sense.

Problem 9 (Banakh). *Investigate the class of non-expandable left topological groups. In particular, is every metrizable non-expandable left topological group Čech-complete?*

According to a remarkable theorem of [14], the countable power X^ω of a zero-dimensional metrizable space X is homogeneous provided X is meager or almost Čech-complete. This theorem in combination with [18] allows us to construct simple examples of homogeneous almost Čech complete metrizable spaces which are not Čech-complete and thus fail to support the structure of a semitopological group. On the other hand, by the technique of [13] and [15] it can be shown that the countable power X^ω of any meager separable metrizable space X is homeomorphic to a Boolean topological group.

There are also simple examples of homogeneous compact first countable spaces homeomorphic to no semitopological groups. Such spaces can be simply constructed with help of the Motorov Theorem asserting that the countable power K^ω of each first-countable zero-dimensional compact space K is homogeneous, see [2] and [12]. Observe that such a power K^ω admits the structure of a semitopological group if and only if the compactum K is metrizable (in this case K^ω is homeomorphic to the Cantor set).

In constrast, the structure of a left-topological group does not impose so strict restrictions on the topological structure of homogeneous compacta. For example, the Aleksandrov "two-arrows" space T is a non-metrizable zero-dimensional first-countable compactum carrying the structure of a left-topological group (algebraically isomorphic to the semi-direct product $S^1 \rtimes \mathbb{Z}_2$ of the circle and $\mathbb{Z}_2 = \{0,1\}$). It follows that T^ω carries a structure of a left-topological group.

Problem 10 (Banakh). *Does the countable power K^ω of a compact first-countable zero-dimensional space K carry a structure of a left topological group?*

Note that the affirmative answer to Problem 6 would follow from the affirmative answer of

Problem 11 (Protasov). *Does every compact left topological group G support a strictly positive probability Borel measure?*

There exists a compact left-topological group admitting no *invariant* probability Borel measure, see [19].

Problem 12 (Protasov). *Is every topological group algebraically generated by a nowhere dense subset?*

Let us mention that each countable topological group is algebraically generated by some closed discrete subset (see reference in [24]) while every left topological group is algebraically generated by some subset with empty interior [24].

It is known that each regular countably compact paratopological group is a bounded topological group, see [21]. I. Guran [17] asked if any Hausdorff countably compact paratopological group is a topological group. O. Ravsky and E. Reznichenko [26] (see also [27]) observed that this question is equivalent to the problem of the ω-boundedness of any Hausdorff countably compact paratopological group. In the same paper [26] an example of a Hausdorff countably compact paratopological group which is not a topological group was constructed under Martin Axiom.

Problem 13 (Guran). *Is there a ZFC-example of a Hausdorff countably compact paratopological group which is not a topological group?*

Problem 14 (Guran). *Is a paratopological group G right ω-bounded if it is left ω-bounded?*

The answer to the last question is positive provided G is *saturated* (in the sense that the inverse U^{-1} of any neighborhood $U \subset G$ of the unit has non-empty interior in G) or *quasi-balanced* (in the sense that for each neighborhood $U \subset G$ of the unit there is a countable family $\mathcal{U}$ of neighborhoods of the unit such that for any $a \in G$ there are a neighborhood $U \in \mathcal{U}$ and an element $b \in G$ with $Wb \subset aU$), see [4] and [5].

A subset A of a topological group G is called *o-bounded* if for any sequence $(U_n)_{n\in\omega}$ of neighborhoods of the origin of G there is a sequence $(F_n)_{n\in\omega}$ of finite subsets of G such that $A \subset \bigcup_{n\in\omega} F_n U_n$. It is clear that each σ-compact topological group is o-bounded while each o-bounded group is ω-bounded. Next, given a subset A of a topological group G, consider the following game OF(A) (abbreviated from Open-Finite). Two players, I and II, choose at every step $k \in \omega$ a neighborhood $U_n \subset G$ of the origin, and a finite subset F_n of G, respectively. At the end of the game, player II is declared the winner if $A \subset \bigcup_{n\in\omega} F_n U_n$. It is easy to see that for a σ-compact group G player II has a winning strategy in the game $\mathrm{OF}(G)$. On the other hand, if a topological group G is not o-bounded, then player I has a winning strategy in $\mathrm{OF}(G)$.

Problem 15 (Banakh). *Is there a (metrizable) o-bounded topological group G such that player I has a winning strategy in the game OF(G)?*

Such a group, if exists, cannot be analytic and abelian (more generally, cannot be an analytic SIN-group). We remind that a topological space X is analytic if it is a metrizable continuous image of a separable complete metric space. On the other hand, the group $\mathbb{Z}^\omega$ contains a dense o-bounded G_δ-subset A such that the first player has a winning strategy in the game $\mathrm{OF}(A)$, see [7].

Problem 16 (Banakh). *Let n be a positive integer. Is there a compact subset K of the real line $\mathbb{R}$ such that the difference $K - K = \{x - y : x, y \in K\}$ is a neighborhood of zero in $\mathbb{R}$ but the sum $\underbrace{K + \cdots + K}_{n}$ is nowhere dense in $\mathbb{R}$?*

For $n = 2$ the answer is affirmative: By computer calculations, S. Ravsky has found that the compact subset

$$K = \Big\{ \sum_{n=1}^{\infty} \frac{x_n}{19^n} : x_n \in \{0, 1, 2, 10, 13, 16, 17, 18\} \Big\}$$

of the closed interval $[0, 1]$ has the following properties $K - K \supset (-1, 1)$ but $K + K \subset \big\{ \sum_{n=1}^{\infty} \frac{x_n}{19^n} : x_n \in \{0, \dots, 18\} \setminus \{6\} \big\}$ and thus is nowhere dense in $\mathbb{R}$.

Added in proofs. Recently T. Banakh and O. Hryniv [3] have answered Problems 3, 4 and 7 in negative. A suitable counterexample is supplied by the van Douwen homogeneous compactum Ξ constructed in [11] (see also [20]). The space has a very simple description: $\Xi = (A \times \{-1, 1\}) \cup ([0, 1] \setminus A) \times \{0\}$ for a suitable subset $A \subset (0, 1)$ and Ξ is endowed with the interval topology generated by the natural lexicographic order. The space Ξ has many surprising properties, in particular: 1) Ξ is a first-countable linearly ordered zero-dimensional homogeneous compactum admitting a continuous map $\pi : \Xi \to [0, 1]$ such that $|\pi^{-1}(x)| \leqslant 2$ for each $x \in [0, 1]$; 2) Ξ possesses a unique Borel probability measure μ that projects onto the Lebesgue measure by the map π; 3) two closed-and-open subsets of Ξ are homeomorphic if and only if their μ-measures coincide; 4) each homeomorphism of Ξ is measure preserving and each continuous self-mapping of Ξ does not increase the measure; 5) Ξ contains a countable dense subset $\mathbb{Q}$ such that $\mathbb{Q} \cap h(\mathbb{Q}) \neq \emptyset$ for any homeomorphism of Ξ 6) Ξ is homeomorphic to no left-topological group; 7) the group of homeomorphisms $H(\Xi)$ of Ξ is neither left nor right-bounded in the semigroup $C(\Xi)$ of continuous self-mappings of Ξ.

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ДЕЯКІ ВІДКРИТІ ПРОБЛЕМИ З ТОПОЛОГІЧНОЇ АЛГЕБРИ

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ON CLOSED EMBEDDINGS OF FREE TOPOLOGICAL ALGEBRAS

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Let $\mathcal{K}$ be a complete quasivariety of completely regular universal topological algebras of continuous signature $\mathcal{E}$ (which means that $\mathcal{K}$ is closed under taking subalgebras, Cartesian products, and includes all completely regular topological $\mathcal{E}$-algebras algebraically isomorphic to members of $\mathcal{K}$). For a topological space X by $F(X)$ we denote the free universal $\mathcal{E}$-algebra over X in the class $\mathcal{K}$. Using some extension properties of the Hartman-Mycielski construction we prove that for a closed subspace X of a metrizable (more generally, stratifiable) space Y the induced homomorphism $F(X) \to F(Y)$ between the respective free universal algebras is a closed topological embedding. This generalizes one result of V.Uspenskiĭ [11] concerning embeddings of free topological groups.

Key words: universal topological algebra, free universal algebra, closed embedding, stratifiable space, metrizable space, Hartman-Mycielski construction

One of important recent achievements in the theory of free topological groups is a charming theorem by O. Sipacheva [9] asserting that the free topological group $F(X)$ of a subspace X of a Tychonov space Y is a topological subgroup of $F(Y)$ if and only if any continuous pseudometric on X can be extended to a continuous pseudometric on Y, see [9]. The "only if" part of this theorem was proved earlier by V. Pestov [8] while the "if" part was proved by V. V. Uspenskiĭ [11] for the partial case of metrizable (or more generally, stratifiable) Y. To prove their theorems both Uspenskiĭ and Sipacheva used a rather cumbersome technique of pseudonorms on free topological groups which makes their method inapplicable for studying some other free objects.

In this paper, using a categorial technique based on extension properties of the Hartman-Mycielski construction we shall generalize the Uspenskiĭ theorem and prove some general results concerning embeddings of free universal algebras. It should be mentioned that the Hartman-Mycielski construction has been exploited in [2] for proving certain results concerning embeddings of free topological inverse semigroups.

Now we remind some notions of the topological theory of universal algebras developed by M. M. Choban and his collaborators, see [5]. Under a *continuous signature* we shall understand a sequence $\mathcal{E} = (E_n)_{n\in\omega}$ of topological spaces. A *universal topological algebra of continuous signature* $\mathcal{E}$ or briefly a *topological* $\mathcal{E}$*-algebra* is a topological space X endowed with a sequence of continuous maps $(e_n : E_n \times X^n \to X)_{n\in\omega}$ called *algebraic operations* of X. A subset $A \subset X$ is called a *subalgebra* of X if

$e_n(E_n \times A^n) \subset A$ for all $n \in \omega$. Under *a homomorphism* between topological $\mathcal{E}$-algebras $(X, (e_n^X)_{n\in\omega})$ and $(Y, (e_n^Y)_{n\in\omega})$ we understand a map $h : X \to Y$ such that

$$e_n^Y(c, h(x_1), \ldots, h(x_n)) = h(e_n^X(c, x_1, \ldots, x_n))$$

for any $n \in \omega$, $c \in E_n$, and points $x_1, \ldots, x_n \in X$. Two topological $\mathcal{E}$-algebras X, Y are *(algebraically) isomorphic* if there is a bijective homomorphism $h : X \to Y$. If, in addition, h is a homeomorphism, then X and Y are *topologically isomorphic*.

Under a *free universal algebra* of a topological space X in a class $\mathcal{K}$ of topological $\mathcal{E}$-algebras we understand a pair $(F(X), i_X)$ consisting of a topological $\mathcal{E}$-algebra $F(X) \in \mathcal{K}$ and a continuous map $i_X : X \to F(X)$ such that for any continuous map $f : X \to K$ into a topological $\mathcal{E}$-algebra $K \in \mathcal{K}$ there is a unique continuous homomorphism $h : F(X) \to K$ such that $f = h \circ i_X$. It follows that for any continuous map $f : X \to Y$ between topological spaces there is a unique continuous homomorphism $F(f) : F(X) \to F(Y)$ such that $F(f) \circ i_X = i_Y \circ f$. Our aim in the paper is to find conditions on f guaranteeing that the homomorphism $F(f)$ is a topological embedding.

According to [5], a free universal algebra $(F(X), i_X)$ of a topological space X exists (and is unique up to a topological isomorphism) provided $\mathcal{K}$ is *a quasivariety*, which means that the class $\mathcal{K}$ is closed under taking subalgebras and arbitrary Cartesian products. A quasivariety $\mathcal{K}$ of topological $\mathcal{E}$-algebras is called *a complete quasivariety* if any completely regular topological $\mathcal{E}$-algebra X, algebraically isomorphic to a topological algebra $Y \in \mathcal{K}$, belongs to the class $\mathcal{K}$.

Finally we remind that a regular topological space X is called *stratifiable* if there exists a function G which assigns to each $n \in \omega$ and a closed subset $H \subset X$, an open set $G(n, H)$ containing H so that $H = \bigcap_{n\in\omega} \overline{G(n, H)}$ and $G(n, K) \supset G(n, H)$ for every closed subset $K \supset H$ and $n \in \omega$. It is known that the class of stratifiable spaces includes all metrizable spaces and is closed with respect to many countable operations over topological spaces, see [3], [6].

Now we are able to state one of our main results.

Theorem 1. *Let $\mathcal{K}$ be a complete quasivariety of completely regular topological $\mathcal{E}$-algebras of continuous signature $\mathcal{E}$. For any closed topological embedding $e : X \to Y$ between stratifiable spaces the induced homomorphism $F(e) : F(X) \to F(Y)$ between the corresponding free algebras is a closed topological embedding.*

In fact, Theorem 1 follows from a more general result involving the construction of Hartman and Mycielski. This construction appeared in [7] and was often exploited in topological algebra, see [4]. For a topological space X let $HM(X)$ be the set of all functions $f : [0; 1) \to X$ for which there exists a sequence $0 = a_0 < a_1 < \cdots < a_n = 1$ such that f is constant on each interval $[a_{i-1}, a_i)$, $1 \leqslant i \leqslant n$. A neighborhood subbase of the Hartman-Mycielski topology of $HM(X)$ at an $f \in HM(X)$ consists of sets $N(a, b, V, \varepsilon)$, where

1) $0 \leqslant a < b \leqslant 1$, f is constant on $[a; b)$, V is a neighbourhood of $f(a)$ in X and $\varepsilon > 0$;
2) $g \in N(a, b, V, \varepsilon)$ means that $|\{t \in [a; b) : g(t) \notin V\}| < \varepsilon$, where $|\cdot|$ denotes the Lebesgue measure on $[0, 1)$.

If X is a Hausdorff (Tychonov) space, then so is the space $HM(X)$, see [7], [4]. The construction HM is functorial in the sense that for any continuous map $p: X \to Y$ between topological spaces the map $HM(p): HM(X) \to HM(Y)$, $HM(p): f \mapsto p \circ f$, is continuous, see [7], [4], [10].

The space X can be identified with a subspace of $HM(X)$ via the embedding $hm_X: X \to HM(X)$ assigning to each point x the constant function $hm_X(x): t \mapsto x$. This embedding $hm_X: X \to HM(X)$ is closed if X is Hausdorff. It is easy to see that for any continuous map $f: X \to Y$ we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle hm_X}\downarrow & & \downarrow{\scriptstyle hm_Y} \\ HM(X) & \xrightarrow{HM(f)} & HM(Y). \end{array}$$

Our interest in the Hartman-Mycielski construction is stipulated by the following important extension result proven in [1].

Proposition 1. *For a closed subspace X of a stratifiable space Y there is a continuous map $r: Y \to HM(X)$ extending the embedding $hm_X: X \subset HM(X)$.*

It will be convenient to call a subspace X of a space Y *an HM-valued retract of* Y if there is a continuous map $r: Y \to HM(X)$ extending the canonical embedding $hm_X: X \subset HM(X)$. In these terms, Proposition 1 asserts that each closed subspace of a stratifiable space Y is an HM-valued retract of Y.

As a set, the space $HM(X)$ can be thought of as a subset of the Cartesian power $X^{[0,1)}$. Moreover, if $(X, (e_n)_{n\in\omega})$ is a topological $\mathcal{E}$-algebra then $HM(X)$ is a subalgebra of $X^{[0,1)}$. Let $\{e_n^{HM}: E_n \times HM(X)^n \to HM(X)\}_{n\in\omega}$ denote the induced algebraic operations on $HM(X)$. That is, $e_n^{HM}(c, f_1, \dots, f_n)(t) = e_n(c, f_1(t), \dots, f_n(t))$ for $n \in \omega$, $(c, f_1, \dots, f_n) \in E_n \times HM(X)^n$, and $t \in [0,1)$. It is easy to verify that the continuity of the operation e_n implies the continuity of the operation e_n^{HM} with respect to the Hartman-Mycielski topology on $HM(X)$). Thus we get

Proposition 2. *If $(X, (e_n)_{n\in\omega})$ is a topological $\mathcal{E}$-algebra, then $(HM(X), (e_n^{HM})_{n\in\omega})$ is a topological $\mathcal{E}$-algebra too. Moreover the embedding $hm_X: X \to HM(X)$ is a homomorphism of topological $\mathcal{E}$-algebras.*

Since $HM(K)$ is algebraically isomorphic to a subalgebra of $X^{[0,1)}$, we conclude that for each completely regular topological $\mathcal{E}$-algebra X belonging to a complete quasivariety $\mathcal{K}$ of topological $\mathcal{E}$-algebras the $\mathcal{E}$-algebra $HM(X)$ also belongs to the quasivariety $\mathcal{K}$. Now we see that Theorem 1 follows from Proposition 1 and

Theorem 2. *Let $\mathcal{K}$ be a quasivariety of (Hausdorff) topological $\mathcal{E}$-algebras of continuous signature $\mathcal{E}$. Then for a subspace X of a topological space Y the homomorphism $F(e): F(X) \to F(Y)$ induced by the natural inclusion $e: X \to Y$ is a (closed) topological embedding provided X is an HM-valued retract of Y and $HM(F(X)) \in \mathcal{K}$.*

Proof. Suppose that $HM(F(X)) \in \mathcal{K}$ and X is a HM-valued retract of Y. The latter means that there is a continuous map $r: Y \to HM(X)$ such that $hm_X = r \circ e$ where $hm_X: X \to HM(X)$ and $e: X \to Y$ are natural embeddings. Applying to the maps hm_X, e and r the functor F of the free universal $\mathcal{E}$-algebra in the quasivariety $\mathcal{K}$, we get the equality $F(hm_X) = F(r) \circ F(e)$.

Applying the functor HM to the canonical map $i_X : X \to F(X)$ of X into its free universal algebra, we get a continuous map $HM(i_X) : HM(X) \to HM(F(X))$. Taking into account that the $\mathcal{E}$-algebra $HM(F(X))$ belongs to the quasivariety $\mathcal{K}$, by the definition of the free algebra $(F(HM(X)), i_{HM(X)})$, we can find a unique continuous homomorphism $h : F(HM(X)) \to HM(F(X))$ such that $h \circ i_{HM(X)} = HM(i_X)$. Let us show that $h \circ F(hm_X) = hm_{F(X)}$. Since the maps $h \circ F(hm_X)$ and $hm_{F(X)}$ are homomorphisms from the free algebra $F(X)$ of X, to prove the equality $h \circ F(hm_X) = hm_{F(X)}$ it suffices to verify that $h \circ F(hm_X) \circ i_X = hm_{F(X)} \circ i_X$.

By the definition of the homomorphism $F(hm_X)$, we get the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & F(X) \\ {\scriptstyle hm_X}\downarrow & & \downarrow{\scriptstyle F(hm_X)} \\ HM(X) & \xrightarrow{i_{HM(X)}} & F(HM(X)) \end{array}$$

which implies that $h \circ F(hm_X) \circ i_X = h \circ i_{HM(X)} \circ hm_X = HM(i_X) \circ hm_X$ by the choice of the homomorphism h.

On the other hand, by the naturality of the transformations $\{hm_Z\}$, we get the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & F(X) \\ {\scriptstyle hm_X}\downarrow & & \downarrow{\scriptstyle hm_{F(X)}} \\ HM(X) & \xrightarrow{HM(i_X)} & HM(F(X)) \end{array}$$

which implies that $HM(i_X) \circ hm_X = hm_{F(X)} \circ i_X$. Thus $h \circ F(hm_X) \circ i_X = hm_{F(X)} \circ i_X$ which just yields $hm_{F(X)} = h \circ F(hm_X) = h \circ F(r) \circ F(e)$. Observe that the map $hm_{F(X)}$ is an embedding. Moreover, it is closed if $F(X)$ is Hausdorff (which happens if the quasivariety $\mathcal{K}$ consists of Hausdorff $\mathcal{E}$-algebras). Now the following elementary lemma implies that $F(e)$ is a (closed) embedding. □

Lemma. *Let $f : X \to Y$, $g : Y \to Z$ be continuous maps. If $g \circ f : X \to Z$ is a (closed) topological embedding, then so is the map f.*

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ПРО ЗАМКНЕНІ ВКЛАДЕННЯ ВІЛЬНИХ ТОПОЛОГІЧНИХ АЛГЕБР

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Нехай $\mathcal{K}$ – повний квазімноговид цілком регулярних універсальних топологічних алгебр неперервної сигнатури $\mathcal{E}$. Для топологічного простору X через $F(X)$ позначимо вільну універсальну $\mathcal{E}$-алгебру над X в класі $\mathcal{K}$. Використовуючи конструкцію Гартмана-Мицельського, доводимо, що для замкненого підпростору X метризовного простору Y індукований гомоморфізм $F(X) \to F(Y)$ вільних універсальних алгебр – замкнене вкладення.

Ключові слова: універсальна топологічна алгебра, вільна топологічна алгебра, замкнене вкладення, стратифікований простір, метризовний простір, конструкція Гартмана-Мицельського.

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GENERATING PROPERTIES OF INVERTIBLE POLYNOMIAL MAPS IN THREE VARIABLES, WHICH HAVE A SMALL COMPOSITIONAL-TRIANGULAR LENGTH

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It is shown that each k-triangular invertible map (choosing in advance) for $k = 1, 2, 3$ with linear maps generate the group of tame polynomial automorphisms in three variables.

Key words: invertible polynomial map, affine space, affine group, infinitely dimensional algebraic group.

Invertible polynomial maps of an affine space A^n over a field K form a group GA_n (see [1],[2]), which sometimes is called the affine Cremona group. Elements of GA_n can be written down as tuples of polynomials

$$\langle f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)\rangle, \tag{1}$$

$f_i \in K[x]$, with composition of tuples as group operation and $X = \langle x_1, \dots, x_n\rangle$ as the unit of GA_n. It is useful to introduce vectors of the standard basis $\vec{e_i} = (0, \dots, 0, 1, 0, \dots, 0), i = 1, 2, \dots, n$ and represent the elements (1) in the form

$$g = \sum_{i=1}^{n} f_i(x_1, \dots, x_n)\vec{e}_i \tag{2}$$

For such a polynomial map $g \in GA_n$ let $\deg g = \max_i \deg f_i$. It is evident that maps with $\deg g \leqslant 1$ form an isomorphic copy of the affine group in GA_n. In particular, the elements $c_i = X + \vec{e}_i$ form a basis of A_n^+ as a vector space over K . Everywhere below we will identify AGL_n with the image of this standard enclosure. In this sense we shall also understand the matrix and permutational notations. For instance, the cycle $(1, 2, 3)$ in the dimension 3 means the transformation $\langle x_2, x_3, x_1\rangle$ in the form (1) and $A_{ij} = X + x_j\vec{e}_i \in GL_n$ in the form (2). It is easy to check that tuples (1) of the kind

$$\langle x_1 + h_1, x_2 + h_2(x_1), \dots, x_i + h_i(x_1, \dots, x_{i-1}), \dots x_n + h_n(x_1, \dots x_{n-1})\rangle \tag{3}$$

or

$$X + \sum_{i=1}^{n} h_i(x_1, \dots, x_{i-1})\vec{e}_i,$$

in the form (2), are invertible polynomial maps for arbitrary polynomials h_i and make up the subgroup U_n of the unitriangular transformations. U_n can be considered as iterated algebraic wreath product of K^+. There is a semidirect decomposition $U_n = U_{n-1}F_n$, where U_{n-1} (F_n) consist of the elements of form (3) with $h_n \equiv 0$ $(h_i \equiv 0,\ i = 1, \ldots, n-1)$. For $n = 3$ we have $U_3 = U_2F_2$. There is a partial order $\prec$ on U_n, which is the extension of the inverse lexicographical order of monomials. We will say that an element u_1 has height less than u_2, for $u_1, u_2 \in U_n$ if $u_1 \prec u_2$.

The normalizer $B_n = N_{GA_n}(U_n) = T_n \cdot U_n$ is a subgroup of triangular automorphisms whose elements have the form

$$\langle \alpha_1 x_1 + h_1, \alpha_2 x_2 + h_2(x_1), \ldots, \alpha_i x_i + h_i(x_1, \ldots, x_{i-1}), \ldots \alpha_n x_n + h_n(x_1, \ldots x_{n-1}) \rangle, \quad (4)$$

where T_n is an algebraic torus, $\alpha_i \in K^*$.

The subgroup GA_n^0 is a stabilizer of zero $(f_i(0) = 0)$ and contains a descended chain of the normal subgroups $GA_n^m, m = 0, 1, 2, 3, \ldots$, whose elements have the form $\langle x_1 + H_1^{m+1} + \ldots, x_2 + H_2^{m+1} + \ldots, \ldots, x_n + H_n^{m+1} + \ldots \rangle$, where H_2^{m+1} are homogeneous forms of degree $m+1$ and the dots mean items of higher degrees. Here is a simplest example of the element from GA_n^m : $\sigma^{(m)} = \langle x_1, x_2, \ldots, x_n + x_1^{m+1} \rangle$. There is a series of natural epimorphisms $\phi_k : GA_n^0 \to GA_n^0/GA_n^k$, moreover the corresponding quotient-group is a finite-dimensional algebraic groups. In particular, we have the semidirect decomposition , $GA_n^0 = GL_n \cdot GA_n^1$. Let us recall next definitions from [3]

Definition 1. An *elementary polynomial map* is defined as a transformation of kind

$$\langle x_1, x_2, \ldots, x_{i-1}, x_i + a(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \ldots, x_n \rangle.$$

Definition 2. A polynomial map, which is a finite composition of elementary or linear maps is called a *tame* map.

Tame automorphisms form group which can be defined as $TGA_n = \langle AGL_n, B_n \rangle$. Indeed, each elementary polynomial map is conjugate by a transposition (i, n) with an unitriangular one. On the other hand, triangular elements of kind $\langle x_1, x_2, \ldots, x_{i-1}, x_i + a(x_1, \ldots, x_{i-1}), x_{i+1}, \ldots, x_n \rangle$ are elementary maps and generate the group U_n.

One of the most difficult questions about affine Cremona groups is (see [2]): is TGA_n a proper subgroup of GA_n (the answer is negative for $n = 2$)?

Theorem 1. ([4]). *$GA_2 = B_2 * AGL_2$, where $*$ stands for the amalgamated product with the intersection $B_2 \cap AGL_2$, consisting of linear triangular maps (4).*

Corollary 1. $TGA_2 = GA_2$.

Corollary 2. *Let $\sigma^{(m)} = \langle x_1, x_2 + x_1^{m+1} \rangle \in GA_2^{(m)} \cap U_2$. Then the groups $Q_m = < AGL_2, \sigma^{(m)} >$ form an ascending chain $AGL_2 = Q_0 < Q_1 < \ldots Q_m < Q_{m+1}, \ldots$ with $GA_2 = \cup_m Q_m$.*

Proof. It follows from a uniqueness of element's decomposition in amalgamated products. Hence, the element $\sigma^{(k)}, k > m$ can not belong to Q_m. On the other hand, given element $\sigma^{(k)}$, one can calculate commutators with translations from A_n^+ and gets all elements $\sigma^{(m)}, m < k$. Thus $Q_m < Q_k$.

In particular, this means that in the dimension 2 the affine group is not maximal.

In the dimension more then 2 we have a more complicated situation. The question about the coincidence TGA_n with GA_n is open even in the dimension 3.

Conjecture (Nagata [2]) *The automorphism*

$$\langle x_1 - 2x_2(x_2^2 + x_3x_1) - x_3(x_2^2 + x_3x_1)^2, x_2 + x_3(x_2^2 + x_3x_1), x_3\rangle$$

is wild.

On the other hand, as follows from [5], the affine group is a maximal closed algebraic subgroup of GA_n as an ∞-dimensional group, correspondent definition was introduced in [1]. Comparing results with Nagata's conjecture, it is natural to pose a question: *is does AGL_n a maximal subgroup of TGA_n?*

As was proved in [6], one can replace the whole subgroup B_3 in the equality $TGA_3 = \langle AGL_3, B_3\rangle$ by any its nonlinear element. More precise,

Theorem 2. *([6]) Let $t \in B_3$ be an arbitrary nonlinear triangular map. Then*

$$TGA_3 = \langle t, AGL_3\rangle.$$

The aim of this paper is to show that bitriangular and three-triangular elements $g \in TGA_3$ have similar property $TGA_3 =< AGL_3, g >$.

Definition 3. A map $q \in GA_n$ is called k-triangular if it can be presented in the form

$$q = A_1 \cdot t_1 \cdot A_2 \cdot t_2 \cdot, \cdots, \cdot A_k \cdot t_k \cdot A_{k+1}, \tag{5}$$

where $A_i \in AGL_n, t_i \in B_n$. The smallest number k satisfying (5) is called *the triangular-compositional length of q.*

The term bitriangular map means that triangular-compositional length of the map equals 2.

Theorem 3. *Let q be an arbitrary bitriangular transformation. Then*

$$TGA_3 = \langle q, AGL_3\rangle.$$

Let $G = \langle q, AGL_3\rangle$. Without lost of generality, we may assume that $q = t^A \cdot t'$, $t, t' \in U_3 \cap GA_3^0, A \in GL_3$. Let $A = B_1WB_2$ be a Brua decomposition, where $B_1, B_2 \in GL_3 \cap U_3$ and W is a permutational matrix. Then we have the bitriangular element $q^{B_2^{-1}} = t_1^W \cdot t_2 \in G$, where $t_1 = t^{B_1}, t_2 = (t')^{B_2^{-1}}$. Let us preserve the notation q for this new element of G

$$q = t_1^W \cdot t_2, \tag{6}$$

and represent the triangular elements t_1, t_2 in the the form

$$t_1 = X + a_1(x_1)\vec{e}_2 + a_2(x_1, x_2)\vec{e}_3, \quad t_2 = X + b_1(x_1)\vec{e}_2 + b_2(x_1, x_2)\vec{e}_3 \in GA_3^1 \cap U_1.$$

Without lost of generality, one can suppose that $t_1, t_2 \in GA_n^0$ and polynomials a_1, a_2, b_1, b_2 have no linear parts. The idea of the proof is simple - to find an affine map a which permutable with t_2 (or t_1^W), calculate a commutator $[a, q^{-1}]$ (or $[a, q]$) and get a triangular element from G. In particular, one can use the element $a = c_3$ from the center of U_3 to get an 1-triangular element $t_1^{-Wc_3}t_1^W$ and apply Theorem 2.

Unfortunately, for some kind of triangular elements t_1, t_2 the correspondent commutators will be linear triangular elements and direct application of Theorem 2 is impossible. The situations, when it can be happened, is described in the next proposition.

Proposition 1. *If* $L = \{l(x_1, x_2) = \alpha x_1 + \beta x_2 + \gamma\}$ *is a set of linear polynomials, then*

1^0. $\Delta h = h(x_1, x_2 + a(x_1) + 1) - h(x_1, x_2 + a(x_1)) \in L$ *for some polynomial* $a(x_1)$, $\deg a > 1$ *iff polynomial* h *has the form* $h = h_{02}x_2^2 - 2h_{02}a(x_1)x_2 + h_{11}x_1x_2 + r(x_1) + l$, *where* $l \in L$.

2^0. $\Delta h = h(x_1 + 1, x_2 - \alpha x_1^2) - h(x_1, x_2 - \alpha x_1^2) \in L$ *iff* $h = r(x_2) + h_{11}x_1x_2 + \frac{1}{3}h_{11}\alpha x_1^3 + h_{20}x_1^2 + l$.

3^0. $\Delta h = h(x_1+1, x_2 - \beta(2x_1+1)) - h(x_1, x_2) \in L$ *iff* $h = f(x_2 - \beta x_1^2) + h_{11}x_1(x_2 - \beta x_1^2) - \frac{1}{3}h_{11}\alpha x_1^3 + h_{20}x_1^2 + l$, *where* $f = f(x_2)$ *is an arbitrary polynomial.*

Proof. 1^0. For $h = \sum_{i=0}^m r_i(x_1)x_2^i$ we have $\Delta h = mr_l(x_1)(x_2^{m-1}) + \ldots$, where dots mean items of lower degree than x_2. From this, it follows, that if $\Delta h \in L$ then $\deg r_l + m - 1 \leqslant 1$. Thus, we have the form of $h = r_2x_2^2 + r_l(x_1)x_2 + r_0(x_1)$, where $r_2 \in K, \deg r_1 \leqslant 1$. For such a form of h we have $\Delta h = 2r_2(x_2 + a(x_1)) + r_1(x_1) + r_2$. Since $\deg a > 1$, this polynomial can be linear when $2r_2a(x_1) + r_1(x_1) \in L$. Hence we get the form of h, pointed in 1^0.

In the case 2^0 select a highest monomial (in the sense $\prec$) containing x_1 : $h = f(x_2) + x_1^kx_2^m + \ldots (k > 0)$. Then we get $\Delta h = x_1^{k-1}x_2^l + \cdots \in L \to k + l \leqslant 2$. If $m = 0$ then $h = f(x_2) + g(x_1)$, $\deg g \leqslant 2$. If $f \neq 0$, then $h = h_{11}x_1x_2 + g_1(x_1) + f(x_2)$ and $\Delta h = h_{11}x_2 - \alpha x_1^2 + g_1(x_1 + t) - g_1(x_1)$. This polynomial can be linear if $\deg g_1 = 3$ and its highest item have the form $\frac{1}{3}h_{11}\alpha x_1^3$. In that way this case is exhausted. The case 3^0 can be proved by analogy.

Now we are ready to prove Theorem 3.

Proof. Let us analyze five cases corresponding to the forms of permutation matrix W in the formula (6).

Case 1. $W = (1,2)$ is a transposition. In this case we get
$t_1^W = X + a_1(x_1)\vec{e_1} + a_2(x_1, x_2)\vec{e_3}$ and

$$q = X + a_1(x_2 + b_1(x_1))\vec{e_1} + b_1(x_1)\vec{e_2} + (b_2(x_1, x_2) + a_2(x_2 + b(x_1), x_1)\vec{e_3}. \quad (7)$$

Consider the linear transformation $A_{32} = X + x_2\vec{e_3} \in GL_n$ and obtain a map $q^{A_{32}} \cdot q^{-1} = X - b_1(x_1 - a_1(x_2))\vec{e_5}$. This element can be linear only if $b_1(x_1) \equiv 0$. If this happens then (6) implies that the element q is not bitriangular.

Case 2. $W = (2,3)$.

Remark. A polynomial a_2 (b_2) can not be independent of x_2, because, in this case t_1^W (t_2^W) and then q^W is triangular.

Let us calculate the commutator with the translation c_2, which is a triangular map

$$q^{-1}q^{c_2} = X + (b_2(x_1, x_2 + t) - b_2(x_1, x_2))\vec{e_3}.$$

It will be linear iff $b_2 = b_{02}x_2^2 + b(x_1)$. In this situation let us consider the element

$$q^{c_1} \cdot q^{-1} = X + (a_2(x_1, x_3 - a_1(x_1 + 1)) - a_2(x_1, x_3 - a_1(x_1)))\vec{e_5}.$$

Accordance to Proposition 1 it will be linear iff $a_2 = a_{02}x_2^2 + 2a_{02}a_{11}(x_1)x_2 + a_{11}x_1x_2 + a(x_1)$. Let us use $A_{31} = X + x_1\vec{e_3} \in U_3 \cap GL_3$ and calculate the double commutator $r = [q, c_3], r_1 = [A_{31}, r^{-1}]$. Next element will be obtained

$$r_1 = X + 2a_{02}x_1\vec{e_2} - 2(2b_{02}a_{02}x_1x_2 + (b_{02}a_{02} - b_{11})a_{02}x_1^2)\vec{e_3}.$$

This element can be linear in the cases
[2.1] $b_{02} = b_{11} = 0$;
[2.2] $a_{02} = 0$,

In both cases the element r will be triangular, with $x_3 + (b_{11}a_{11} + b_{02}a_{11}^2)x_1^2 - 2b_{02}a_{11}x_1x_2$ as a third coordinate and a linear second coordinate. In the case 2.1 the polynomial $b_2 = b_2(x_1)$ does not depend on the variable x_2. As was pointed in the remark at the beginning of this case, it contradicts to the assumption that q is bitriangular element. In the case 2.2, if 2.1 is not hold, then the element r is linear when $a_{11} = 0$. The equalities $a_{02} = a_{11} = 0$ imply that a_2 does not depend on x_2, which (by the remark) leads to a contradiction also.

Case 3. $W = (1,3)$. This is the hardest case in which we have

$$q = X+a_2(x_1+b_2(x_1,x_2),x_2+b_2(x_1))\vec{e}_1+(b_1(x_1)+a_1(x_3+b_2(x_1,x_2)))\vec{e}_2+b_2(x_1,x_2)\vec{e}_3.$$

Let us calculate the commutator

$$q^{c_3}q^{-1} = X + (a_1(x_1+1) - a_1(x_1))\vec{e}_2 + (a_2(x_1+1, x_2 - a(x_1)) - a_2(x_1, x_2 - a(x_1)))\vec{e}_3$$

Clearly that this triangular element can be linear only if $a_1(x_1) = \alpha x_1^2$ and (by Proposition 1) $a_2(x_1,x_2) = a(x_2) + a_{11}x_1x_2 + \frac{1}{3}a_{11}\alpha x_1^3 + a_{20}x_1^2$.

One the other hand, we have

$$q^{-c_1}q^{-1} = X+(b_1(x_1)-b_1(x_1+1))\vec{e}_2+(b_2(x_1,x_2)-b_2(x_1+1,x_2+b(x_1)-b(x_1+1)))\vec{e}_3,$$

which implies $b_1 = \beta x_1^2$ and (by p. 3^0 of Proposition 1) $b_2(x_1,x_2) = b(x_2+\beta x_1^2) + b_{11}x_1(x_2+\beta x_1^2) - \frac{1}{3}b_{11}\beta x_1^3 + b_{20}x_1^2$

Let us consider the

Case 3.1 when $\alpha = \beta = 0$. In this situation we can use the linear element $A_{32} = X + x_2\vec{e}_3$ again and get the element

$$r = q^{A_{32}} \cdot q^{-1} = X + [(a_{11}+a_{20})x_2^2 + 2a_{20}x_2x_3]\vec{e}_1.$$

Clearly, that $r^{(1,3)}$ is a triangular element and it will be linear if $a_{11} = a_{20} = 0$ only. If we replace the element q with q^{-W} then the similar procedure leads to the conclusion $b_{11} = b_{20} = 0$. So, the situation when $t_1 = X+a(x_1)\vec{e}_3, t_2 = X+b(x_1)\vec{e}_3$, $\deg a, \deg b > 1$ and $q = X+a(x_2)\vec{e_1}+b(x_2)\vec{e}_3$ should be considered. In this way we get the nonlinear triangular element $q^{(1,2)} = X + a(x_1)\vec{e_2} + b(x_1)\vec{e}_3$.

Case 3.2, in which $\alpha \neq 0, \beta = 0$. Let us put $Q_1 = X - b_{11}x_1\vec{e}_3 \in GL_3$ and calculate the element $r = q^{c_2} \cdot Q_1 \cdot q^{-1} \in G$. The element $A_{13} = X + x_3\vec{e}_1$ belongs to the center of $U_3^W \cap GA_3^0$ and so, one can get the element

$$r_1 = r^{A_2} \cdot r^{-1} = X + (b(x_2 - \alpha x_3^2) - b(x_2 + 1 - \alpha x_3^2))\vec{e_1}.$$

Thus we can get the triangular element r_1^W moreover, since $\alpha \neq 0$, and $\deg a \neq 1$ it can be linear only if $b \equiv 0$, i.e. $r_1 = X$. In this situation the element r has form

$$r = t_1^{Wc_2} \cdot t_1^{-W} = X + (a(x_2+1-\alpha x_3^2) - a(x_2 - \alpha x_3^2) + a_{11}x_2)\vec{e}_1.$$

Similarly we can get $a \equiv 0$. Let us use the element $Q_2 = X - 2b_{20}x_1\vec{e}_3 \in AGL_3$ permutable with t_1, t_2 and get the element

$$r_3 = q^{c_1}Q_2q^{-1} \in G.$$

The calculation of the commutator with a linear element A_{13}, which was used above leads to the equality $r_4 = r_3^{A_{13}} r^{-1} = X + (b_{11}\alpha x_3^2 - b_{11}x_2 - b_{20})\vec{e_1}$. The triangular element r_4^W can be linear if $b_{11} = 0$. If this holds, then the element of torus $\tau = \langle x_1, sx_2, x_3\rangle$ is permutable with t_2 for any $s \in K^*$. Therefore one can get the triangular element $(q^\tau q^{-1})^W = t_1^\tau t_1^{-1}$, whose second coordinate equals $x_2 + \alpha\frac{1-s}{s}x_1^2$. Since $\alpha \neq 0$ this element is nonlinear for $s \neq 1$. This completes the analysis of case 3.2 .

Case 3.3. $\alpha = 0, \beta \neq 0$. Replacing q with q^{-W} it can be reduced to the previous case.

Case 3.4. $\alpha, \beta \neq 0$. This item is central in the case 3. Put

$$Q_3 = X - \frac{sb_{20} + a_{20}}{\alpha} x_2 \vec{e}_1 \in GL_3$$

where s is a parameter. It is convenient to introduce the element $q_1 = Q_3 t_1^W t_2$. The reason will be explained bellow. It is easy to see that monomial structures of the elements t_2^{-1} and t_1 are similar. Taking this in account, we could choose an element of the torus $T = \langle \lambda x_1, \mu x_2, \nu x_3\rangle$ in such a manner that the element $r_5 = t_2^W (Q_3 t_1^W)^T$ has the form

$$X + f(x_2)\vec{e}_1. \tag{8}$$

If we succeed in this then, without special difficulties we could derive a triangular nonlinear element from G. For first coordinates of the elements $(Q_3 t_1^W)^T$ and t_2^{-W} let us equate coefficients of their monomials x_3^2, x_3^3, x_2x_3. In this way we get three equations with unknown λ, μ, ν

$$\frac{a_{11}\mu\nu}{\lambda} = -b_{11}; \ \frac{a_{11}\alpha\nu^3}{\lambda} = b_{11}\beta; \ \frac{\nu^2 s b_{20}}{\lambda} = b_{20}. \tag{9}$$

It is evident that the solution $\lambda, \mu, \nu \in K^*$ exists if either both a_{11}, b_{11} equal to zero or both not. Now let us remark that the element Q_5 was introduced to avoid the same problem with the coefficients a_{02}, b_{02}.

In the Case 3.4.1 $a_{11}, b_{11} \neq 0$ we have the solution (9)

$$\lambda = \frac{b_{11}^2\beta^2 s^3}{a_{11}^2\alpha^2}; \quad \mu = -\frac{b_{11}^2\beta s^2}{a_{11}^2\alpha}; \quad \nu = \frac{b_{11}\beta s}{a_{11}\alpha}.$$

After the substitution of these values in $t_2^W (Q_3 t_1^W)^T$ we get an element of the necessary form (8), where

$$f(x_2) = s^{-3}\left(\frac{a_{11}\alpha}{b_{11}\beta}\right)^2 a\left(-\frac{b_{11}^2\beta s^2 x_2}{a_{11}^2\alpha}\right) + b(x_2) + (b_{20} + a_{20}s^{-1})\frac{x_2}{\beta}.$$

Since A_{31} is permutable with the unitriangular elements t_1, t_2, for the element $g = q_1^W (q_1)^T = t_1 r t_2^T$, we get a triangular element $g^{A_{31}} g^{-1} = \tilde{t}_1 r_5^{A_{31}} \tilde{t}^{-1}$, where $\tilde{t}_1 = (A_{31})^W t_1 \in U_3$. Direct calculations lead to the following formula

$$g^A g^{-1} = X +$$

$$(s^{-3}\left(\frac{a_{11}\alpha}{b_{11}\beta}\right)^2 a\left(-\frac{b_{11}^2\beta s^2 (x_2 - \alpha x_1^2)}{a_{11}^2\alpha}\right) +$$

$$b(x_2-\alpha x_1^2)+(b_{20}+a_{20}s^{-1})\frac{x_2-\alpha x_1^2}{\beta})\vec{e}_3 \qquad (10)$$

Let us analyze conditions under which it will be linear. If a_k, b_k are coefficients at k-degrees of polynomials a, b then the coefficient of a monomial $x_2^k, k \geqslant 2$ is a polynomial from s

$$\left(\frac{a_{11}\alpha}{b_{11}\beta}\right)^2\left(\frac{b_{11}^2\beta}{a_{11}^2\alpha}\right)^k a_k s^{2k-3}+b_k.$$

It can be equal to zero if $a_k=b_k=0$. Hence, the element $g^A g^{-1}$ is linear iff $a\equiv b\equiv 0, a_{20}=b_{20}=0$, i.e. $t_2^W(Q_3t_1^W)^T=X$. In this situation we can put $s=1$ in the polynomial f_2 and suppose that ν is a parameter. Then we have a triangular element $t=t_1t_2^Wt_1^{WT}t_2^T=t_1t_2^T$ whose second coordinate is equal to

$$x_2-(\nu^4\left(\frac{a_{11}}{b_{11}}\right)^2-1)\alpha x_1^2.$$

Since $\alpha, a_{11}\neq 0$, one can choose a value of ν in such a manner that t is a nonlinear element.

Case 3.4.2. $a_{11}=0, b_{11}\neq 0$.

Let us put $g=q^{-c_2}q$ and use A_{31} again. Since A_{31} is permutable with t_1 it is easy to calculate

$$g^{-A_{31}}g=t_2^{-1}\left(t_1^{-W}t_1^{Wc_2}\right)^{A_{31}}t_1^{-Wc_2}t_1^Wt_2=X+(a(x_2+\beta x_1^2)-a(x_2+\beta x_1^2+1))\vec{e}_3.$$

Since $\beta\neq 0$, it follows that this triangular element can be linear only if $a\equiv 0$. In this situation the element $t_1^W=\langle x_1+a_{20}x_3^2, x_2+\alpha x_3^2, x_3\rangle$ is permutable with c_2, hence,

$$q^{-c_2}q=t_2^{-c_2}t_2=X+(b(x_2+\beta x_1^2)-b(x_2+\beta x_1^2+1)-b_{11}x_1)\vec{e}_3.$$

This element can be linear if $b\equiv 0$.

In this case it is easy to verify that the linear transformation $Q_4=\langle x_1,(1-\frac{ca_{20}}{\alpha})x_2+cx_1,x_3\rangle$ is permutable with t_1^W for each value $c\neq\frac{\alpha}{a_{20}}$, hence, one can get the element $q^{-1}q^{Q_4}$. It turned out, that its second coordinate $x_2+\frac{\beta a_{20}c}{\alpha-a_{20}c}x_1^2$ is a nonlinear polynomial if $a_{20}\neq 0$. In the opposite case $a_{20}=0$ is the third coordinate of this element $x_3+b_{11}x_1^2c$ is a nonlinear polynomial.

Case 3.4.3. $a_{11}\neq 0, b_{11}=0$ can be reduced to the previous one replacing q by q^{-W}.

Case 3.4.4. $a_{11}=b_{11}=0$. In this case we use $\lambda=\nu^2 s, \mu=\alpha\nu^2/\beta$ and parameter ν as a solution of equations (9) and get the element

$$r_5=t_2^W(Q_4t_1^W)^T=X+\left(\frac{a\left(-\frac{\alpha\nu^2(x_2-\alpha x_1^2)}{\beta}\right)}{\nu^2 s}+b(x_2-\alpha x_1^2)+\frac{sb_{20}+a_{20}}{s\beta}x_2\right)\vec{e}_1.$$

One can repeat the argument done after the formula (10) and conclude that r_5 can be linear only if $a\equiv b\equiv 0$ and $a_{20}=b_{20}=0$. But under these conditions the element q is triangular and we get a contradiction.

Case 4. $W=(1,2,3)$. Our standard procedure leads to equalities :

$$(q^{c_3}q^{-1})^{W^2}=X+(a_2(x_1,x_2+a_1(x_1)+1)-a_2(x_1,x_2+a_1(x_1),))\vec{e}_3,$$

$q^{c_1}q^{-1} = X+(b_1(x_1+1)-b_1(x_1))\vec{e}_2+(b_2(x_1+1,x_2)-b_2(x_1,x_2+b_1(x_1+1)+b_1(x_1)))\vec{e}_3$

and the next conditions on the polynomials a_1, a_2, b_1, b_2, under which the obtained triangular elements are linear. $b_2(x_1) = \beta x_1^2$, and (by Proposition 1)

$$a_2 = a_{02}x_2^2 + 2a_{02}a_1(x_1)x_2 + a_{11}x_1x_2 + r(x_1)$$

$$b_2 = f(x_2 + \beta x_1^2) + b_{11}x_1(x_2 + \beta x_1^2) + \frac{1}{3}b_{11}\beta x_1^3 + b_{20}x_1^2,$$

where $r(x_1), f(x_1)$ are polynomials.

At the same time, together with the element $q = t_1^W t_2$ the group G contains the element $q_1 = q^{-W^{-1}} = t_2^{-W^2}t_1^{-1}$, for which one can to calculate the commutator $q_2 = q_1^{-1}q_1^{c_1}$ and check that it is not triangular. But the element

$$q_3 = q_2^{A_{31}}q_2^{-1} = X - b_{11}x_1\vec{e}_2 - b_{11}(2a_{02}x_1x_2 + (a_{11} - a_{02})x_1^2)\vec{e}_3 \in G$$

is triangular. Thus we have the alternative cases 4.1 and 4.2

Case 4.1. $b_{11} \neq 0, a_{02} = a_{11} = 0$, where we have

$$q = X + r(x_2+\beta x_1^2)\vec{e}_1 + \beta x_1^2\vec{e}_2 + [f(x_2+\beta x_1^2) + b_{11}x_1x_2 + \frac{2}{3}b_{11}\beta x_1^3 + a_1(x_2+\beta x_1^2)]\vec{e}_3.$$

One can get a triangular element $q^{-1}q^{A_{13}} = X - r(x_2 + \beta x_1^2)\vec{e}_3$, which can be linear under condition $r \equiv 0$. This yields a contradiction that q is triangular.

Case 4.2. $b_{11} = 0$ Direct calculation of a commutator leads to the triangular element

$$q_1^{-1}q_1^{c_1} = X - 2b_{20}(2x_1 + 1)\vec{e}_2 - (4a_{02}b_{20}x_1x_2 + \dots)\vec{e}_3,$$

which could be linear if $a_{02}b_{20} = 0$. Here dots mean items of the lower height.

Case 4.2.1. $b_{20} = 0$. The element $q^{-1}q^{A_{31}} = X + \beta(2x_1x_3 + x_3^2)$ is 1-triangular and can be linear only if $\beta = 0$.

Case 4.2.2. $a_{02} = 0$. In this case one can use $A_{21} = X + x_1\vec{e}_2$ and calculate the double commutator $g = q_1^{A_{31}}q_1^{-1}, g_1 = g^{A_{21}}g^{-1}$, which conjugate by the transposition $(2,3)$ with a triangular element of the form $X - (\beta^3x_3t + \dots)\vec{e}_5$. Just as in the the previous case the case $\beta = 0$ should be treated. But in both cases the equality $\beta = 0$ gives contradiction: q isn't bitriangular.

Case 5. $W = (1,3,2)$. The group G contains the element q together with $q_1 = q^{-W^{-1}} = t_2^{-W^{-1}}t_1^{-1}$. Since $W^{-1} = (1,2,3)$ this case is reduced to the previous one and this completes the proof of the theorem.

The previous proof was based on calculations with commutators $[q,c], c \in A_3^+$, which have the compositional-triangular length less then the element q. For 3-triangular maps, as a rule, we will get 3-triangular elements also. But the height of the intermediate triangular element of the new elements will decrease. The proof of the next theorem is based on this simple remark.

Theorem 4. *Let q be a 3-triangular element of GA_3. Then*

$$TGA_3 = \langle AGL_3, q\rangle.$$

Proof. If $G = \langle AGL_3, q\rangle$, then without loss of generality one can suppose that q has a form $q = t_1^{A_1} t_2 t_3^{A_3}$. The Brua decomposition leads to the equality

$$q = B_1^{-1} \cdot {t'_1}^{W_1} \cdot B_1 \cdot t_2 \cdot B_3^{-1} \cdot {t'_3}^{W_3} \cdot B_3$$

and we get an element

$$B_1 \cdot q \cdot B_3 = {t'_1}^{W_1} \cdot t'_2 \cdot {t'_3}^{W_3} \in G,$$

where $t'_1 = B_1 \cdot t_1 \cdot B_1^{-1}, t'_3 = B_3 \cdot t_3 \cdot B_3^{-1}, t'_2 = B_1 \cdot t_2 \cdot B_3^{-1}$. Bellow we will preserve notations and suppose that the group G contains a map of the form

$$q = {t_1}^{W_1} \cdot t_2 \cdot {t_3}^{W_3}. \tag{11}$$

Case 1. $W_1 = W_3 = (1,2)$.
Put $A_{32} = X + x_2\vec{e}_3$. Since it is permutable with ${t_i}^{W_i}$, $i = 1,2$, we get a triangular element

$$q^{A_{32}} \cdot q^{-1} = X - b_1(x_1 - a_1(x_2))\vec{e}_3,$$

which could be linear only if $b_1 \equiv 0$. But in this situation q is a bitriangular element, because the element

$${t_1}^{(1,2)} \cdot t_2 = \left(t_1 \cdot t_2^{(1,2)}\right)^{(1,2)}$$

is 1-triangular. This contradiction completes analysis of this case.

Case 2. $W_1 = W_3 = (2,3)$.
Since $c_2 = X + \vec{e}_2$ is permutable with ${t_i}^{W_i}$, $i = 1,2$ we get the element

$$q_1 = q^{c_2} \cdot q^{-1} = {t_1}^{W_1} \cdot {t_2}^{c_2} \cdot {t_2}^{-1} \cdot {t_1}^{-W_1} \in G.$$

If a_2 is independent of x_2, then ${t_1}^{W_1}$ is triangular and q isn't 3-triangular, hence, a_2 depends on x_2.

Therefore one can proceed to calculate commutators $q_{i+1} = {q_i}^{c_2} q^{-1} = {t_1}^{W_1}\tau_i \cdot {t_1}^{-W_1}$ till τ_i will be of the form

Case 2.1. $\tau_i = X + \alpha x_1^k \vec{e}_3, \quad k > 0$;

Case 2.2. $\tau_i = X + (\beta x_2 + \gamma)\vec{e}_3$.

In the case 2.1 we get a 1-triangular element $q_i = \left(t_1 \tau_i^{W_1} t_1^{-1}\right)^{W_1}$, where $\tau_i^{W_1} = X + x_1^k \vec{e}_2$ is a triangular element and hence the element q_i is a nonlinear 1– triangular element.

In the case 2.2, τ_i is a linear element and hence the element q_i is bitriangular.

Case 3. $W = (1,3)$. Consider the case when t_2 doesn't depend on x_1, i.e. $t_2 = X + b_2(x_2)\vec{e}_3$. Then the commutator $q^{A_{13}} \cdot q^{-1} = X - b_2(x_2 - a_1(x_3))\vec{e}_1$ is a nonlinear 1-triangular element. In a similar way let us consider the case when t_1 doesn't depend on x_1, i.e. $t_2 = X + a_2(x_2)\vec{e}_3$. Remark that in this case $b_1 \not\equiv 0$. Indeed, if $b_1 \equiv 0$, then $t_1^{W_1} t_2$ is 1-triangular and q is not 3-triangular. Therefore the element $q^{A_{31}} q^{-1} = X + (a_2(x_2 - b_1(x_1 - a_2(x_2))) - a_2(x_2))\vec{e}_3$ is a nonlinear triangular one.

Let us consider the general case and calculate the commutator

$$q_1 = q^{c_1} \cdot q^{-1} = {t_1}^{W_1} \cdot {t_2}^{c_1} \cdot {t_2}^{-1} \cdot {t_1}^{-W_1} \in G,$$

where

$t_2^{c_1} \cdot t_2^{-1} = X + (b_1(x_1+1) - b_1(x_1))\vec{e}_2 + (b_2(x_1+1, x_2 - b_1(x_1)) - b_2(x_1, x_2 - b_1(x_1)))\vec{e}_3.$

Let us put $q_{i+1} = q_i^{c_1} \cdot q_i^{-1} = t_1^{W_1} \cdot \tau_i \cdot (t_1^{-W_1}) \in G$. One can proceed the process till the $\deg_{x_1} \tau_1 = 1$ (the case when $\deg_{x_1} t_2 = 0$, was considered above). Thus we will stop a process when the element $\tau_i = X + (\alpha x_1 + \beta)\vec{e}_2 + (x_1 r(x_2) + r_0(x_2))\vec{e}_3$ will be obtained. If $\deg r \leqslant 0$, then one can pick out the linear part

$$\tau_i = L \cdot \tau_i' = (X + (\alpha x_1 + \beta)\vec{e}_2 + (\alpha_1 x_1 + \beta_1)\vec{e}_3) \cdot (X + r_0(x_2)\vec{e}_3)$$

and to join it to $t_1^{W_1}$. It could be done by replacing q_{i+1} with $L^{-1} \cdot q_{i+1} \in G$. In this way we get the element $\bar{q}_{i+1} = t_1^{W \cdot L} \tau_i' t_1^{-W}$. It is easy to check that the map

$$\bar{q}_{i+1}^{c_1} q_{i+1}^{-1} = t_1^{W \cdot L} \cdot t_1^{-W}$$

is a nonlinear bitriangular one. If $\deg r > 0$ then the element

$$\tau_i = X + (\alpha x_1 + \beta)\vec{e}_2 + (x_1 r_1 + r_0)\vec{e}_3$$

is linear and q_{i+1} is a bitriangular or 1-triangular. It is easy to check that the last case can be realized only if $a_2 = a_2(x_1), r_1 = 0$. Then it can be linear if $a_2 \equiv 0$, but it yields the contradiction that q is 1-triangular map.

In the case when $\deg r > 0$ one can proceed the process of the calculations τ_i until an element of the kind $\tau_i = X + (\alpha x_1 + \beta)\vec{e}_3$ appears. Similarly to the case of $\deg r < 1$ one can get a nonlinear bitriangular element.

The case when $W_1 = W_2 = (1, 2, 3)$ can be investigated by the previous procedure of an iterated commutators with c_1.

In the case $W_1 = W_2 = (1, 3, 2)$ we can calculate commutators $q_1 = q^{c_2} \cdot q^{-1} = t_1^{W_1} \cdot \tau_1 \cdot (t_1^{-W_1}) \in G$, where τ_1 has the form $X + r(x_1, x_2)\vec{e}_3$. If $\deg r > 1$, then one can consider the element $q_1^{(1,2)} = t_1^{(1,3)} \tau_1 t_1^{-(1,3)} \in G$ and reduce this case to the previous one. Let us investigate the situations, when $\deg \tau_1 \leqslant 1$. If $\deg \tau_1 = 1$, then we have the map q_1 which is bitriangular unless the case when a_1, a_2 doesn't depend on x_1. But the last case is impossible because it contradicts to the suggestion that q is 3-triangular. We can obtain the element q_1 with $\deg \tau_1 = 0$, when b_2 doesn't depend on x_2. In this case the element $q^{(2,3)} = t_1^{(1,2)} t_2^{(1,2)} t_1^{-(1,2)} \in G$ have the form of the case 1.

If q has the form (11), where $W_1 \neq W_2$ one can choose the linear element A_{ij}, permutable with $t_1^{W_1}$ or $t_2^{W_2}$ but not permutable with t_2. Then the map $q^{A_{ij}} \cdot q^{-1}$ or $q^{-1} \cdot q^{A_{ij}}$ will be 3-triangular and has the form (11) with $W_1 = W_2$.

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ПОРОДЖУЮЧІ ВЛАСТИВОСТІ ОБОРОТНИХ ПОЛІНОМІАЛЬНИХ ВІДОБРАЖЕНЬ ВІД ТРЬОХ ЗМІННИХ, ЩО МАЮТЬ МАЛУ КОМПОЗИЦІЙНО-ТРИКУТНУ ДОВЖИНУ

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Показано, що кожне k-трикутне оборотне відображення (наперед вибране) для $k = 1, 2, 3$ разом з лінійними відображеннями породжує групу ручних поліноміальних автоморфізмів від трьох змінних.

Ключові слова: оборотне поліноміальне перетворення, афінний простір, афінна група, афінна група Кремони, нескінченновимірна алгебрична група.

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PSEUDOCOMPACTENESS OF THE SPACES OF ALMOST CONTINUOUS MAPPINGS

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Given two Hausdorff spaces we study properties of the function space $AC_p(X,Y) \subset Y^X$ consisting of all almost continuous mappings $f : X \to Y$ (the almost continuity of f means that any nonempty subspace $A \subset X$ contains a point of continuity of the mapping $f|A : A \to Y$). We prove that for infinite Hausdorff spaces X, Y the space $AC_p(X,Y)$ is pseudocompact iff $AC_p(X,Y)$ is σ-pseudocompact iff Y^ω is pseudocompact and X each countable subspace of X is scattered.

Key words: almost continuous mappings, topology of pointwise convergence, pseudocompact space, scattered space.

In the paper we detect pseudocompact spaces $AC_p(X,Y) \subset Y^X$ consisting of all almost continuous mappings $f : X \to Y$. We remind that a mapping $f : X \to Y$ between topological spaces is called *almost continuous* if every non-empty subspace $A \subset X$ of X contains a continuity point of the map $f|A : A \to Y$. By $AC_p(X,Y) \subset Y^X$ we denote the space of all almost continuous functions $f : X \to Y$, endowed with the topology of point-wise convergence, see [1].

Observe that $AC_p(X,Y) = Y^X$ for any scattered space X. We recall that a topological space X is *scattered* if each subspace of X has an isolated point (equivalently, the identity map of X into X endowed with the discrete topology is almost continuous). We define a space X to be *ω-scattered* if each countable subspace of X is scattered. We shall prove that for an ω-scattered space X the subset $AC_p(X,Y)$ of Y^X still is very massive.

Given a function $f \in Y^X$ let $\Sigma(f) = \{g \in Y^X : |\{x \in X : f(x) \neq g(x)\}| \leqslant \aleph_0\}$. We call a subset $F \subset Y^X$ an *ω-tail set* in Y^X if $\Sigma(f) \subset F$ for any $f \in F$. Observe that each non-empty ω-tail subset $F \subset Y^X$ is *G_δ-dense* in the sense that $G \cap F \neq \emptyset$ for each non-empty G_δ-subset G of Y^X.

We shall say that a subspace Y of a space X is *C-embedded* into X if each continuous function $f : Y \to \mathbb{R}$ can be continuously extended over all X.

Proposition. *For a Hausdorff topological space X and an infinite Hausdorff space Y the following conditions are equivalent:*

1) X is ω-scattered;

2) $AC_p(X,Y)$ is an ω-tail subset of Y^X;

3) $AC_p(X,Y)$ is G_δ-dense in Y^X.

Moreover, if any finite power of Y is regular and Lindelöf, then the conditions (1)–(3) are equivalent to:

4) $AC_p(X,Y)$ is C-embedded into Y^X.

Proof. (1) $\Rightarrow$ (2) Assume that X is ω-scattered, $f : X \to Y$ is an almost continuous function and $g : X \to Y$ is a function such that the set $Z = \{x \in X : f(x) \neq g(x)\}$ is at most countable. We have to prove that g is almost continuous. Take any subset $A \subset X$. We consider two cases:

a) $A \cap Z$ is not dense in A. Then we can find a continuity point $a \in A \setminus \bar{Z}$ of the function $f|A \setminus \bar{Z}$ which also is a continuity point of the function $g|A$.

b) $A \cap Z$ is dense in A. The space $A \cap Z$, being a countable subspace of the ω-scattered space X, is scattered. Consequently, $A \cap Z$ contains an isolated point z which by the density of $A \cap Z$ in A is also isolated in A. Then z is a continuity point of the function $g|A$.

The implication (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Assume that $AC_p(X,Y)$ is G_δ-dense in Y^X but X contains a countable non-scattered subspace $A = \{a_n\}_{n\in\omega}$. Without loss of generality we can assume that A has no isolated points. The space Y, being infinite and Hausdorff, contains a countable collection $\{U_n\}_{n\in\omega}$ of non-empty pair-wise disjoint open subsets. Observe that the G_δ-subset $G = \{f \in Y^X : f(a_n) \in U_n \text{ for all } n \in \omega\}$ of Y^X misses the set $AC_p(X,Y)$ since G consists of functions everywhere discontinuous on A.

(2) $\Rightarrow$ (4) This implication follows from [2, 3.12.23(a)] asserting that for any Hausdorff space Y with Lindelöf finite powers Y^n and any $f \in Y^X$ the Σ-product $\Sigma(f)$ is C-embedded into Y^X.

The implication (4) $\Rightarrow$ (3) follows from the well-known fact asserting that each C-embedded subspace of a Tychonov space is G_δ-dense. □

Next we find conditions on infinite Hausdorff spaces X, Y under which the space $AC_p(X,Y)$ is (σ-)pseudocompact. We remind that a Hausdorff space X is *pseudocompact* if each locally finite collection of open subsets of X is finite. For Tychonov spaces this is equivalent to saying that each continuous real-valued function on X is bounded. A Hausdorff space X is defined to be *σ-pseudocompact* if X is the countable union of pseudocompact subspaces. It is easy to see that each dense pseudocompact subspace of a Hausdorff space X is G_δ-dense in X.

Theorem. *For infinite Hausdorff spaces X and Y the following conditions are equivalent:*

1) $AC_p(X,Y)$ is pseudocompact;

2) $AC_p(X,Y)$ is σ-pseudocompact;

3) X is ω-scattered and Y^ω is pseudocompact.

Proof. The implication (1) $\Rightarrow$ (2) is trivial.

(3) $\Rightarrow$ (1) Suppose X is ω-scattered and Y^ω is pseudocompact. To show that the space $AC_p(X,Y)$ is pseudocompact, assume that $\{U_n\}_{n\in\omega}$ is a locally finite collection of non-empty open subsets of $AC_p(X,Y)$. Without loss of generality, we can assume that for each set U_n there are a finite subset $C_n \subset X$ and an open set $W_n \subset Y^{C_n}$ such that $U_n = \pi_{C_n}^{-1}(W_n)$. The countable subspace $C = \bigcup_{n\in\omega} C_n$ of the ω-scattered space X is scattered. Consequently, the restriction operator $\pi_C : AC_p(X,Y) \to Y^C$, $\pi_C : f \mapsto f|C$, is surjective. This implies that $\{\pi_C(U_n)\}_{n\in\omega}$ is a locally finite collection of open sets in $\pi_C(AC_p(X,Y)) = Y^C$ which is not possible

because of the pseudocompactness of the space Y^ω. This contradiction shows that the space $AC_p(X,Y)$ is pseudocompact.

(2) $\Rightarrow$ (3) Suppose that the space $AC_p(X,Y)$ is σ-pseudocompact. First we show that the space Y^ω is pseudocompact.

The space X, being infinite and Hausdorff, contains a countable discrete subspace Z. Observe that the restriction operator $\pi_Z : AC_p(X,Y) \to Y^Z$, $\pi_Z : f \mapsto f|Z$, is surjective which implies that the space Y^ω is σ-pseudocompact. Using the fact that the spaces Y^ω and $(Y^\omega)^\omega$ are homeomorphic by the standard diagonal procedure it can be shown that the space Y^ω is pseudocompact.

Next we show that the space X is ω-scattered. Assume conversely that the space X contains a countable non-scattered subspace Z. Write $AC_p(X,Y) = \bigcup_{n\in\omega} B_n$, where B_n is pseudocompact for every $n \in \omega$.

Put $F_0 = \emptyset$ and $C_0 = Z$. By induction we shall construct countable sequences of function $(f_n)_{n\in\mathbb{N}} \in Y^X$, finite subsets $(F_n)_{n\in\omega}$ of Y and closed non-scattered subspaces $(C_n)_{n\in\omega}$ of Y such that

(a) $F_{n+1} \subset C_n$, $C_{n+1} \subset C_n \setminus F_{n+1}$;

(b) $g \notin \pi_{C_n}(B_n)$ for each function $g \in Y^{C_n}$ with $g|F_{n+1} = f_{n+1}|F_{n+1}$.

Assume that a non-scattered closed subspace C_n has been constructed. First we show that the projection $\pi_{C_n}(B_n)$ is not dense in Y^{C_n}. Assuming the converse we will get that the space $AC_p(C_n,Y)$ is pseudocompact since it contains a dense pseudocompact space $\pi_{C_n}(B_n)$. The pseudocompactness of $AC_p(C_n,Y)$ implies that it is G_δ-dense in Y^{C_n}. Applying the implication (3) $\Rightarrow$ (1) we conclude that the space C_n is ω-scattered which contradicts to the choice of C_n.

Hence $\pi_{C_n}(B_n)$ is not dense in Y^{C_n} and there are a function $f_{n+1} \in Y^X$ and a finite subset $F_{n+i} \subset C_n$ such that $g \notin \pi_{C_n}(B_n)$ for each $g \in Y^{C_n}$ with $g|F_{n+1} = f_{n+1}|F_{n+1}$. Finally take any non-scattered closed subspace $C_{n+1} \subset C_n$, disjoint with F_{n+1}. This completes the inductive step.

It follows that the subspace $F = \bigcup_{n\in\omega} F_n$ of X is scattered. Fix any point $y_0 \in Y$ and observe that the function $f : X \to Y$ defined by $f|X\setminus F \equiv y_0$ and $f|F_n = f_n|F_n$ for all n is almost continuous. On the other hand, by (b) $f \notin \bigcup_{n\in\omega} B_n = AC_p(X,Y)$ which is a contradiction. □

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ПСЕВДОКОМПАКТНІСТЬ ПРОСТОРІВ МАЙЖЕ НЕПЕРЕРВНИХ ВІДОБРАЖЕНЬ

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Для заданих двох топологічних просторів X і Y вивчають властивості простору $AC_p(X,Y)$ майже неперервних відображень з простору X у простір Y в топології поточкової збіжності (відображення $f : X \to Y$ називається майже неперервним, якщо в кожному непорожньому підпросторі $A \subset X$ існує точка неперервності відображення $f|_A : A \to Y$). Доведено, що для нескінченних гаусдорфових просторів X і Y такі умови еквівалентні: 1) $AC_p(X,Y)$ – псевдокомпактний; 2) $AC_p(X,Y)$ є σ-псевдокомпактний; 3) кожний зліченний підпростір простору X є розрідженим і Y^ω – псевдокомпактним.

Ключові слова: майже неперервне відображення, топологія поточкової збіжності, псевдокомпактний простір, розріджений простір.

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GENERALIZED NILPOTENT GROUPS WITH THE WEAK π-MINIMAL AND THE WEAK π-MAXIMAL CONDITIONS

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Locally nilpotent and generalized radical groups with the weak π-minimal and the weak π-maximal conditions are investigated.

Key words: weak π-minimal condition, weak π-maximal condition, soluble minimax group, locally nilpotent group, locally finite group, Chernikov group.

Below π is some set of primes. Recall that a group G satisfies the *π-minimal* condition or, briefly, the condition π-min if G has no infinite chains $G_1 \supset G_2 \supset \ldots \supset G_i \supset G_{i+1} \supset \ldots$ of subgroups such that for each i the difference $G_i \setminus G_{i+1}$ contains a π-element (S.N. Chernikov, 1958). Recall that a group G satisfies the *weak π-minimal* (resp., the *weak π-maximal*) condition or, briefly, the π-min-∞ (resp., π-max-∞) condition if it has no infinite chains $G_1 \supset G_2 \supset \ldots \supset G_i \supset G_{i+1} \supset \ldots$ (resp., $G_1 \subset G_2 \subset \ldots \subset G_i \subset G_{i+1} \subset \ldots$) of subgroups such that for each i the index $|G_i : G_{i+1}|$ (resp., $|G_{i+1} : G_i|$) is infinite and the difference $G_i \setminus G_{i+1}$ (resp., $G_{i+1} \setminus G_i$) contains some π-element (N.S. Chernikov, see [1]).

The main results of the present paper are as follows.

Theorem 1 [1]. *For a locally nilpotent group G the following assertions are equivalent:*

1. *G satisfies the* π-min *condition.*
2. *G satisfies the* π-min-∞ *condition.*
3. *The Sylow π-subgroup P of G is Chernikov.*

Theorem 2 [2]. *Let G be a locally nilpotent group and P is the Sylow π-subgroups of G. The group G satisfies the* π-max-∞ *condition iff P is finite or G is a soluble minimax group.*

Theorem 3 [3]. *Let a group G have an infinite normal π-subgroup and possess an ascending series with locally nilpotent and locally finite factors. Then G satisfies the* π-max-∞ *condition iff it is almost soluble minimax.*

Lemma 1. *Let a group G satisfy the π-min-∞ or the π-max-∞ condition. Then in arbitrary direct decomposition of G the number of multipliers with nontrivial π-elements is finite.*

Proof is analogous to the proof of Lemma from [4].

Proposition 1. *Let a locally finite π-group G satisfy the π-min-∞ or the π-max-∞ condition. Then G is Chernikov.*

Proof. Let $G \neq 1$, A be a nontrivial abelian p-subgroup of G and B be the subgroup of all its elements of the orders $\leqslant p$. By the First Prufer's theorem A is a direct product of subgroups of order p. Therefore Lemma 1 implies $|B| < \infty$ and by Lemma 1.10 of [5], A is Chernikov. Further, any abelian subgroup $K \neq 1$ of G is a direct product of its nonidentity Sylow p-subgroups. By Lemma 1 the number of direct multipliers is finite. Consequently K is Chernikov. Then by results from [6, 7], G is Chernikov.

Note that according to Lemma 1.2 of [8], an abelian group G satisfies the min-∞ condition (i.e. the weak minimal condition for subgroups) or the max-∞ condition (i.e. the weak maximal condition for subgroups) iff G is minimax (i.e. G has a finite series such that each its factor satisfies the minimal or maximal condition for subgroups).

Proposition 2. *Let a group G satisfy the π-max-∞ condition and has some infinite normal locally finite π-subgroup H. Then G satisfies the* max$-\infty$ *condition for abelian subgroups (or, equivalently, all abelian subgroups of G are minimax).*

Proof. In view of Proposition 1, H is Chernikov. Let $K \leqslant H$, $|H : K| < \infty$ and K is a direct product of quasicyclic subgroups; K_i is the subgroups of K which consists of all its elements with orders $\leqslant i$, $i \in \mathbb{N}$. Then $K \trianglerighteq G$, and $|K_i| < \infty$, $K_{i+1} \supseteq K_i \triangleright G$ and $K = \bigcup_{i \in \mathbb{N}} K_i$. Let some abelian subgroup $A \subseteq G$ is not minimax. Since $A \cap K$ is Chernikov, it is easy to see that there exists some nonminimax subgroup $L \subseteq A$ such that $L \cap K = 1$. By Lemma 1.2 of [8] there is some ascending chain $L_1 \subset L_1 \subset \ldots \subset L_\omega$ of subgroups of L such that each index $|L_{i+1} : L_i|$ is infinite. Then for the chain $K_1L_1 \subset K_2L_2 \subset \ldots \subset KL_\omega$ of subgroups in G every index $|K_{i+1}L_{i+1} : K_iL_i|$ is infinite and, also, the set of all differences $K_{i+1}L_{i+1} \setminus K_iL_i$ possessing π-elements is infinite. Thus G does not satisfy the π-max condition, a contradiction.

Proposition 3. *Let a group G with minimax abelian subgroups has an ascending series with locally nilpotent factors and locally finite factors. Then G is minimax and almost soluble.*

Proof. Let H be a subgroup of G generated by all its normal radical (in the sense of B. I. Plotkin) subgroups. It is easy to see that H is radical. Obviously, G/H has the same series as G has, and, also, the locally nilpotent radical of G/H is identity. Let L/H be the locally finite radical of G/H, and A/H be arbitrary abelian subgroup of G/H. Then A is radical. Therefore in view of Theorem 4.2 of [8], A is minimax. Consequently, A/H is minimax too. Since A/H is periodic, it follows that A/H is Chernikov. Then by results from [6,7] L/H is Chernikov. Let $R \leqslant L/H$,

$|L/H : R| < \infty$ and R is a direct product of quasicyclic subgroups or $R = 1$. Then R is contained in the locally nilpotent radical of G/H. Consequently, $R = 1$ and $|L/H| < \infty$.

Thus, if $G/L = 1$, we have $|G : H| < \infty$.

Let $G/L \neq 1$. According to Theorem 1.2 from [9, Chapter V, §5] the locally finite radical of arbitrary group X contains all ascendant locally finite subgroups of X. Therefore G/L has no nonidentity ascendant locally finite subgroups. Then G/L has some nontrivial ascendant locally nilpotent subgroup. Therefore by the same theorem from [10] the locally nilpotent radical S/L of the group G/L is nontrivial. Further, obviously, $C_{S/H}(L/H) \trianglerighteq G/H$ and $C_{S/H}(L/H)$ is locally nilpotent. Therefore $C_{S/H}(L/H) = 1$. Since $|L/H| < \infty$ it follows that $|S/H : C_{S/H}(L/H)| < \infty$. So $|S/H| < \infty$. Then $S/H = L/H$, a contradiction.

Thus $G/L = 1$ and $|G : H| < \infty$. In view of Theorem 4.2 from [8], H is minimax and soluble. Consequently, G is minimax and almost soluble. Proposition is proven.

Proof of Theorem 1. Obviously, the assertion 1 is as a consequence of the assertion 2. Suppose the assertion 2 holds and $P \neq 1$. Since group G is locally nilpotent, P is a direct product of nontrivial Sylow p-subgroups by some primes $p \in \pi$. By Proposition 1 these Sylow p-subgroups are Chernikov, and by Lemma 1 their number is finite. Consequently P is Chernikov. Let the assertion 3 hold. Then by Lemma 1 of [10] G satisfies the π-min.

Proof of Theorem 2. Necessity. Let G satisfy the π-max-∞ condition and P is infinite. Then by Proposition 2 and Theorem 4.2 of [8], G is soluble minimax.

Sufficiency. Let G be soluble minimax. Then by Lemmas 1.1 and 1.2 from [8], G satisfies the max-∞ condition. Further, let $|P| < \infty$. Then the set of all π-elements of G is finite and, obviously, G satisfies the π-max-∞.

Proof of Theorem 3. Let G satisfy the π-max-∞. Then by Propositions 2 and 3, G is minimax and almost soluble.

Let G be minimax and almost soluble. Then by Lemmas 1.1, 1.2 of [8], G satisfies the max-∞.

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УЗАГАЛЬНЕНО НІЛЬПОТЕНТНІ ГРУПИ ЗІ СЛАБКИМИ УМОВАМИ π-МІНІМАЛЬНОСТІ ТА π-МАКСИМАЛЬНОСТІ

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Досліджено локально нільпотентні та узагальнено радикальні групи зі слабкими умовами π-мінімальності та π-максимальності.

Ключові слова: слабка умова π-мінімальності, слабка умова π-максимальності, розв'язна мінімальна група, локально нільпотентна група, локально скінченна група, група Чернікова.

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IDEALS OF GORENSTEIN TILED ORDERS WHOSE FACTOR RINGS ARE QUASI-FROBENIUS

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The Gorenstein tiled orders whose the exponent matrices are the Cayley tables of finite groups are studied. The ideals of orders such that the factor rings modulo these ideals being quasi-Frobenius, are described.

Key words: Gorenstein tiled order, exponent matrix, adjacency matrix, quasi-Frobenius ring.

1. Preliminaries. The following result giving a constructive description of one class of semidistributive rings was proved in [5]:

1.1. Theorem. *A right Noetherian semiperfect semiprime and semidistributive ring is isomorphic to a finite direct product of the full matrix rings $M_{m_k}(D_k)$ over skew fields D_k, and rings of the form:*

$$\Lambda = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix}, \qquad (1)$$

where $n \geqslant 1$ and $\mathcal{O}$ is a discrete valuation ring with a prime element π, α_{ij} are integers, moreover, $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ for all i, j, k and $\alpha_{ii} = 0$ for all i.

Conversely, all such rings are Noetherian semiperfect semiprime and semidistributive ones.

Any ring of form (1) is called a semimaximal order (a tiled order). It is a prime two-sided Noetherian semiperfect ring.

We shall use the following notation: $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E}(\Lambda) = (\alpha_{ij})$ is the exponent matrix of a ring Λ. If a tiled order is reduced then $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \dots, n$.

Every tiled order Λ can be embedded in a simple Artinian ring $Q = \sum_{i,j=1}^{n} e_{ij}D$, where D is the division ring of fractions of a ring $\mathcal{O}$. Here Q is both the left and right classical quotient ring of the ring Λ. Let I be a two-sided ideal of a tiled order Λ. Obviously, $I = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\beta_{ij}}\mathcal{O}$, where e_{ij} are the matrix units. Denote by $\mathcal{E}(I) = (\beta_{ij})$

the exponent matrix of an ideal I. Let I and J be two-sided ideals of the ring Λ, $\mathcal{E}(I) = (\beta_{ij})$ and $\mathcal{E}(J) = (\gamma_{ij})$. We have $\mathcal{E}(IJ) = (\delta_{ij})$, where $\delta_{ij} = \min_k(\beta_{ik} + \gamma_{kj})$. If R is the Jacobson radical of a reduced tiled order Λ then $\mathcal{E}(R) = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for $i = 1, \dots, n$. Let $Q(\Lambda)$ be the quiver of a reduced tiled order Λ [6] and $[Q(\Lambda)]$ the adjacency matrix of the quiver $Q(\Lambda)$. Evidently, $[Q(\Lambda)]$ is a (0, 1)-matrix and $[Q(\Lambda)] = \mathcal{E}(R^2) - \mathcal{E}(R)$.

1.2 Definition. A tiled order Λ will be called a Gorenstein tiled order if Λ is a bijective Λ-lattice, i.e. Λ^* is a projective left Λ-lattice (see [7]).

Further the Gorenstein tiled order will be often called the Gorenstein order.

1.3. Lemma. *[1, Lemma 3.2] The following conditions for a tiled order $\Lambda = \{\mathcal{O}, E(\Lambda) = (\alpha_{pq})\}$ are equivalent:*

a) there exists a bijective Λ-lattice;

b) there exist indices i, j such that $\alpha_{ik} + \alpha_{kj} = \alpha_{ij}$ for $k = 1, \dots, n$.

1.4. Theorem. *[2] The following conditions for a reduced tiled order $\Lambda = \{\mathcal{O}, E(\Lambda) = (\alpha_{pq})\}$ are equivalent:*

a) Λ is a Gorenstein order;

b) there exists a permutation $\sigma = \{i \to \sigma(i)\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i = 1, \dots, n$; $k = 1, \dots, n$.

Proof follows immediately from Lemma 1.3.

Below the permutation σ will be called a Kirichenko permutation.

1.5. Theorem. *[4] Let A_A be Noetherian.*

a) The following conditions are equivalent:

1) A_A is injective;

2) A_A is cogenerating;

3) ${}_AA$ is injective;

4) ${}_AA$ is cogenerating;

5) $\forall M \subset A_A\ [rl(M) = M] \wedge \forall N \subset {}_AA\ [lr(N) = N]$.

b) If the conditions from (p) hold then A is a two-sided Artinian ring.

(By $l(M)$ and $r(N)$ the annihilators of modules M and N are denoted, i.e. $l(M) = \{a \in A \mid aM = 0\}$, $r(N) = \{b \in A \mid Nb = 0\}$).

1.6 Definition. [4] A ring is called quasi-Frobenius (a QF-ring) if the conditions from the previous theorem are satisfied.

1.7 Theorem. *[2] Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a prime reduced Gorenstein tiled order with the Jacobson radical R and J be a two-sided ideal Λ such that $\Lambda \supset R^2 \supset J \supset R^n$ ($n \geqslant 2$). The factor ring Λ/J is quasi-Frobenius (QF) if and only if there exists $p \in R^2$ such that $J = p\Lambda = \Lambda p$.*

2. Finite groups and Gorenstein orders. Put $G_0 = \{0\}$. Denote by Γ_0 a Gorenstein tiled order with the exponent matrix $\mathcal{E}(\Gamma_0) = (0)$.

The matrix $\mathcal{E}(\Gamma_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Cayley table of a cyclic group (2) and also the exponent matrix of the Gorenstein tiled order Γ_1 with the Kirichenko permutation $\sigma = (12)$.

Denote by $U_n = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$ the square matrix of order n.

Clearly, the Cayley table of the Klein Viergruppe $(2) \times (2)$ can be written in the form

$$\mathcal{E}(\Gamma_2) = \begin{pmatrix} \mathcal{E}(\Gamma_1) & \mathcal{E}(\Gamma_1) + 2U_2 \\ \mathcal{E}(\Gamma_1) + 2U_2 & \mathcal{E}(\Gamma_1) \end{pmatrix}.$$

Let us consider the matrix

$$\mathcal{E}(\Gamma_k) = \begin{pmatrix} \mathcal{E}(\Gamma_{k-1}) & \mathcal{E}(\Gamma_{k-1}) + 2^{k-1}U_{2^{k-1}} \\ \mathcal{E}(\Gamma_{k-1}) + 2^{k-1}U_{2^{k-1}} & \mathcal{E}(\Gamma_{k-1}) \end{pmatrix}. \tag{2}$$

2.1. Proposition. *$\mathcal{E}(\Gamma_k)$ is the exponent matrix of a tiled order.*

Evidently,

$$\Gamma_k = \begin{pmatrix} \Gamma_{k-1} & \pi^{2^{k-1}}\Gamma_{k-1} \\ \pi^{2^{k-1}}\Gamma_{k-1} & \Gamma_{k-1} \end{pmatrix}. \tag{3}$$

Induction on k easily yields that Γ_k is a tiled order.

Let $G = H \times \langle g \rangle$ be a finite Abelian group, $H = \{h_1, \dots, h_n\}$, $g^2 = e$. We shall consider the Cayley table of the group H as the matrix $K(H) = (h_{ij})$ with the entries in H, where $h_{ij} = h_i h_j$. The following proposition is obvious.

2.2. Proposition. *The Cayley table of the group G is of the form*

$$K(G) = \begin{pmatrix} K(H) & gK(H) \\ gK(H) & K(H) \end{pmatrix}.$$

2.3. Proposition. *$\mathcal{E}(\Gamma_k)$ is the Cayley table of a group G_k of order 2^k.*

The proof is based on induction on k. The basis of induction have been already considered. If $\mathcal{E}(\Gamma_{k-1})$ is the Cayley table of a group of order 2^{k-1} then by 2.2. Proposition $\mathcal{E}(\Gamma_k)$ is the Cayley table of a group G_k of order 2^k.

2.4. Proposition. *A tiled order Γ_k is Gorenstein with the Kirichenko permutation* $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & 2^k-1 & 2^k \\ 2^k & 2^k-1 & 2^k-2 & \dots & 2 & 1 \end{pmatrix}$.

Let us prove by induction on k that the tiled order Γ_k is Gorenstein. For $k = 1$ this is obvious. Let the tiled order Γ_k be Gorenstein with the exponent matrix $\mathcal{E}(\Gamma_k) = (\alpha_{ij}^k)$ ($i,j = 1,2,\dots,2^k$) and the Kirichenko permutation $\sigma = \sigma_k$, where $\sigma_k(i) = 2^k + 1 - i$. Then $\alpha_{ij}^k + \alpha_{j\sigma_k(i)}^k = \alpha_{i\sigma_k(i)}^k$ for all $i,j = 1,2,\dots,2^k$. Since

$$\alpha_{2^k+i,j}^{k+1} = \alpha_{i,2^k+j}^{k+1} = \alpha_{ij}^k + 2^k, \quad \alpha_{2^k+i,2^k+j}^{k+1} = \alpha_{ij}^{k+1} = \alpha_{ij}^k \text{ for all } i,j = 1,2,\dots,2^k, \tag{4}$$

we have $(\alpha_{ij}^k + 2^k) + \alpha_{j\sigma_k(i)}^k = \alpha_{ij}^k + (2^k + \alpha_{j\sigma_k(i)}^k) = \alpha_{i\sigma_k(i)}^k + 2^k$. Thus, taking into account (4) we obtain that

$$\alpha_{ij}^{k+1} + \alpha_{j,2^k+\sigma_k(i)}^{k+1} = \alpha_{i,2^k+\sigma_k(i)}^{k+1}, \quad \alpha_{i,2^k+j}^{k+1} + \alpha_{2^k+j,2^k+\sigma_k(i)}^{k+1} = \alpha_{i,2^k+\sigma_k(i)}^{k+1},$$

$$\alpha_{2^k+i,2^k+j}^{k+1} + \alpha_{2^k+j,\sigma_k(i)}^{k+1} = \alpha_{2^k+i,\sigma_k(i)}^{k+1}, \quad \alpha_{2^k+i,j}^{k+1} + \alpha_{j\sigma_k(i)}^{k+1} = \alpha_{2^k+i,\sigma_k(i)}^{k+1},$$

$i,j = 1,2,\dots,2^k$. Putting $\sigma_{k+1}(i) = 2^k + \sigma_k(i)$, $\sigma_{k+1}(2^k+i) = \sigma_k(i)$, we have $\alpha^{k+1}_{pq} + \alpha^{k+1}_{q\sigma_{k+1}(p)} = \alpha^{k+1}_{p\sigma_{k+1}(p)}$ for all $p,q = 1,2,\dots,2^{k+1}$, i.e. the tiled order Γ_{k+1} is Gorenstein with the Kirichenko permutation $\sigma = \sigma_{k+1}$, where $\sigma_{k+1}(i) = 2^{k+1}+1-i$.

2.5. Theorem. *The Cayley table of a finite group G is the exponent matrix of a reduced Gorenstein tiled order if and only if $G = G_k = (2)\times\dots\times(2)$.*

Proof. The Cayley table of an Abelian 2-group G_k has form (2) and is the exponent matrix of a tiled order Γ_k.

Conversely, let G be a finite group and its Cayley table be the exponent matrix of a reduced Gorenstein tiled order. Then for any $g \in G$ we obtain $g^2 = e$ and G is an elementary Abelian 2-group. The theorem is proved.

Let us calculate the adjacency matrix of the quiver $Q(\Gamma_k)$. For this aim we present the tiled order Γ_k in form (3).

Let $R_k = rad\Gamma_k$ be the Jacobson radical of the ring Γ_k and $\mathcal{E}(\Gamma_k) = (\alpha^k_{ij})$, $\mathcal{E}(R_k) = (r^k_{ij})$, $\mathcal{E}(R_k^2) = (\beta^k_{ij})$. Then

$$R_k = \begin{pmatrix} R_{k-1} & \pi^{2^{k-1}}\Gamma_{k-1} \\ \pi^{2^{k-1}}\Gamma_{k-1} & R_{k-1} \end{pmatrix}, R_k^2 = \begin{pmatrix} R_{k-1}^2 + \pi^{2^k}\Gamma_{k-1} & \pi^{2^{k-1}}R_{k-1}\Gamma_{k-1} \\ \pi^{2^{k-1}}R_{k-1}\Gamma_{k-1} & R_{k-1}^2 + \pi^{2^k}\Gamma_{k-1} \end{pmatrix}.$$

As $r^{k-1}_{ij} \leqslant 2^{k-1}$ so $\beta^{k-1}_{ij} \leqslant 2^k \leqslant 2^k + \alpha^{k-1}_{ij}$. Therefore $R_{k-1}^2 + \pi^{2^k}\Gamma_{k-1} = R_{k-1}^2$.

The equality $(radA)A = A(radA) = radA$ holds for any prime tiled order A. Hence $\pi^{2^{k-1}}R_{k-1}\Gamma_{k-1} = \pi^{2^{k-1}}R_{k-1}$. Since $\mathcal{E}(\pi^{2^{k-1}}R_{k-1}) - \mathcal{E}(\pi^{2^{k-1}}\Gamma_{k-1}) = (2^{k-1} + \mathcal{E}(R_{k-1}) - (2^{k-1} + \mathcal{E}(\Gamma_{k-1})) = E$, we obtain

$$\mathcal{E}(R_k^2) - \mathcal{E}(R_k) = \begin{pmatrix} \mathcal{E}(R_{k-1}^2) - \mathcal{E}(R_{k-1}) & E \\ E & \mathcal{E}(R_{k-1}^2) - \mathcal{E}(R_{k-1}) \end{pmatrix}.$$

From this follows that

$$[Q(\Gamma_k)] = \begin{bmatrix} [Q(\Gamma_{k-1})] & E \\ E & [Q(\Gamma_{k-1})] \end{bmatrix}.$$

Let us compute the characteristic polynomial $\chi_k(x) = \chi_{[Q(\Gamma_k)]}(x)$.

$$\chi_{k+1}(x) = |xE - [Q(\Gamma_{k+1})]| = \begin{vmatrix} xE - [Q(\Gamma_k)] & -E \\ -E & xE - [Q(\Gamma_k)] \end{vmatrix} =$$

$$= \begin{vmatrix} xE - [Q(\Gamma_k)] - E & 0 \\ -E & xE - [Q(\Gamma_k)] + E \end{vmatrix} =$$

$$= |(x-1)E - [Q(\Gamma_k)]| \cdot |(x+1)E - [Q(\Gamma_k)]|$$

Therefore

$$\chi_{k+1}(x) = \chi_k(x-1)\cdot\chi_k(x+1). \tag{5}$$

Since $\chi_1(x) = \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix} = x(x-2)$, we have $\chi_2(x) = (x-3)(x-1)(x-1)(x+1) = (x-3)(x-1)^2(x+1)$, $\chi_3(x) = (x-4)(x-2)^2x(x-2)x^2(x+2) = (x-4)(x-2)^3x^3(x+2)$.

2.6. Proposition. $\chi_m(x) = \prod_{i=0}^{m}(x - m - 1 + 2i)^{C_m^i}$.

We shall prove this proposition by induction. The basis of induction is clear. Suppose that the formula is true for $m = k$. Then by formula (5) we have

$$\chi_{k+1}(x) = \prod_{i=0}^{k}(x - k - 2 + 2i)^{C_k^i} \cdot \prod_{j=0}^{k}(x - k + 2j)^{C_k^j} =$$

$$= (x - k - 2)\prod_{i=1}^{k}(x - k - 2 + 2i)^{C_k^i} \cdot \prod_{j=0}^{k-1}(x - k + 2j)^{C_k^j}(x + k) =$$

$$= (x - k - 2)\prod_{i=0}^{k-1}(x - k + 2i)^{C_k^{i+1}} \cdot \prod_{j=0}^{k-1}(x - k + 2j)^{C_k^j}(x + k) =$$

$$= (x - k - 2)\prod_{i=0}^{k-1}(x - k + 2i)^{C_k^i + C_k^{i+1}}(x + k).$$

As $C_k^i + C_k^{i+1} = C_{k+1}^{i+1}$, then $\chi_{k+1}(x) = (x - k - 2)\prod_{i=0}^{k-1}(x - k + 2i)^{C_{k+1}^{i+1}}(x + k) =$
$= (x - k - 2)\prod_{j=1}^{k}(x - k + 2(j - 1))^{C_{k+1}^j}(x + k) = \prod_{j=0}^{k+1}(x - (k + 1) - 1 + 2j)^{C_{k+1}^j}$.

By induction it is easily to prove that $\sum_{i=1}^{2^k} q_{ij}(\Gamma_k) = k + 1$, $\sum_{j=1}^{2^k} q_{ij}(\Gamma_k) = k + 1$. So $[Q(\Gamma_k)] = (k + 1)P_k$, where P_k is a twice stochastic matrix.

3. On quasi-Frobenius factor rings of Gorenstein tiled orders. In this section we describe all two-sided ideals I of a Gorenstein prime tiled order $A = \Gamma_k$ such that I alies in square of the Jacobson radical of the ring A and the factor ring A/I is quasi-Frobenius.

Recall that there is a one-to-one correspondence between a two-sided ideal I and the exponent matrix $\mathcal{E}(I)$ also, and , besides, the inequalities $i_{pq} + \alpha_{qt} \geqslant i_{pt}$ and $\alpha_{pq} + i_{qt} \geqslant i_{pt}$ hold.

From the results of paper [2] it follows that the factor ring A/I is quasi-Frobenius if and only if there are isomorphisms $I \simeq A_A$, $I \simeq_A A$.

3.1. Proposition. *Let* $I_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix}$ *be an ideal of the ring* $\Gamma_{k+1} = \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix}$. *Then* L, M, N, T *are ideals of the ring* Γ_k, $\hat{I}_{k+1} = \begin{pmatrix} N & T \\ L & M \end{pmatrix}$ *and* $\check{I}_{k+1} = \begin{pmatrix} M & L \\ T & N \end{pmatrix}$ *are ideals of the ring* Γ_{k+1}, *too.*

Proof. Since I_{k+1} is an ideal of the ring Γ_{k+1}, we have

$$I_{k+1}\Gamma_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix}\begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} =$$

$$= \begin{pmatrix} L\Gamma_k + M\pi^{2^k}\Gamma_k & L\pi^{2^k}\Gamma_k + M\Gamma_k \\ N\Gamma_k + T\pi^{2^k}\Gamma_k & N\pi^{2^k}\Gamma_k + T\Gamma_k \end{pmatrix} = I_{k+1},$$

$$\Gamma_{k+1}I_{k+1} = \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} \begin{pmatrix} L & M \\ N & T \end{pmatrix} =$$

$$= \begin{pmatrix} \Gamma_k L + \pi^{2^k}\Gamma_k N & \Gamma_k M + \pi^{2^k}\Gamma_k T \\ \pi^{2^k}\Gamma_k L + \Gamma_k N & \pi^{2^k}\Gamma_k M + \Gamma_k T \end{pmatrix} = I_{k+1}.$$

We obtain the following equalities: $L\Gamma_k + M\pi^{2^k}\Gamma_k = L$,' $L\pi^{2^k}\Gamma_k + M\Gamma_k = M$, $N\Gamma_k + T\pi^{2^k}\Gamma_k = N$, $N\pi^{2^k}\Gamma_k + T\Gamma_k = T$, $\Gamma_k L + \pi^{2^k}\Gamma_k N = L$, $\Gamma_k M + \pi^{2^k}\Gamma_k T = M$, $\pi^{2^k}\Gamma_k L + \Gamma_k N = N$, $\pi^{2^k}\Gamma_k M + \Gamma_k T = T$.

Thus, L, M, N, T are ideals of the ring Γ_k. Taking into account these equalities we have

$$\hat{I}_{k+1}\Gamma_{k+1} = \begin{pmatrix} N & T \\ L & M \end{pmatrix} \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} =$$

$$= \begin{pmatrix} N\Gamma_k + T\pi^{2^k}\Gamma_k & N\pi^{2^k}\Gamma_k + T\Gamma_k \\ L\Gamma_k + M\pi^{2^k}\Gamma_k & L\pi^{2^k}\Gamma_k + M\Gamma_k \end{pmatrix} = \begin{pmatrix} N & T \\ L & M \end{pmatrix} = \hat{I}_{k+1},$$

$$\Gamma_{k+1}\hat{I}_{k+1} = \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} \begin{pmatrix} N & T \\ L & M \end{pmatrix} = \begin{pmatrix} \Gamma_k N + \pi^{2^k}\Gamma_k L & \Gamma_k T + \pi^{2^k}\Gamma_k M \\ \pi^{2^k}\Gamma_k N + \Gamma_k L & \pi^{2^k}\Gamma_k T + \Gamma_k M \end{pmatrix} =$$

$$= \begin{pmatrix} N & T \\ L & M \end{pmatrix} = \hat{I}_{k+1}.$$

Therefore $\hat{I}_{k+1}$ is an ideal of the ring Γ_{k+1}. Analogously, it can be proved that $\check{I}_{k+1}$ is an ideal of the ring Γ_{k+1}. The proposition is proved.

3.2. Proposition. *Let I_{k+1} be an ideal of the ring Γ_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is quasi-Frobenius. Then $I_{k+1} = \begin{pmatrix} I_k & \pi^{2^k}I_k \\ \pi^{2^k}I_k & I_k \end{pmatrix}$ or*

$I_{k+1} = \begin{pmatrix} \pi^{2^k}I_k & I_k \\ I_k & \pi^{2^k}I_k, \end{pmatrix}$ where I_k is an ideal of the ring Γ_k such that the factor ring Γ_k/I_k is quasi-Frobenius.

Proof. The entries of the rows of the matrix $\mathcal{E}(\Gamma_{k+1})$ have such a property: if $i,l \leqslant 2^k$ or $i,l > 2^k$ then $|\alpha_{ij} - \alpha_{lj}| < 2^k$ for any j; if $i \leqslant 2^k, l > 2^k$ or $i > 2^k, l \leqslant 2^k$ then there exist at the least two values j and s such that $\alpha_{ij} - \alpha_{lj} \geqslant 2^k$ and $\alpha_{is} - \alpha_{ls} \leqslant 2^k$. The elements of the columns of the matrix $\mathcal{E}(\Gamma_{k+1})$ possess this property, too .

Let I_{k+1} be an ideal of the ring Γ_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is quasi-Frobenius, $1 = e_1 + e_2 + \cdots + e_{2^{k+1}}$ be a decomposition of the identity of the ring into a sum of local pairwise orthogonal idempotents. Hence by [2] for any i there exists t such that the right module e_iI_{k+1} is isomorphic to the indecomposable projective right module $e_t\Gamma_{k+1} = P_t$. Analogously, for any j there exists m such that the left module $I_{k+1}e_j$ is isomorphic to the indecomposable projective left module $\Gamma_{k+1}e_m = Q_m$. Note that if $e_iI_{k+1} \simeq P_t$, $I_{k+1}e_j \simeq Q_m$ and $\mathcal{E}(I_{k+1}) = (\kappa_{uv})$ so by [1] $\kappa_{iv} = \alpha_{tv} + a_i$ for all v, $\kappa_{uj} = \alpha_{um} + b_j$ for all u (a_i, b_j are some integers).

Let $e_iI_{k+1} \simeq P_t$ and $e_jI_{k+1} \simeq P_m$. Assume that $i,j \leqslant 2^k, t \leqslant 2^k$ and $m > 2^k$. Then $\kappa_{iv} = \alpha_{tv} + a_i, \kappa_{jv} = \alpha_{mv} + a_j$. Let $a_i \geqslant a_j$. By the property of the rows of the matrix $\mathcal{E}(\Gamma_{k+1})$ there exists w such that $\alpha_{tw} - \alpha_{mw} \geqslant 2^k$. Hence $\kappa_{iw} - \kappa_{jw} = \alpha_{tw} + a_i - (\alpha_{mw} + a_j) \geqslant 2^k$. Since $0 \leqslant \alpha_{lw} < 2^k$ for all $0 < l \leqslant 2^k$, or $2^k \leqslant \alpha_{lw} < 2^{k+1}$for all $0 < l \leqslant 2^k$ then if $i, j \leqslant 2^k$ we have $\kappa_{iw} - \kappa_{jw} = \alpha_{iz} + b_w - (\alpha_{jz} + b_w) < 2^k$, where z satisfies the condition $I_{k+1}e_w \simeq Q_z$. We obtain contradiction.

Analogously, the case $t > 2^k$ and $m \leqslant 2^k$ is impossible for $i, j \leqslant 2^k$.

So for $i, j \leqslant 2^k$ two cases are possible : a) $t, m \leqslant 2^k$, b) $t, m > 2^k$.

By Proposition 3.1, if $I_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix}$ is an ideal of the order Γ_{k+1} then L, M, N, T are ideals of the order Γ_k.

In case a) for every $i \leqslant 2^k$ there exists $t \leqslant 2^k$ such that $e_i I_{k+1} \simeq P_t$.

Since $\alpha_{t,2^k+v} = \alpha_{t,v} + 2^k$ for $v \leqslant 2^k$, we have $\kappa_{i,2^k+v} = \alpha_{t,2^k+v} + a_i = \alpha_{tv} + 2^k + a_i = \kappa_{iv} + 2^k$, that is we have $M = \pi^{2^k} L$.

For every $j > 2^k$ there exists $m > 2^k$ such that $e_j I_{k+1} \simeq P_m$. Since $\alpha_{m,v} = \alpha_{m,2^k+v} + 2^k$ for $v \leqslant 2^k$, hence $\kappa_{jv} = \alpha_{mv} + a_j = \alpha_{m,2^k+v} + 2^k + a_j = \kappa_{j,2^k+v} + 2^k$, i.e. $N = \pi^{2^k} T$.

In case b) we obtain analogously $L = \pi^{2^k} M$, $T = \pi^{2^k} N$.

Now let us examine the entries of the columns of the matrix $\mathcal{E}(\Gamma_{k+1})$. Let $I_{k+1}e_i \simeq Q_p$ and $I_{k+1}e_j \simeq Q_r$, $i, j \leqslant 2^k$. For these entries as well as for the elements of the rows of this matrix there are two possible cases: c) $p, r \leqslant 2^k$, d) $p, r > 2^k$.

Suppose that in case a) for the entries of the rows we have case d) for the elements of the columns. Then we obtain $L = \pi^{2^k} N$ and $T = \pi^{2^k} M$.

Since $M = \pi^{2^k} L$ and $N = \pi^{2^k} T$, we obtain

$$L = \pi^{2^k} N = \pi^{2^k} \pi^{2^k} T = \pi^{2^{k+1}} T \text{ and } T = \pi^{2^k} M = \pi^{2^k} \pi^{2^k} L = \pi^{2^{k+1}} L.$$

Thus, $L = T = N = M = 0$ and $I_{k+1} = 0$.

In case c) for the entries of the columns we obtain $N = \pi^{2^k} L$, $M = \pi^{2^k} T$. As $M = \pi^{2^k} L$ and $N = \pi^{2^k} T$ so $L = T$, and therefore $M = N$.

Denote $L = I_k$, then $T = I_k$, $M = N = \pi^{2^k} I_k$. Thus, in case a) we have

$$I_{k+1} = \begin{pmatrix} I_k & \pi^{2^k} I_k \\ \pi^{2^k} I_k & I_k \end{pmatrix}.$$

Analogously in case b)

$$I_{k+1} = \begin{pmatrix} \pi^{2^k} I_k & I_k \\ I_k & \pi^{2^k} I_k \end{pmatrix}.$$

It remains to prove only the factor ring Γ_k / I_k is quasi-Frobenius. In case a) we have

$$\begin{aligned} \kappa_{iv} &= \alpha_{tv} + a_i \quad \text{for all } v = 1, \dots, 2^{k+1}, \\ \kappa_{uj} &= \alpha_{um} + b_j \quad \text{for all } u = 1, \dots, 2^{k+1}. \end{aligned} \tag{6}$$

Since $\mathcal{E}(I_k) = (\kappa_{uv})$, where $u, v \leqslant 2^k$, and $\mathcal{E}(\Gamma_k) = (\alpha_{ij})$, $i, j \leqslant 2^k$, the equalities (6) are true for the entries of the matrices $\mathcal{E}(I_k)$ and $\mathcal{E}(\Gamma_k)$. Therefore by [1] $e_i I_k \simeq e_t \Gamma_k$ and $I_k e_j \simeq \Gamma_k e_m$. In view of [2] we obtain that the factor ring Γ_k / I_k is quasi-Frobenius.

This completes the proof of the proposition.

3.3. Proposition. *There exist 2^{k+1} essentially different ideals I_{k+1} of the ring Γ_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is quasi-Frobenius.*

Proof. By Proposition 3.2, it follows that on the basis of any ideal I_k such that Γ_k/I_k is a QF-ring one can construct two ideals of the ring Γ_{k+1} such that Γ_{k+1}/I_{k+1} is a QF-ring. So the number of the ideals of the ring Γ_{k+1} is twice as many as the number of the ideals of the ring Γ_k (with the property Γ_k/I_k being a QF-ring). As $\Gamma_0 = \mathcal{O}$ has the unique non-isomorphic ideal, by induction it is easily to obtain the proposition.

Let I_{k+1} be an ideal of the ring Γ_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is quasi-Frobenius. Therefore by Proposition 3.2 the transformation $I_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix} \rightarrow \hat{I}_{k+1} = \begin{pmatrix} N & T \\ L & M \end{pmatrix}$ is equivalent to the transformation $I_{k+1} = \begin{pmatrix} L & M \\ N & T \end{pmatrix} \rightarrow \check{I}_{k+1} = \begin{pmatrix} M & L \\ T & N \end{pmatrix}$. By such transformations we obtain again an ideal $\overline{I}_{k+1} = \hat{I}_{k+1} = \check{I}_{k+1}$ such that the factor ring $\Gamma_{k+1}/\overline{I}_{k+1}$ is quasi-Frobenius.

3.4. Proposition. *By the above-indicated transformations any ideal I_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is a QF-ring can be obtained from the principal ideal $\tilde{I}_{k+1} = \pi^p\Gamma_{k+1}$.*

Proof. The transformation $I_{k+1} \rightarrow \overline{I}_{k+1}$ is invertible, hence it is sufficient to show that any ideal I_{k+1} such that the factor ring Γ_{k+1}/I_{k+1} is a QF-ring can be reduced to the principal ideal $\tilde{I}_{k+1} = \pi^p\Gamma_{k+1}$.

Suppose that an ideal I_{s+1} has the form $I_{s+1} = \begin{pmatrix} \pi^{2^k} I_s & I_s \\ I_s & \pi^{2^k} I_s \end{pmatrix}$ for some $s \leqslant k$. Then $\overline{I}_{s+1} = \begin{pmatrix} I_s & \pi^{2^k} I_s \\ \pi^{2^s} I_k & I_s \end{pmatrix}$. Along with this, the ideal I_{k+1} can be transformed into an ideal I'_{k+1} but the factor ring Γ_{k+1}/I'_{k+1} still remains quasi-Frobenius. Therefore by the indicated transformations the ideal I_{k+1} is reduced to the ideal $\tilde{I}_{k+1}$ for which $\tilde{I}_{s+1} = \begin{pmatrix} \tilde{I}_s & \pi^{2^k} \tilde{I}_s \\ \pi^{2^s} \tilde{I}_k & \tilde{I}_s \end{pmatrix}$ for all $s \leqslant k$.

Let $\mathcal{E}(\tilde{I}_{k+1}) = (\tilde{\kappa}_{uv})$ and $\tilde{\kappa}_{11} = p$. Then $\tilde{I}_1 = \begin{pmatrix} \tilde{I}_0 & \pi^{2^0}\tilde{I}_0 \\ \pi^{2^0}\tilde{I}_0 & \tilde{I}_0 \end{pmatrix} = \begin{pmatrix} \tilde{I}_0 & \pi\tilde{I}_0 \\ \pi\tilde{I}_0 & \tilde{I}_0 \end{pmatrix} = \begin{pmatrix} \pi^p\mathcal{O} & \pi^{p+1}\mathcal{O} \\ \pi^{p+1}\mathcal{O} & \pi^p\mathcal{O} \end{pmatrix} = \pi^p \begin{pmatrix} \mathcal{O} & \pi\mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix} = \pi^p\Gamma_1$.

Assume that $\tilde{I}_k = \pi^p\Gamma_k$. Hence

$$\tilde{I}_{k+1} = \begin{pmatrix} \tilde{I}_k & \pi^{2^k}\tilde{I}_k \\ \pi^{2^k}\tilde{I}_k & \tilde{I}_k \end{pmatrix} = \begin{pmatrix} \pi^p\Gamma_k & \pi^{2^k}\pi^p\Gamma_k \\ \pi^{2^k}\pi^p\Gamma_k & \pi^p\Gamma_k \end{pmatrix} = \pi^p \begin{pmatrix} \Gamma_k & \pi^{2^k}\Gamma_k \\ \pi^{2^k}\Gamma_k & \Gamma_k \end{pmatrix} = \pi^p\Gamma_{k+1}.$$

By induction we have proved that $\tilde{I}_{k+1} = \pi^p\Gamma_{k+1}$.

Obviously, $e_i\tilde{I}_{k+1} \simeq P_i$, $\tilde{I}_{k+1}e_j \simeq Q_j$. We note that by the indicated transformations the first column of the matrix $\mathcal{E}(\Gamma_k)$ (and the others, too) always turns into another column of this matrix. If the first column turns into l-th column then for any $i \leqslant 2^k$ there exists an integer r such that $\alpha_{i1} = \alpha_{lr}$. So by such transformations every principal ideal turns into an ideal I_k for which $e_lI_k \simeq P_i$, $I_ke_l \simeq Q_i$. As

$\alpha_{i1} = i - 1 = \alpha_{lr}$ so $i = \alpha_{lr} + 1$. Therefore

$$e_l I_k \simeq P_{1+\alpha_{lr}}, \; I_k e_l \simeq Q_{1+\alpha_{lr}}. \tag{7w}$$

The exponent matrix $\mathcal{E}(\Gamma_k)$ possesses 2^k columns. Let us enumerate 2^k ideals in such a way that every ideal for which (3) holds has number r. Thus, we have proved the following theorem.

3.5. Theorem. *There exist exactly 2^k essentially different ideals I_{kr}, $r = 1, 2, \ldots, 2^k$, such that the factor rings Γ_k/I_{kr} are quasi-Frobenius. Besides, $e_l I_k \simeq P_{1+\alpha_{lr}}$, $I_k e_l \simeq Q_{1+\alpha_{lr}}$.*

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ІДЕАЛИ ГОРЕНШТЕЙНОВИХ ЧЕРЕПИЧНИХ ПОРЯДКІВ, ФАКТОР-КІЛЬЦЯ ЗА ЯКИМИ КВАЗІФРОБЕНІУСОВІ

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Вивчено горенштейновий порядок, матриця показників якого є таблицею Келі скінченної групи. Описано ідеали цього порядку, фактор-кільця за якими є квазіфробеніусовими.

Ключові слова: горенштейновий черепичний порядок, матриця показників, матриця суміжності, квазіфробеніусове кільце.

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FROBENIUS RINGS

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We prove that a finite dimensional algebra A is a weakly symmetric if and only if when every algebra C which is Morita equivalent to a Frobenius algebra A is Frobenius. We give a description of serial rings the square of Jacobson radical of which is zero.

Key words: quasi-Frobenius ring, Frobenius ring, serial ring.

1. Let A be a two-sided artinian ring and R be its Jacobson radical. For a (right) A-module M we denote by M^n the direct sum of n copies of M and we set $M^0 = 0$. Then A can be represented as a direct sum of right ideals: $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$, where $P_1, \ldots, P_s$ are pairwise non-isomorphic indecomposable right A-modules, which are called the *principal right A-modules.* Set $U_i = P_i/P_iR, i = 1, \ldots, s$. It is well-known that $P_1, \ldots, P_s$ represent up to isomorphism all indecomposable projective A-modules, while $U_1, \ldots, U_s$ form a representative set of isomorphism classes of all simple right A-modules. Let M be a right A-module and N be a left A-module. We set $top\ M = M/MR$ and $top\ N = N/RN$. We denote by $soc\ M$ (respectively $soc\ N$) the largest semisimple right (respectively left) submodule of M (respectively N). Since A is artinian, *soc* exists for all A-modules. Let $1 = f_1 + \ldots + f_s$ be a decomposition of the identity element of A into a sum of idempotents such that $f_iA = P_i^{n_i}$ $(i = 1, \ldots, s)$. Then $Af_i = Q_i^{n_i}$, where $Q_1, \ldots, Q_s$ are the pairwise non-isomorphic indecomposable projective left A-modules (the *principal left A-modules*). Set $A_{ij} = f_iAf_j$ $(i, j = 1, \ldots, s)$. Then A has the following *canonical Peirce decomposition*

$$A = \bigoplus_{i,j=1}^{s} A_{ij}. \tag{1}$$

Denote by R_i the radical of A_{ii}, $(i = 1, \ldots, s)$. Obviously, A_{ii} is artinian. Since $Hom(P_j^{n_j}, P_i^{n_i}) \cong A_{ij}$, then $A_{ij} \subset R$ if $i \neq j$. The radical R of A has the following Peirce decomposition:

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$$R = \bigoplus_{i,j=1}^{s} f_i R f_j, \tag{2}$$

where $f_i R f_i = R_i$ and $f_i R f_j = A_{ij}, i \neq j \, (i = 1, \dots, s)$.

Observe that two principal A-modules P and P' are isomorphic if and only if $top\, P \simeq top\, P'$.

We recall now the classical definition of Frobenius and quasi-Frobenius rings as given by Tadasi Nakayama (see [13, p.8], [9, Section 13.4]).

Definition 1.1. A two sided artinian ring A is called *quasi-Frobenius*, if there exists a permutation ν of $\{1, 2, \dots, s\}$ such that for each $k = 1, \dots, s$ we have

(qf1) $soc\ P_k \cong top\ P_{\nu(k)}$,

(qf2) $soc\ Q_{\nu(k)} \cong top\ Q_k$.

A quasi-Frobenius ring A is called *Frobenius*, if $n_{\nu(i)} = n_i$ for all $i = 1, \dots, s$. This permutation ν is called the *Nakayama permutation* of A. Clearly, ν is determined up to conjugation in the symmetric group on s letters, and conjugations correspond to renumberings of the principal modules $P_1, \dots, P_s$.

We construct now some examples of quasi-Frobenius rings. Recall that a local ring $\mathcal{O}$ with non-zero unique maximal right ideal $\mathcal{M}$ is called a discrete valuation ring, if it has no zero divisors, the right ideals of $\mathcal{O}$ form the unique chain:

$$\mathcal{O} \supset \mathcal{M} \supset \mathcal{M}^2 \supset \dots \supset \mathcal{M}^n \supset \dots,$$

and, moreover, this chain is also the unique chain of left ideals of A. Then, obviously, $\mathcal{O}$ is noetherian, but not artinian, all powers of $\mathcal{M}$ are distinct and $\bigcap_{k=1}^{\infty} \mathcal{M}^k = 0$. Moreover, $\mathcal{M}$ is principal as a right (left) ideal.

Example. Denote by $H_s(\mathcal{O})$ the ring of all $s \times s$ matrices of the following form:

$$H = H_s(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \end{pmatrix}.$$

It is easily seen that the radical R of $H_s(\mathcal{O})$ is

$$R = \begin{pmatrix} \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{M} \end{pmatrix} \text{ and } R^2 = \begin{pmatrix} \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}^2 & \mathcal{M} & \cdots & \mathcal{M} \end{pmatrix}.$$

The principal right modules of H are the "row-ideals" of H and the submodules of each of them form a chain. In particular, the submodules of the "first-row-ideal" form the following chain:

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \begin{pmatrix} \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \begin{pmatrix} \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \ldots .$$

It is easy to see that each other row-ideal of H is isomorphic to a submodule of the above module. In a similar fashion, the principal left H-modules are the column-ideals, whose submodules form corresponding chains. Thus, H is a serial ring in the sense of [5, p. 224]. Let $P_1, \ldots, P_s$ be the principal right modules of the quotient ring $A = H_s(\mathcal{O})/R^2$ and $Q_1, \ldots, Q_s$ be the principal left A-modules numbered such that $P_i = e_{ii}A, Q_i = Ae_{ii}, (i = 1, \ldots, s)$, where e_{ij} denote the $s \times s$ matrix whose (i,j)'s entry is 1 and all other entries are zero. Then the submodules of every P_i and Q_i form finite chains, and a direct verification show that

$$soc\ P_1 \cong top\ P_2, soc\ P_2 \cong top\ P_3, \ldots, soc\ P_s \cong top\ P_1$$

and

$$top\ Q_1 \cong soc\ Q_2, top\ Q_2 \cong soc\ Q_3, \ldots, top\ Q_s \cong soc\ Q_1.$$

Moreover, each of these modules is a one-dimensional vector space over $\mathcal{O}/\mathcal{M}$. Hence, A is a quasi-Frobenius ring whose Nakayama permutation is $(1, 2, ..., s)$.

More in general, the quotient ring $A = H_s(\mathcal{O})/R^m$ $(m \geqslant 2)$ is a quasi-Frobenius ring whose Nakayama permutation is $(1, 2, ..., s)^{m-1}$. It follows, in particular, that the Nakayama permutation of A is identical if and only if $m \equiv 1 (mod\ s)$.

We shall use the next two results.

Lemma 1.1. [4, Lemma 6.3.12.]. *Let $1 = e_1 + \ldots + e_m = h_1 + \ldots + h_n$ be two decompositions of $1 \in A$ into a sum of pairwise orthogonal primitive idempotents. Then $m = n$ and there exists an invertible element $a \in A$ and a permutation $i \to \sigma(i)$ such that $e_i = ah_{\sigma(i)}a^{-1}$ for each $i = 1, \ldots, n$.*

Lemma 1.2. *For every simple right A-module U_i and for each f_j we have $U_i f_j = \delta_{ij} U_i$, $(i,j = 1, \ldots, s)$. Similarly, for every simple left A-module V_i and for each f_j, $f_j V_i = \delta_{ij} V_i$, $(i,j = 1, \ldots, s)$.*

Proof. Go modulo R and apply the Wedderburn-Artin Theorem.

This lemma will be a useful tool in our further considerations and we shall refer to it as to *Lemma on annihilation of simple modules.* An idempotent $f \in A$, which is central modulo R, shall be called *minimal modulo* R if f can not be decomposed into a sum of two orthogonal idempotents, which are central modulo R. For two idempotents e and g of A we shall write $e \in g$, if $g = e + e'$, where $ee' = e'e = 0$. Clearly, e' is also an idempotent in A.

Theorem 1.3. *Let $1 = f_1 + \ldots + f_s = g_1 + \ldots + g_t$ be two decompositions of $1 \in A$ into a sum of pairwise orthogonal idempotents, which are minimal central modulo R. Then $s = t$ and there exist an invertible element $a \in A$ and a permutation $i \to \tau(i)$ of $\{1, \ldots, s\}$ such that $f_i = ag_{\tau(i)}a^{-1}$ for each $i = 1, \ldots, s$.*

Proof. Applying the Wedderburn-Artin Theorem to $\bar{A} = A/R$, we get immediately that $s = t$. Let $f_i = e_1^{(i)} + \ldots + e_{n_i}^{(i)}$ be a decomposition of f_i into a sum of pairwise orthogonal local idempotents. Then, obviously, $U_i e_k^{(i)} \neq 0$ for $k = 1, \ldots, n_i$. It follows from the Lemma on annihilation of simple modules that $U_i g_{\sigma(i)} = U_i$ for some $g_{\sigma(i)}$ and, moreover, $U_i g_j = 0$ if $j \neq \sigma(i)$. Renumber the idempotents $g_1, \ldots g_s$ such that $U_i g_i = U_i$ $(i = 1, \ldots, s)$. Take a decomposition $g_i = h_1^{(i)} + \ldots + h_{n_i}^{(i)}$ into a sum of pairwise orthogonal local idempotents. Then we obtain two decompositions of $1 \in A$, which satisfy the assumptions of Lemma 1.1. Hence, there exits a conjugating element $a \in A$ which transforms one decomposition into the other, up to a permutation. It follows from our numeration of idempotents $g_1, \ldots g_s$ that $a\{h_1^{(i)}, \ldots, h_{n_i}^{(i)}\}a^{-1} = \{e_1^{(i)}, \ldots, e_{n_i}^{(i)}\}$ for each $i = 1, \ldots, s$ and, consequently, $ag_i a^{-1} = f_i$ $(i = 1, \ldots, s)$.

Set $A_{ij} = f_i A f_j$. Then

$$A = \bigoplus_{i,j=1}^{s} A_{ij}, \quad R = \bigoplus_{i,j=1}^{s} R_{ij},$$

where $R_{ij} = f_i R f_j = A_{ij}$ for $i \neq j$ and R_{ii} is the Jacobson radical of A_{ii} $(i, j = 1, \ldots, s)$.

Such two-sided Peirce decompositions of A and R shall be called *canonical.* It follows from Theorem 1.3. that every other canonical Peirce decomposition of A can be obtained from

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix}$$

by a simultaneous permutation of lines and columns and the substitution of all Peirce components A_{ij} by $aA_{ij}a^{-1}$.

2. MONOMIAL IDEALS. Let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents. By an ideal we mean a two-sided ideal. For an ideal I of A the abelian group $e_i I e_j$ $(i, j = 1, \ldots, n)$ obviously lies in I, and $I = \oplus_{i,j=1}^{n} I_{ij}$ is a decomposition of I into a direct sum of abelian subgroups. Such decomposition is called the *two-sided Peirce decomposition* of I corresponding to $1 = e_1 + \ldots + e_n$. It has a natural matrix form:

$$I = \begin{pmatrix} I_{11} & I_{12} & \cdots & I_{1n} \\ I_{21} & I_{22} & \cdots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n1} & I_{n2} & \cdots & I_{nn} \end{pmatrix}.$$

If $J = \oplus_{i,j=1}^{n} J_{ij}$ is also an ideal, then

$$I + J = \begin{pmatrix} I_{11} + J_{11} & I_{12} + J_{12} & \cdots & I_{1n} + J_{1n} \\ I_{21} + J_{21} & I_{22} + J_{22} & \cdots & I_{2n} + J_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n1} + J_{n1} & I_{n2} + J_{n2} & \cdots & I_{nn} + J_{nn} \end{pmatrix},$$

and each Peirce component $(IJ)_{ij}$ of the product IJ is given by
$(IJ)_{ij} = \sum_{k=1}^{n} I_{ik} J_{kj} \quad (i, j = 1, \dots, n)$,
so that addition and multiplication of elements from I and J can be done by the addition and multiplication of corresponding matrices.

Let A be a two-sided artinian ring and $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents. Then $I = \oplus_{i,j=1}^{s} I_{ij}$ with $I_{ij} = f_i I f_j$ $(i, j = 1, \dots, s)$ is called the *canonical Peirce decomposition* of I. As above, it is easily seen that one canonical Peirce decomposition of I can be obtained from another one by a simultaneous permutation of lines and columns and the substitution of each Peirce component I_{ij} by $aI_{ij}a^{-1}$.

Definition 2.1. An ideal I of a artinian ring A shall be called *monomial* if each line and each column of a canonical Peirce decomposition of I contains exactly one non-zero Peirce component.

If I is a monomial ideal, then there exists a permutation $\nu \to \nu(i)$ of $\{1, \dots, s\}$ such that $I_{i\nu(i)} \neq 0$. Clearly, ν is determined up to conjugation in the symmetric group on s letters. We denote this permutation by $\nu(I)$.

Lemma 2.1. *Let A be a artinian ring. If I is a monomial ideal of A then each canonical Peirce component of I is an ideal in A.*

Proof. Let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents. Write $\nu = \nu(I)$, then $I = \oplus_{i,j=1}^{s} f_i I f_\nu(i)$. Obviously $f_i I f_{\nu(i)} f_k A f_l = 0$ if $k \neq \nu(i)$. Moreover, $f_i I f_{\nu(i)} f_{\nu(i)} A f_l \subseteq f_i I f_l$ which is non-zero if and only if $l = \nu(i)$, as I is monomial. Similarly, $f_k A f_l f_i I f_{\nu(i)} \neq 0$ if and only if $k = l = i$. It follows that $f_i I f_{\nu(i)}$ is an ideal in A for each $i = 1, \dots, n$.

Lemma 2.2. *Let A be a artinian ring. Then soc A_A coincides with the left annihilator $l(R)$ of $R = R(A)$, whereas soc ${}_AA$ coincides with the right annihilator $r(R)$. In particular, soc${}_AA$ and soc A_A are two-sided ideals.*

Proof. If U is a simple right A-module, then, obviously, $UR = 0$ and, consequently, *soc* $A_A \subseteq l(R)$. On the other hand, the equality $l(R)R = 0$ implies that $l(R)$ is a semisimple right A-module, so it has to be contained in the right socle of A, hence, $l(R) =$ *soc* A_A. Similarly, $r(R) =$ *soc* ${}_AA$.

The first statement of the next theorem is well known (see [1]), however, we include a proof in order to show that the whole result is a consequence of the Lemma on annihilation of simple modules.

Theorem 2.3. *Let A be a quasi-Frobenius ring. Then $soc\ _AA = soc\ A_A$. Moreover, $\mathcal{Z} = soc\ _AA$ is a monomial ideal and $\nu(\mathcal{Z})$ coincides with the Nakayama permutation $\nu(A)$ of A.*

Proof. Denote by $\mathcal{Z}_l$ (respectively $\mathcal{Z}_r$) the left (respectively right) socle of A. It follows from the definition of quasi-Frobenius rings and from the Lemma on annihilation of simple modules that $f_i\mathcal{Z}_l \neq 0$ for each $i = 1, \dots, s$. Then for every local idempotent $e \in f_i$ the set $ef_i\mathcal{Z}_l = e\mathcal{Z}_l$ is different from 0. Therefore, the right ideal $e\mathcal{Z}_l$ is a non-zero submodule of the principal module P_i and, consequently, $e\mathcal{Z}_l$ contains $soc\ P_i$, which implies that $\mathcal{Z}_l \supseteq \mathcal{Z}_r$. Since the Nakayama's definition of quasi-Frobenius rings is left-right symmetric, it follows that $\mathcal{Z}_r \supseteq \mathcal{Z}_l$, and thus, $\mathcal{Z}_l = \mathcal{Z}_r = \mathcal{Z}$.

It remains to show that $\mathcal{Z}$ is monomial and $\nu(\mathcal{Z}) = \nu(A)$. Write $\nu = \nu(A)$ and consider the canonical Peirce decomposition of $\mathcal{Z}$: $\mathcal{Z} = \oplus_{i,j=1}^{s} f_i\mathcal{Z}f_j$. Since $A_A = \oplus_{i=1}^{s} f_iA = \oplus_{i=1}^{s} P_i^{n_i}$, we have that $\mathcal{Z} = \oplus_{i=1}^{s} soc\ f_iA$ and $f_i\mathcal{Z} = soc\ f_iA = soc\ P_i^{n_i}$. It follows from Definition 1.1. that $soc\ P_i^{n_i} \cong U_{\nu(i)}^{n_i}$, so $f_i\mathcal{Z} \cong U_{\nu(i)}^{n_i}$, and the Lemma on annihilation of simple modules implies that $f_i\mathcal{Z}f_j = 0$ if and only if $j \neq \nu(i)$. Hence, $\mathcal{Z}$ is monomial and $\nu(\mathcal{Z})$ coincides with $\nu(A)$.

3. FROBENIUS RINGS. In [9] a ring A was called Frobenius if it is quasi-Frobenius and $soc\ A_A \cong top\ A_A$, $soc\ _AA \cong top\ _AA$. We want to point out that one of these isomorphisms can be ommited, namely:

Proposition 3.1. *A quasi-Frobenius ring A is Frobenius if and only if*

$$soc\ A_A \cong top\ A_A.$$

Proof. Suppose that $soc\ A_A \cong top\ A_A$. Since $top\ A_A \cong \oplus_{k=1}^{s} U_{\nu(k)}^{n_{\nu(k)}}$ and $soc\ A_A \cong \oplus_{k=1}^{s} U_{\nu(k)}^{n_k}$, it follows from the Jordan-Hölder Theorem that $n_k = n_{\nu(k)}$ for all k.

We have that $top\ A_A \cong \oplus_{k=1}^{s} top\ P_{\nu(k)}^{n_{\nu(k)}} \cong \oplus_{k=1}^{s} U_{\nu(k)}^{n_{\nu(k)}} \cong soc\ A_A$. Then $top\ _AA \cong \oplus_{k=1}^{s} top\ Q_{\nu(k)}^{n_{\nu(k)}} \cong \oplus_{k=1}^{s} V_{\nu(k)}^{n_{\nu(k)}} \cong soc\ _AA$.

Proposition 3.2. *A reduced QF-ring is Frobenius.*

Proof. Immediately follows from Definition 1.1.

Lemma 3.3. *If A is a Frobenius ring and $\nu(A)$ is a cycle then $A = M_n(B)$, where B is a reduced Frobenius ring with cyclic Nakayama permutation.*

Proof. Let $A = P_1^{n_1} \oplus \dots \oplus P_s^{n_s}$ be a decomposition of a Frobenius ring A into a direct sum of principal A-modules. We can suppose that $\nu(A) = (1 \dots s)$. Then by Definition 1.1. $n_1 = n_2 = \dots = n_s$ and $A = (P_1 \oplus P_2 \oplus \dots \oplus P_s)^n$ which yields $A = M_n(B)$ where $B = E(P_1 \oplus \dots \oplus P_s)$ and $\nu(B) = (1 \dots s)$.

Recall that a ring A is indecomposable if A cannot be decomposed into a direct product of two rings.

Proposition 3.4. *If A is a QF-ring and $\nu(A)$ is a cycle, then A is indecomposable.*

Proof. We can obviously suppose that $\nu(A) = (12\dots s)$. Then $\mathcal{Z} = soc\ A$ is a monomial ideal and $\nu(A) = \nu(\mathcal{Z})$. Therefore the canonical Peirce components $A_{ii+1}(i = 1, \dots, s-1)$ and A_{s1} are different from zero, by implies that A is indecomposable.

Definition 3.1. [2] A ring A is called *weakly prime* if the product of any two ideals that are not in the Jacobson radical R of A is non-zero.

Obviously, any prime ring is weakly prime.

Proposition 3.5. [2] *Let $1 = e_1 + \dots + e_n$ be a decomposition of the identity of semi-perfect ring A into a sum of mutually orthogonal local idempotents and $A_{ij} = e_iAe_j(i,j = 1,\dots,n)$. Then A is weakly prime if and only if $A_{ij} \neq 0$ for all i,j.*

In [14] QF-rings A are considered which satisfy the following conditions:

a) A is reduced;

b) $\nu(A)$ is a cycle;

c) for any non-trivial idempotent $e \in A$ eAe is a QF-ring and $\nu(eAe)$ is a cycle.

Proposition 3.6. *If a Frobenius ring A satisfies conditions (a), (b), (c) then A is weakly prime and every local ring e_iAe_i is Frobenius.*

Proof. Since A is reduced, the local idempotents coincide with the canonical idempotents. Let $A_{ij} = f_iAf_j$ for $i = 1, 2 \dots s$. If $A_{ij} = 0$ then $eAe(e = f_i + f_j)$ is a QF-ring. Obviously, $eAe = \begin{pmatrix} A_{ii} & 0 \\ A_{ji} & A_{jj} \end{pmatrix}$ and $\nu(eAe)$ is a cycle.

By Proposition 3.4., eAe is an indecomposable ring. Let $\mathcal{Z} = soc\, eAe$. The local ring e_iAe_i are Frobenius by condition (c).

Let $A = P_1^{n_1} \oplus \dots \oplus P_s^{n_s}$ be a decomposition of an artinian ring A into a direct sum of principal right A-modules and let $1 = f_1 + \dots + f_s$ be the corresponding decomposition of identity of the ring A into a sum of pairwise orthogonal idempotents, i.e., $f_iA = P_i^{n_i}$.

Definition 3.2. An idempotent $g \in A$ will be called *standard* if $g = f_{i_1} + \dots + f_{i_k}$, where $n_{i_1} = \dots = n_{i_k}$, in particular, if $f \in \{f_1, \dots, f_s\}$ and $fA = P^{n_{i_1}}$ then $f \in \{f_{i_1}, \dots, f_{i_k}\}$.

Definition 3.3. Let $1 = g_1 + \dots + g_m$ be a decomposition of $1 \in A$ into a sum of pairwise orthogonal standard idempotents. Put $A_{ij} = g_iAg_j\ (i,j = 1,\dots,m)$. The decomposition $A = \oplus_{i,j=1}^{m} A_{ij}$ will be called *the standard Peirce decomposition* of artinian ring A.

Theorem 3.7. *Let* $A = \bigoplus_{i,j=1}^{m} A_{ij}$ *be a standard Peirce decomposition of a Frobenius ring* A, *then* $A_{ii} = M_{k_i}(B_i)$ *where all rings* B_i *are reduced Frobenius rings.*

Proof. By Lemma on annihilation of simple modules the socle of arbitrary principal A-module $P = eA$ with $e \in g_i$ is annihilated by every standard idempotent $g_j \neq g_i$. Hence, $g_i \mathcal{Z} g_i$ is the socle of $M_{n_i}(B_i)$ and for each local idempotent $e \in g_i$ the A_{ii}-module $e\mathcal{Z}q_i$ is simple as a simple left A_{ii}-module and it is also simple as a right A_{ii}-module. Therefore A_{ii} is quasi-Frobenius.

The multiplicities of all principal A_{ii}-modules are k_i and consequently A_{ii} is Frobenius for all $i = 1 \dots , s$.

Theorem 3.8. *Let* A *be a* QF*-ring and the Nakayama permutation* $\nu(A)$ *of* A *is identical. Then* A *is Frobenius and every ring* C *which is Morita equivalent to* A *is also Frobenius. Conversely, if every ring* C *which is Morita equivalent to a Frobenius ring* A *is Frobenius, then* $\nu(A)$ *is identity.*

Proof. By Definition 1.1. every QF-ring with identical Nakayama permutation is automatically Frobenius. Clearly, every ring which is Morita equivalent to a Frobenius ring with identical Nakayama permutation is Frobenius.

Let A is a Frobenius ring and $\nu(A)$ is not identity. Then we can assume that $soc\ P_1 = top\ P_2$. Let $A = P_1^{n_1} \oplus P_2^{n_2} \oplus \dots \oplus P_s^{n_s}$ be a canonical decomposition of A into a direct sum of principal A-modules. It follows from the Definition 1.1., $n_2 = n_1$. Set $P = P_1^2 \oplus P_2 \oplus \dots \oplus P_s$. Then $C = End_A P$ is a QF-ring, and $\nu(A) = \nu(C)$, and multiplicity of the first principal C-module is 2 and does not coincide with the multiplicity of the second principal C-module. Therefore, C is not Frobenius.

Finite-dimensional Frobenius algebras with identical Nakayama permutation were called by [11, p. 444] weakly symmetric algebras. So from Theorem 3.8. we have such theorem.

Theorem 3.9. *Let* A *be a weakly symmetric algebra. Then* A *is Frobenius and every algebra* C *which is Morita equivalent to* A *is also Frobenius. Conversely, if every finite dimensional algebra* C *which is Morita equivalent to a Frobenius algebra* A *is Frobenius, then* A *is a weakly symmetric algebra.*

4. SERIAL QUASI-FROBENIUS RINGS.

Definition 4.1. A module is called *uniserial* if the lattice of its submodules is a chain, i.e. the set if all its submodules is linearly ordered by inclusion. A module is said to be serial of it is a finite direct sum of uniserial submodules.

Definition 4.2. A ring A is called *right* (resp. *left*) *uniserial* if A_A (resp. $_AA$) is an uniserial A-module. A ring which is right and left uniserial is called *uniserial.* A ring A is *right* (resp. *left*) *serial* if A_A (resp. $_AA$) is a serial A-module. A right serial and left serial ring shall be called *serial.*

Theorem 4.1. [10, Theorem 2.1] *The quiver* $Q(A)$ *of a serial two-sided Noetherian ring* A *is a disconnected union of cycles and chaines (i.e. of quivers corresponding to finite lineary ordered sets).*

Proposition 4.2. *Let $Q(A)$ be a quiver of a quasi-Frobenius ring A. If there is a vertex $i \in Q(A)$ which is either a sink (i is not the tail of any arrow) or a source (i is not the head of any arrow) then $A \simeq A_1 \times A_2$, where $A_1 \simeq M_n(\mathcal{D})$ with a division algebra $\mathcal{D}$.*

Proof. Let i is a sink. Then indecomposable projective A-module P_i is simple. Therefore $\nu(i) = i$, where $\nu = \nu(A)$ is Nakayama permutation of a ring A, and $A_{ij} \neq 0$ for $j = 1, \dots, i-1, i+1, \dots, n$.

Now we shall show that $A_{ki} = 0$ for $k = 1, \dots, i-1, i+1, \dots, s$. Let $A_{ki} \neq 0$. Then, because $A_{ki} \simeq Hom\,(P_i^{n_i}, P_k^{n_k})$ we obtain by the Lemma of Shur that the simple module U_i appears in a direct decomposition of $soc\,P_k$. So $\nu(k) = \nu(i) = i$ and $A_{ki} = 0$. Analogously, if i is source, then the left indecomposable projective A-module Q_i is simple and $A_{ki} = 0$, $A_{ij} = 0$ for $j, k = 1, \dots, i-1, i+1, \dots, s$.

As a corollary of this result and Theorem 4.1 we obtain the description of the quivers of serial QF-rings.

Theorem 4.3. *The quiver $Q(A)$ of a serial QF-ring A is a disconnected union of cycles and one-point quivers without arrows.*

Definition 4.3. A local serial (=uniserial) ring is called *a Köthe ring*.

Proposition 4.5. *A Köthe ring is Frobenius.*

Proof. Immediatly follows from Definition 1.1.

Let A be a Köthe ring. Then the length $l(A_A)$ of the right regular A-module coincides with the length $l({}_AA)$ of the left regular A-module. Then $l = l(A_A) = l({}_AA)$ shall be called the length of a Köthe ring A and denoted by $l(A)$.

A Köthe ring of length 1 is a division ring.

A Köthe ring of length m has a unique chain of ideals (right, left, two-sided):

$$A \supset R \supset R^2 \supset \dots \supset R^{m-1} \supset 0.$$

Lemma 4.6. *A local Frobenius ring A with $R^2 = 0$ is either a division ring or a Köthe ring of length 2. In the second case $Q(A)$ is a loop.*

The proof follows from Definition 1.1.

We give the description of serial reduced rings, the square of Jacobson radical of which is zero. Such rings are two-sided artinian, since the length of every right and every left principal module is less or equal than 2.

Lemma 4.7. *If A is an artinian indecomposable reduced serial non-local ring with $R^2 = 0$ then there is a subring A_0 in A such that $A = A_0 \oplus R$ (direct sum of abelian groups).*

Proof. Obviously, $Q(A)$ has more than one vertex. We have two cases:

a) $Q(A) = \{1 \to 2 \to \dots \to s-1 \to s\}$ is a chain;

b) $Q(A) = \{1 \to 2 \to \dots \to s-1 \to s \to 1\}$ is a cycle.

Suppose (a). By [6, p.287] $A \simeq T_s(\mathcal{D})/I$, where I is two-sided ideal of the ring $T_s(\mathcal{D})$ of all upper-triangular $s \times s$-matrices over $\mathcal{D}$.

Clearly, we can take A_0 be equal to the subring of all diagonal $s \times s$-matrices over $\mathcal{D}$.

In case (b) suppose first that $s = 2$. Then $1 = e_1 + e_2$. Put $A_i = e_iAe_i$, R_i is the Jacobson radical of $A_i (i = 1, 2), X = e_1Ae_2$ and $Y = e_2Ae_1$.

By formula 2 (see §1) we have $R = \begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix}$.

Clearly,

$$R^2 = \begin{pmatrix} R_1^2 + XY & R_1X + XR_2 \\ YR_1 + R_2Y & R_2^2 + YX \end{pmatrix}.$$

Since $Q(A)$ is two-pointed cycle by the Lemma on annihilation of simple modules we have that $XY = R_1$ and $YX = R_2$, which implies that $XYX = XR_2 = R_1X$ and $YXY = YR_1 = R_2Y$. Since $R^2 = 0$ it follows that $R_1 = 0$ and $R_2 = 0$. Hence, $A_1 = \mathcal{D}_1$ and $A_2 = \mathcal{D}_2$ are division rings and $A_0 = \begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix}, R = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$, i.e. $A = A_0 \oplus R$.

Let $Q(A) = \{1 \to 2 \to \ldots \to s-1 \to s \to 1\}$ be a cycle which contains at least three vertices. Let $1 = e_1 + \ldots + e_s$ be a decomposition of $1 \in A$ in a sum of mutually orthogonal idempotents.

Set $A_i = e_iAe_i$ where R_i is the Jacobson radical of $A_i (i = 1, \ldots, s)$. Let $A_{ij} = e_iAe_j (i \neq j; i, j = 1, \ldots, s)$. By the definition of $Q(A)$ we have that $A_{ii+1} \neq 0$ for $i = 1, \ldots, s-1$ and $A_{s1} \neq 0$.

We show that $R_i = 0$ for all i. Then by formula 2 of §1 we obtain that $A = A_0 \oplus R$ where

$$A_0 = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{pmatrix}.$$

In fact, let $R_k \neq 0$ for some $1 \leqslant k \leqslant s$. applying a cyclic renumbering of principal modules we may assume that $R_1 \neq 0$.

Consider the ring $B = (e_1 + e_2)A(e_1 + e_2)$. By [3] B is a serial ring. Clearly, B is reduced and $(R(B))^2 = 0$ where $R(B)$ is radical of B. Since $A_{12} \neq 0$, then if $A_{21} \neq 0$ we obtain that $Q(B)$ is a two-pointed cycle. But then it follows from the above arguments that $R_1 = 0$.

If $A_{21} = 0$ then $Q(B) = \{1 \to 2\}$ and $B \simeq T_2(\mathcal{D})$. Again $R_1 = 0$. Lemma is proved.

Let $\mathcal{O} = \mathcal{D}[[x, \sigma]]$ be an augmented Ore domain (see [18], Chapter VII, §14). The ring $\mathcal{D}[[x, \sigma]]$ is the set of formal power series $\sum_0^\infty \alpha_i x^i, \alpha_i \in \mathcal{D}$; σ is an automorphism of the division ring $\mathcal{D}$. Addition and equality is defined in the usual way. Multiplication is defined by the formula $\alpha x = x\alpha^\sigma$ and its consequences. Then $\mathcal{O}$ is a discrete valuation ring with unique maximal ideal $\mathcal{M} = x\mathcal{O} = \mathcal{O}x$. Denoted by $H_s^{(m)}(\mathcal{O})$ the quotient ring $H_s(\mathcal{O})/R^m$, where R is Jacobson radical of $H_s(\mathcal{O})$ (see Example from §1).

It follows from Lemma 4.7. and [10, §4] that every serial non-local reduced ring whose quiver is a cycle, the square of whose Jacobson radical is zero, is isomorphic to $H_s^{(2)}(\mathcal{O})$.

Set $T_s^{(2)}(\mathcal{D}) = T_s(\mathcal{D})/R^2$ where R is radical of $T_s(\mathcal{D})$. It follows from Lemma 4.7. and [6] that every serial reduced non-local ring A with $R(A)^2 = 0$ whose quiver is a chain is isomorphic to $T_s^{(2)}(\mathcal{D})$.

Since every serial A with one-point quiver and $R(A)^2 = 0$ is either a division ring or a Köthe ring of length 2, we obtain the following theorem.

Theorem 4.8. *Every indecomposable serial reduced ring A with $R(A)^2 = 0$ is isomorphic to one of the following:*

a) a division ring;

b) a Köthe ring of length 2;

c) $H_s^{(2)}(\mathcal{O})$;

d) $T_s^{(2)}(\mathcal{D})$.

In the cases (c) and (d) we have $s \geqslant 2$. Conversely, all these rings are indecomposable serial reduced rings, the square of the Jacobson radical of which is zero.

Remark. The rings of types (a), (b), (c) are Frobenius. In cases (a) and (b) the Nakayama permutation is identity and in case (c) it is a cycle $(1, 2, \ldots, s)$.

Remark. If the quiver $Q(A)$ of serial ring A is a chain then, there is a subring A_0, such that $A = A_0 \oplus R$ (a direct sum of abelian groups). If $Q(A)$ is a cycle with s vertices and $R^s = o$ then there is a subring A_0 in A such that $A = A_0 \oplus R$ (a direct sum of abelian groups). In the last case if A is reduced then A is isomorphic to a quotient ring of the QF-ring $H_s^{(s)}(\mathcal{O})$, and the Nakayama permutation $\nu(H_s^{(s)}(\mathcal{O}))$ is equal to $(1, s, s-1, \ldots, 2)$.

Proposition 4.9. *Let A be a serial ring, $P_1, \ldots, P_s$ all pairwise non-isomorphic principal A-modules. If $l(P_i) = l_i$ then soc $P_i = U_k$, where $k = i + l_i - 1 \,(mod\, s)$.*

Proof. The proof immediately follows from the definition of $Q(A)$.

This implies the following well-known fact (see [12] and also [8]).

Corollary 4.10. *A serial artinian indecomposable ring A is a QF-ring if and only if the lengthes of all principal A-modules are equal.*

Proof. If the lengthes of all principal A-modules are equal to 1 . Then since A is indecomposable, by the Wedderburn-Artin Theorem A is isomorphic to $M_n(\mathcal{D})$ where $\mathcal{D}$ is a division ring, consequently, A is a QF-ring.

If the length of all principal A-modules are equal to $l \geqslant 2$ then $Q(A)$ is a cycle. The map

$$\nu : i \to \nu(i) = l + i - 1 \,(mod\, s)$$

is a permutation $\{1, \ldots, s\}$.

By Definition 1.1. A is a QF-ring and there exists a principal A-module P which is simple. By Theorem 4.3. and Proposition 4.2. we obtain that $Q(A)$ is a one-point quiver without arrows. Therefore, $A = P^n$ and by Schur's Lemma $A \simeq M_n(E(P))$, where $E(P)$ is a division ring.

Thus, we can assume that if $l(P_i) \geqslant 2$ for all i. By Theorem 4.3. and Proposition 4.2. $Q(A)$ is a cycle.

Let $\varphi : P \to P_iR$ be an epimorphism of the principal A-module P on P_iR. If φ is an isomorphism then $soc\ P \simeq soc\ P_i$ which contradicts the Definition 1.1. Hence, $ker\,\varphi \neq 0$ and $l = l(P) \geqslant l_i = l(P_i)$. Let

$$Q(A) = \{1 \to 2 \to \ldots s-1 \to s \to 1\}.$$

Then $P = P_{i+1}$ for $1 \leqslant i \leqslant s-1$ and $P = P_1$ for $i = s$. Thus $l_1 \leqslant l_2 \leqslant \ldots \leqslant l_s \leqslant l_1$ as required.

Proposition 4.11. *Let A be an indecomposable serial artinian ring and $Q(A)$ is a cycle, J is a two-sided ideal with $J \subset R^2$. The quotient ring A/J is a QF-ring if and only if $J = R^l$ for some l.*

Proof. If $J = R^l$ then, obviously, the lengthes of all principal A/J-modules are equal and A/J is quasi-Frobenius.

Let A/J be a QF-ring. Then the lengthes of all principal modules are equal to $l \geqslant 2$. Thus, $[R(A/J)]^l = 0$ which implies that $J = R^l$.

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ФРОБЕНІУСОВІ КІЛЬЦЯ

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Доведено, що скінченновимірна алгебра A є слабо симетричною тоді і тільки тоді, коли кожна алгебра C , яка є Моріта еквівалентною фробеніусовій алгебрі A також фробеніусова. Наведено опис напівланцюгових кілець, квадрат радикала Джекобсона яких дорівнює нулю.

Ключові слова: квазі-фробеніусове кільце, фробеніусове кільце, напівланцюгове кільце.

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ON POLYNOMIAL FUNCTORS

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This is a survey of the last results of the author on classification of polynomial functors, especially quadratic and cubic.

Key words: polynomial functors, tame and wild algebras, string and band modules.

Polynomial functors appeared in algebraic topology [8] and proved themselves useful in various questions of this theory, especially in studying homotopy types. So their classification is of a definite interest. Some time ago the author noticed that at least the quadratic case can be treated in more or less usual framework of the representation theory. It gave possibility to obtain their complete description [6]. Unfortunately, this is the last case when such a description can be given. The cubic case is already *wild* in the sense of the representation theory [7]. Nevertheless, some special types of cubic functors can be classified. Perhaps, the most important seems the *2-divisible* case, which is completely analogous to the quadratic one [7]. As a consequence, a conjecture appears that the situation is the same for polynomial functors of degree p (prime) if we invert all smaller primes. This survey is mainly devoted to these results. Other special types of cubic functors that have been classified are "*cubic vector spaces*," weakly alternative and torsion free functors, but we only give a brief outlook of their description, since its proper place is still unclear. The author is grateful to Professor H.-J. Baues for his enthusiastic support of this research.

1. Generalities. We suppose all categories *pre-additive*, i.e. all morphism sets endowed with abelian group structure. On the other hand, the *functors* are not supposed additive, though we always suppose that they map zero objects to zero. If $F : \mathcal{A} \to \mathcal{B}$ is any functor, we can measure its non-additivity by its polarizations (or *cross-effects*). The latter are constructed as follows. Let $\mathcal{A}$ be additive (i.e. having finite direct sums) and $\mathcal{B}$ be *fully additive*, i.e. additive category such that every idempotent corresponds to a direct decomposition. For any objects $A_1, \dots, A_n$ from $\mathcal{A}$ consider their direct sum $A = \bigoplus_{k=1}^{n} A_i$ together with the embeddings $i_k : A_k \to A$ and projections $p_k : A \to A_k$.

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Then $e_k = i_k p_k$ are orthogonal idempotent endomorphisms of A, hence $f(k) = F(e_k)$ are orthogonal idempotent endomorphisms of $F(A)$. Define recursively endomorphisms $f(k_1 \dots k_m)$ for each $m \leqslant n,\ 1 \leqslant k_1 < \dots < k_m \leqslant n$ setting .

$$f(k_1 \dots k_m) = F(e_{k_1} + \dots + e_{k_m}) - \sum_{l<m} \sum_{j_1 < \dots < j_l} f(j_1 \dots j_l),$$

for instance $f(kl) = F(e_k + e_l) - F(e_k) - F(e_l)$. Set $F_n(A_1|\dots|A_n) = \mathrm{Im} f(12\dots n)$. Then

$$F(A) = \bigoplus_{m \leqslant n} \bigoplus_{k_1 < \dots < k_m} F_m(A_{k_1}|\dots|A_{k_m}).$$

The functor F is called *polynomial* if there is an integer d such that $F_n = 0$ for $n > d$. The smallest d with this property is called the *degree* of F. Certainly functors of degree 1 are just additive; those of degree 2 are called *quadratic* and of degree 3 *cubic*.

In what follows we consider the case when $\mathcal{A} = \mathbf{fab}$, the category of finitely generated free abelian groups, and $\mathcal{B} = \mathbf{R}\text{-}Mod$, the category of modules over a ring $\mathbf{R}$. As any additive functor $F : \mathbf{fab} \to \mathbf{R}\text{-}Mod$ can be identified with the $\mathbf{R}$-module $F(\mathbb{Z})$, we call polynomial functors $F : \mathbf{fab} \to \mathbf{R}\text{-}Mod$ *polynomial* $\mathbf{R}$*-modules.* Moreover, as a rule we only deal with *finitely generated* polynomial modules, i.e. polynomial functors $F : \mathbf{fab} \to \mathbf{R}\text{-}mod$, the category of finitely generated $\mathbf{R}$-modules. If $\mathbf{R} = \mathbb{Z}$, we simply say "polynomial modules" not precising the ring.

One can show (see [1]) that a polynomial module M of degree d is completely defined by the values $M_n = M_n(\mathbb{Z}|\dots|\mathbb{Z})$ (n times) for $n \leqslant d$ and the homomorphisms $H^n_m : M_n \to M_{n+1}$, $P^n_m : M_{n+1} \to M_n$ for each $n < d,\ m \leqslant n$, which are defined as the following compositions:

$$H^n_m : M_n \to M(\mathbb{Z}^n) \to M(\mathbb{Z}^{n+1}) \to M_{n+1},$$

where the first mapping is just the embedding of the direct summand, the last one is the projection onto the direct summand, and the middle one equals $M(\delta_m)$, where

$$\delta_m : \mathbb{Z}^n \to \mathbb{Z}^{n+1}, \quad \delta_m(z_1, \dots, z_n) = (z_1, \dots, z_{m-1}, z_m, z_m, z_{m+1}, \dots, z_n);$$

and

$$P^n_m : M_{n+1} \to M(\mathbb{Z}^{n+1}) \to M(\mathbb{Z}^n) \to M_n,$$

where the first mapping is the projection, the last one is the embedding, and the middle one equals $M(\gamma_m)$, where

$$\gamma_m : \mathbb{Z}^{n+1} \to \mathbb{Z}^n, \quad \gamma_m(z_1, \dots, z_{n+1}) = (z_1, \dots, z_{m-1}, z_m + z_{m+1}, z_{m+2} \dots, z_n).$$

Certainly, these mappings must satisfy some relations (cf. [1]), which we shall not write in general case.

Important examples of polynomial modules are:

- tensor powers $T^n : A \mapsto A^{\otimes n}$,
- symmetric powers $\mathrm{S}^n : A \mapsto \mathrm{S}^n A$,
- exterior (skew-symmetric) powers $\Lambda^n : A \mapsto \Lambda^n A$.

In particular, tensor power T^n and its polarizations $T^{n,k} : A \to T^n_k(A|\dots|A)$ (k times) are just indecomposable projectives in the category of all polynomial modules of degree n.

2. Quadratic modules. For quadratic modules the previous construction gives two $\mathbf{R}$-modules M_1, M_2, and two mappings $H : M_1 \to M_2$ and $P : M_2 \to M_1$, such that $PHP = 2P$ and $HPH = 2H$.

We consider the "absolute" case, when $\mathbf{R} = \mathbb{Z}$ (it is the most important for topology). Then one can easily see that a quadratic module can be considered as a module over a special ring $\mathbf{A}$, which is the subring in the direct product $\mathbb{Z} \times \mathrm{Mat}(2, \mathbb{Z}) \times \mathbb{Z}$ consisting of the triples

$$(a, b, c), \quad \text{where} \quad b = \begin{pmatrix} b_1 & 2b_2 \\ b_3 & b_4 \end{pmatrix}, \quad b_1 \equiv a, \ b_4 \equiv c \ (\mathrm{mod}\ 2).$$

Namely, if M is an $\mathbf{A}$-module, in the corresponding quadratic module $M_1 = e_1 M$, $M_2 = e_2 M$, H is the multiplication by h and P is the multiplication by p, where

$$e_1 = (1, e_{11}, 0), \ e_2 = (0, e_{22}, 1), \ h = (0, e_{21}, 0), \ p = (0, 2e_{12}, 0)$$

(e_{ij} are the matrix units in $\mathrm{Mat}(2, \mathbb{Z})$). Fortunately, this ring belongs to the class considered by the author in [5]. In particular, it is *tame*; moreover, its representations can be described in a rather usual language of "strings" and "bands." Indeed, this classification is a special case of the so-called *representations of bunches of chains* (cf. [2]). For details of the calculations we refer to [6]; here we only formulate the result in a bit more convenient form.

First, using the common tool of adèles groups, like in [4], we establish a sort of "Hasse principle" for quadratic modules. Remind that we always suppose our modules finitely generated.

2.1. Proposition. *Two quadratic modules M and N are isomorphic if and only if there localizations M_p and N_p are isomorphic for each prime number p.*

If $p > 2$, $\mathbf{A}_p = \mathbb{Z}_p \times \mathrm{Mat}(2, \mathbb{Z}_p) \times \mathbb{Z}_p$, so the description of $\mathbf{A}_p$-modules is quite simple: there are three indecomposable torsion free modules (direct summands of $\mathbf{A}_p$), and every other indecomposable module is isomorphic to $P/p^k P$ for some positive integer k and one of these modules P. The description in case $p = 2$ is more interesting. First introduce some configurations of integers called *strings* and *bands*. Namely, define two symmetric relations on the set $\{1, 2, 3, 4\}$: an equivalence relation $-$ such that the only non-trivial equivalence is $2 - 3$, and $\sim$ (not an equivalence!) such that $1 \sim 2$ and $3 \sim 4$. Now a *string* is a configurations of one of the following sorts:

$$\text{(i)} \quad \begin{array}{ccccccccccc} j_1 & & & & j_2 j_3 & & & & j_{2n-2} j_{2n-1} & & \\ & k_1 & & k_2 & & k_3 & & \cdots & & k_{2n-1} & \\ & & i_1 i_2 & & & & i_3 i_4 & & & & i_{2n-1} \end{array}$$

or

$$\text{(ii)} \quad \begin{array}{cccccccccc} & & j_2 j_3 & & & & j_{2n-2} j_{2n-1} & & \\ & k_2 & & k_3 & & \cdots & & k_{2n-1} & \\ i_2 & & & & i_3 i_4 & & & & i_{2n-1} \end{array}$$

or

$$\text{(iii)}\qquad \begin{matrix} j_1 & & & & j_2 j_3 & & & & j_{2n} \\ & k_1 & & k_2 & & \cdots & & k_{2n} & \\ & & i_1 i_2 & & & & i_{2n-1} i_{2n} & & \end{matrix} \quad .$$

where $i_r, j_r \in \{1,2,3,4\}$, $k_r \in \mathbb{N}$ satisfy the following conditions:

- $i_{2r-1} \sim i_{2r}$ for each $r = 1,2,\dots,n$. This condition is empty for types (i) and (ii) if $r = n$ and for type (ii) if $r = 1$, but in these cases we *define* i_{2n}, respectively i_1 so that it holds.
- $j_{2r+1} \sim j_{2r}$ for each $r = 1,2,\dots,n-1$.
- $i_r - j_r$ for each $r = 1,2,\dots,2n$ (again it is empty in some cases, but here we do not define any extra values).

Consider now the following mappings acting in every quadratic module:

$$\theta(11) = 2\mathrm{id}_{M_1} - PH, \quad \theta(22) = PH, \quad \theta(23) = H,$$

$$\theta(32) = P, \quad \theta(33) = HP, \quad \theta(44) = 2\mathrm{id}_{M_2} - HP.$$

Set also $\nu\{1,2\} = 1$, $\nu\{3,4\} = 2$. Then the quadratic *string module* $M = M^{\mathrm{D}}$ corresponding to a string diagram D is generated by the elements

$$\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n \qquad \mathbf{g}_r \in M_{\nu\{i_{2r-1}, i_{2r}\}}$$

subject to the relations

$$2^{k_{2r}}\theta(i_{2r}j_{2r})\mathbf{g}_r = 2^{k_{2r+1}}\theta(i_{2r+1}j_{2r+1})\mathbf{g}_{r+1} \quad (r = 0,1,\dots,n).$$

We set here $\mathbf{g}_0 = \mathbf{g}_{2n+1} = 0$ and omit the case $r = n$ for diagrams of types (i),(ii) and the case $r = 0$ for diagrams of type (ii).

A *band data* is a pair (D, m, ϕ), where D is a diagram of type (iii) and $\phi = \lambda_1 + \lambda_2 t + \dots + \lambda_m t^{m-1} + t^m$ is a polynomial over the residue field $\mathbb{Z}/2$ such that

- $j_{2n} \sim j_1$.
- D is non-periodic, i.e. cannot be written as a repetition $\mathrm{D}'\mathrm{D}'\dots\mathrm{D}'$ of a shorter diagram D'.
- ϕ is a power of an irreducible polynomial and $\lambda_1 \neq 0$.

The quadratic *band module* $M = M^{\mathrm{D},\phi}$ corresponding to a band data is generated by the elements

$$\mathbf{g}_{rs} \quad (r = 1,2,\dots,n,\ s = 1,2,\dots,m) \quad \mathbf{g}_{rs} \in M_{\nu\{i_{2r-1}i_{2r}\}}$$

subject to the relations

$$2^{k_{2r}}\theta(i_{2r}j_{2r})\mathbf{g}_{rs} = 2^{k_{2r+1}}\theta(i_{2r+1}j_{2r+1})\mathbf{g}_{r+1,s} \quad (r = 0,1,\dots,n) \ \text{ if } 1 \leqslant r < n;$$

$$2^{k_{2n}}\theta(i_{2n}j_{2n})\mathbf{g}_{ns} = 2^{k_1}\theta(i_1j_1)\mathbf{g}_{1,s+1} \ \text{ if } 1 \leqslant s < m;$$

$$2^{k_{2r}}\theta(i_{2n}j_{2n})\mathbf{g}_{nm} = -2^{k_1}\theta(i_1j_1)\sum_{s=1}^{m}\lambda_s\mathbf{g}_{1s}.$$

2.2. Theorem. *1) Every indecomposable quadratic module is isomorphic to one of the string or band modules defined above, or to a module* S^2/p^k, Λ^2/p^k, *or* Id/p^k, *where* p *is an odd prime.*

2) The only isomorphisms between these indecomposable modules are the following:

- $M^{D} \simeq M^{D^*}$, *where* D *is the symmetric diagram to a diagram* D *of type (ii) or (iii).*
- $M^{D,\phi} \simeq M^{D^l,\phi}$, *where* D^l *denotes the* l*-th cyclic shift of the diagram of type (iii), i.e. the configuration*

$$\begin{matrix} j_{2l+1} & & & & j_{2l+2}j_{2l+3} & & & & j_{2l} \\ & k_{2l+1} & & k_{2l+2} & & \cdots & & k_{2l} & \\ & & i_{2l+1}i_{2l+2} & & & & i_{2l-1}i_{2l} & & \end{matrix}$$

- $M^{D,\phi} \simeq M^{D^{*l},\phi^*}$, *where* $\phi^*(t) = \lambda_1^{-1}t^m\phi(1/t)$.

3) Any quadratic module uniquely decomposes into a direct sum of indecomposable ones.

2.3 Corollary. • *Every quadratic module* M *has a periodic projective resolution of period 4, namely*

$$\cdots \to P_n \xrightarrow{\alpha_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\alpha_1} P_0 \to M \to 0$$

with $P_{n+4} = P_n$, $\alpha_{n+4} = \alpha_n$ *for* $n > 2$.

• *The projective dimension of a quadratic module is either 0, or 1, or* ∞. *Hence the finitistic projective dimension of the category of quadratic modules equals 1.*

3. Cubic modules. A cubic module is given by 3 groups M_1, M_2, M_3 and 6 mappings

$$H: M_1 \to M_2,\ P: M_2 \to M_1,\ H_m: M_2 \to M_3,\ P_m: M_3 \to M_2 \quad (m = 1, 2)$$

subject to the conditions:

$$H_1P_2 = H_2P_1 = 0,\ H_1H = H_2H,\ PP_1 = PP_2,$$

$$H_iP_iH_i = 2H_i,\ P_iH_iP_i = 2P_i\ (i = 1, 2),$$

$$HPH = 2(H + (P_1 + P_2)\bar{P}),\ PHP = 2(P + \bar{P}(H_1 + H_2)),$$

$$\bar{H}P + H_1 + H_2 = H_1P_1H_2P_2H_1 + H_2P_2H_1P_1H_2,$$

$$H\bar{P} + P_1 + P_2 = P_1H_2P_2H_2P_1 + P_2H_1P_1H_2P_2,$$

where $\bar{H} = H_1H = H_2H$, $\bar{P} = PP_1 = PP_2$.

We consider the ring $\mathbf{B}$ generated by three orthogonal idempotents e_1, e_2, e_3 such that $e_1 + e_2 + e_3 = 1$ and 6 elements

$$H \in e_2\mathbf{B}e_1,\ P \in e_1\mathbf{B}e_2,\ H_m \in e_3\mathbf{B}e_2,\ P_m \in e_2\mathbf{B}e_3 \quad (m = 1, 2)$$

subject to the above relations. Then any cubic module can be considered as $\mathbf{B}$-module. Set $\mathbf{B}_1 = e_1\mathbf{B}e_1$.

3.1. Proposition. *1. The ring* $\mathbf{B}_1$ *is generated by two elements* $a = PH - \bar{P}\bar{H}$, $b = PH$ *subject to the relations* $a^2 = 2a$, $b^2 = 6b$, $ab = ba = 0$.

2. The ring $\mathbf{B}_1$ *(all the more* $\mathbf{B}$*) is wild.*

Proof. The first claim is verified by straightforward calculations [7]. To prove the second, consider the free (non-commutative) algebra $\Sigma = \mathbb{Z}/4\langle x, y\rangle$ over the residue ring $\mathbb{Z}/4$ and the homomorphism $\sigma : \mathbf{B}_1 \to \Sigma$ mapping $a \mapsto 2x$, $b \mapsto 2y$. For every Σ-module L denote by ${}^{\sigma}L$ the $\mathbf{B}_1$-module obtained from L by the change of rings. Then one easily verifies that for any Σ-modules L, L', which are free as $\mathbb{Z}/4$-modules,

- ${}^{\sigma}L \simeq {}^{\sigma}L'$ if and only if $L/2 \simeq L'/2$;
- ${}^{\sigma}L$ is indecomposable if and only if $L/2$ is indecomposable.

Hence the classification of $\mathbf{B}_1$-modules is at least as complicated as that of modules over $\Sigma/2 \simeq \mathbb{Z}/2\langle x, y\rangle$. It means that $\mathbf{B}_1$ is wild in the sense of the representation theory.

It gives no hope to obtain a good classification of cubic modules. Nevertheless, the situation becomes much better if we "invert 2," that is consider cubic modules over the ring $\mathbb{Z}' = \mathbb{Z}[1/2]$. We call them *2-divisible cubic modules.* Then straightforward, though rather cumbersome, calculations give the following result.

3.2. Proposition. *The ring* $\mathbf{B}[1/2]$ *is Morita equivalent to the direct product* $\mathbb{Z}' \times \mathbb{Z}' \times \mathbf{B}'$, *where* $\mathbf{B}'$ *is the subring of* $\mathbb{Z}' \times \mathrm{Mat}(2, \mathbb{Z}') \times \mathrm{Mat}(2, \mathbb{Z}') \times \mathbb{Z}'$ *consisting of quadruples*

$$(a, b, c, d), \quad where\ b = \begin{pmatrix} b_1 & 3b_2 \\ b_3 & b_4 \end{pmatrix}, \ c = \begin{pmatrix} c_1 & 3c_2 \\ c_3 & c_4 \end{pmatrix},$$

such that $a \equiv b_1$, $b_4 \equiv c_1$, $c_4 \equiv d \pmod 3$.

The cubic modules corresponding to the first two factor $\mathbb{Z}'$ are just S^2/p^k and Λ^2/p^k for odd primes p (they are indeed quadratic modules). The description of $\mathbf{B}'$-modules can be given in the same frames as that of quadratic modules. The corresponding string and bands only differs from those of the preceding section by the features that now the indices i_r, j_r are taken from the set $\{1,2,3,4,5,6\}$ with the relations $2-3$, $4-5$, $1 \sim 2$, $3 \sim 4$, $5 \sim 6$, polynomials ϕ are taken from $\mathbb{Z}/3[t]$, and the mappings $\theta(ij)$ are defined as follows:

$$\theta(11) = 3\mathrm{Id}_{M_1} - \beta_1\alpha_1,\ \theta(22) = \beta_1\alpha_1,\ \theta(23) = \alpha_1,\ \theta(32) = \beta_1,\ \theta(33) = \alpha_1\beta_1,$$

$$\theta(44) = \beta_2\alpha_2,\ \theta(45) = \alpha_2,\ th(54) = \beta_2,\ \theta(55) = \alpha_2\beta_2,\ \theta(66) = 3\mathrm{Id}_{M_3} - \alpha_2\beta_2,$$

where $\alpha_1 : M_1 \to M_2$ corresponds to the quadruple $(0, e_{21}, 0, 0)$, $\beta_1 : M_2 \to M_1$ to the quadruple $(0, 3e_{12}, 0, 0)$, $\alpha_2 : M_2 \to M_3$ to the quadruple $(0, 0, e_{21}, 0)$, and $\beta_2 : M_3 \to M_2$ to the quadruple $(0, 0, 3e_{12}, 0)$.

So we get the following results.

3.3. Theorem. *1) Two cubic 2-divisible modules* M, N *are isomorphic if and only if* $M_p \simeq N_p$ *for each odd prime* p.

2) Every indecomposable 2-divisible cubic module is isomorphic to one of the following:

- *string or band module;*

• S^3/p^k, S^{3*}/p^k, Λ^3/p^k, Id/p^k, *where* $S^{3*}(A) = S^2_3(A|A)$ *and* $p > 3$ *is a prime;*

• S^2/p^k *or* Λ^2/p^k, *where* p *is an odd prime.*

3) The only isomorphisms between these indecomposable cubic modules are:

• $M^{\mathrm{D}} \simeq M^{\mathrm{D}^*}$, *where* D *is the symmetric diagram to a diagram* D *of type (ii) or (iii).*

• $M^{\mathrm{D},\phi} \simeq M^{\mathrm{D}^l,\phi}$, *where* D^l *denotes the l-th shift of the diagram of type (iii), i.e. the configuration*

$$\begin{array}{ccccccccc} j_{2l+1} & & & & j_{2l+2}\,j_{2l+3} & & & & j_{2l} \\ & k_{2l+1} & & k_{2l+2} & & \cdots & & k_{2l} & \\ & & i_{2l+1}\,i_{2l+2} & & & & i_{2l-1}\,i_{2l} & & \end{array}$$

• $M^{\mathrm{D},\phi} \simeq M^{\mathrm{D}^{*l},\phi^*}$, *where* $\phi^*(t) = \lambda_1^{-1} t^m \phi(1/t)$.

4) Any 2-divisible cubic module uniquely decomposes into a direct sum of indecomposable ones.

3.4. Corollary. • *Every 2-divisible cubic module M has a periodic projective resolution of period 6, namely*

$$\cdots \to P_n \xrightarrow{\alpha_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\alpha_1} P_0 \to M \to 0$$

with $P_{n+6} = P_n$, $\alpha_{n+6} = \alpha_n$ *for* $n > 2$.

• *The projective dimension of a 2-divisible cubic module is either 0, or 1, or* ∞. *Hence the finitistic projective dimension of the category of 2-divisible cubic modules equals 1.*

3.5. Conjecture. *Let* p *be a prime,* $\mathbb{Z}^{(p)} = \mathbb{Z}[1/(p-1)!]$. *Then the category of polynomial* $\mathbb{Z}^{(p)}$*-modules of degree* p *is equivalent to the category* $\mathbf{A}^{(p)}$*-modules, where* $\mathbf{A}^{(p)}$ *is a direct product of several copies of* $\mathbb{Z}^{(p)}$ *and of the subring of* $\mathbb{Z}^{(p)} \times \mathrm{Mat}(2,\mathbb{Z}^{(p)})^{p-1} \times \mathbb{Z}^{(p)}$ *consisting of* $(p+1)$*-tuples* $(a, b^1, \ldots, b^{p-1}, c)$, *where*

$$b^m = \begin{pmatrix} b^m_{11} & pb^m_{12} \\ b^m_{21} & b^m_{22} \end{pmatrix} \quad \text{with } b^m_{22} \equiv b^{m+1}_{11} \pmod p \ \text{ for } m = 1, \ldots, p-2,$$

$a \equiv b^1_{11}$, $c \equiv b^{p-1}_{22} \pmod p$.

If this conjecture is true, the description of $\mathbb{Z}^{(p)}$-modules of degree p (we call them ($<p$)-*divisible p-modules*) becomes quite analogous to that of quadratic or 2-divisible cubic modules. Namely:

• Two ($<p$)-divisible p-modules M, N are isomorphic if and only if $M_q \simeq N_q$ for all prime $q \geqslant p$.

• Indecomposable ($<p$)-divisible p-modules, except some "trivial" ones, are string and band modules defined as above. Now i_r, j_r are taken from the set $\{1, 2, \ldots, 2p\}$ with corresponding changes of $-$, $\sim$ and $\theta(ij)$. The isomorphisms between these modules are the same as in Theorems 2.2 and 3.3.

• Every ($<p$)-divisible p-module uniquely decomposes into a direct sum of indecomposable ones.

• Every ($<p$)-divisible p-module has a periodic projective resolution of period $2p$ starting from α_2. Therefore a projective dimension of such a module is $0, 1$ or

∞. In particular, the finitistic projective dimension of the category of ($<p$)-divisible p-modules equals 1.

4. Other classes of cubic modules. We shortly outline three other classes of cubic modules that allow an acceptable description referring for details to [7].

A. Cubic vector spaces. They are functors $\mathbf{fab} \to \mathrm{vect}_{\mathbf{k}}$, the category of vector spaces over a field $\mathbf{k}$. The interesting case is $\mathrm{char}\,\mathbf{k} = 2$, because otherwise such functors are special cases of 2-divisible ones. Rewriting the relations for the mappings H, P, H_i, P_i for this special case gives the following result.

4.1. Proposition. *The category of cubic vector spaces is equivalent to the direct product of a trivial* $\mathbf{k}$*-linear category with one object (it corresponds to the functor* $\mathrm{Id}\otimes\Lambda^2$ *) and the category of modules over the* $\mathbf{k}$*-algebra* $\mathbf{A}$ *generated by three orthogonal idempotents* e_1, e_2, e_3 *such that* $e_1 + e_2 + e_3 = 1$ *and four elements*

$$h \in e_2\mathbf{A}e_1,\ p \in e_1\mathbf{A}e_2,\ h_1 \in e_3\mathbf{A}e_2,\ p_1 \in e_3\mathbf{A}e_2$$

subject to the relations

$$hph = php = h_1p_1h_1 = p_1h_1p_1 = 0, \quad h_1p_1 = h_1hpp_1.$$

We consider $\mathbf{A}$-modules as diagrams of vector spaces

$$M_1 \rightleftarrows M_2 \rightleftarrows M_3,$$

where $M_i = e_iM$ and the arrows correspond to the action of h, p, h_1, p_1. As $hph = php = 0$, the fragment $M_1 \rightleftarrows M_2$ decomposes into blocks of dimension at most 3 (the dimensions of M_1, M_2 at most 2, and only one of them can be 2-dimensional).

Hence the mappings h and p can be chosen in the form

$$h = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad p = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(I denotes the identity matrix). Now, if we reduce the matrix of h_1 to the simplest possible form, the matrix of p_1 splits into 8 horizontal and 10 vertical stripes, which we denote respectively by R_i ($i = 1, \dots, 8$) and S_j ($j = 1, \dots, 10$). Moreover, one can check that the admissible transformations of these stripes can be described as representations of a bunch of semi-chains in the sense of [2], namely, we have two semi-chains

$$\mathcal{E} = \{R_1 > R_2 > R_3 > R_4 > R_6 > R_7 > R_8,\ R_3 > R_5 > R_6\},$$

$$\mathcal{F} = \{S_1 < S_2 < S_3 < S_4 < S_5 < S_7 < S_8 < S_9 < S_{10},\ S_4 < S_6 < S_7\}$$

with the involution σ such that $\sigma(x) = x$ except for the cases

$$\sigma(R_1) = R_8,\ \sigma(R_2) = S_8,\ \sigma(R_6) = S_4,\ \sigma(S_2) = S_9.$$

Hence, the description of cubic vector spaces fits again the frames of strings and bands, though this time they are more complicated than before. We shall not precise their shape (rather complicated) here, referring to [7].

B. Weakly alternative cubic modules. We call a cubic module M *weakly alternative* if $M(\mathbb{Z}) = 0$. Examples of such modules are Λ^3 and $\Lambda^2 \otimes \mathrm{Id}$. For the corresponding diagram it means that $M_1 = 0$. Then, reducing the relations with respect to the conditions $h = p = 0$, one obtains the following result.

4.2. Proposition. *The category of weakly alternative cubic modules is equivalent to the category of* $\mathbf{C}$*-modules, where* $\mathbf{C}$ *is a semi-direct product* $\mathbf{C} = (\mathbb{Z}\times\mathbf{C}_0)\ltimes D$, *where* $\mathbf{C}_0$ *is the subring of* $\mathbb{Z}\times\mathrm{Mat}(2,\mathbb{Z})$ *consisting of pairs* (a, b) *such that* $a \equiv b_{11}$, $b_{12} \equiv 0 \pmod 2$, D *is an elementary abelian 2-group with three generators* ξ, η, θ, *with the multiplication* $\xi\eta = \theta$, $\eta\xi = 0$ *and the* $\mathbf{C}$*-action:*

$$e\xi = \xi,\ \eta e = \eta,\ \xi(0, a, b) = a\xi,\ (0, a, b)\eta = a\eta, \quad \text{where } e = (1, 0, 0),\ (a, b) \in \mathbf{C}_0.$$

The same observations as for quadratic and 2-divisible cubic modules imply

4.3. Proposition. *Two weakly alternative cubic modules* M, N *are isomorphic if and only if* $M_p \simeq N_p$ *for all* p.

The only non-trivial cases are, of course, $p = 2$ and $p = 3$. In the former case $(\mathbf{C}_0)_2 \simeq \mathbb{Z}_2 \times \mathrm{Mat}(2, \mathbb{Z}_2)$ and the second factor acts trivially on $D = D_2$. So the problem reduces to the classification of diagrams of $\mathbb{Z}_2$-modules

$$W_1 \underset{\eta}{\overset{\xi}{\rightleftarrows}} W_2$$

such that $2\xi = 2\eta = \eta\xi = 0$. Splitting each of W_i into direct sum of free modules $C_\infty = \mathbb{Z}_2$ and finite cyclic groups $C_k = \mathbb{Z}/2^k$, one can reduce ξ and η to a normal form. Namely, consider (finite) words ω of the shape

$$\ldots \xi^{i_r}\eta_{j_r}\xi^{i_{r+1}}\eta_{j_{r+1}}\ldots \qquad (i_r, j_r \in \mathbb{Z}\cup\{\infty\}$$

not containing subwords ${}_\infty\xi$, ${}^\infty\eta$, $\eta_1\xi$. Such a diagram gives rise to a weakly alternative module $W = W(\omega)$. Namely,

$$W_1 = \bigoplus_r C_{i_r},\ W_2 = \bigoplus_r C_{j_r},\ \xi(C_{i_r}) \subset C_{j_{r-1}},\ \eta(C_{j_r}) \subset C_{i_r},$$

and the induced mappings are non-zero of period 2. Note that such mappings are unique; we denote them by γ (not precising indices). The modules $W(\omega)$ are called *string* $\mathbf{C}_2$*-modules.* A *band* $\mathbf{C}_2$*-module* depends on a pair (ω, ϕ), where

$$\omega = {}_j\xi^{i_1}\eta_{j_1}\xi^{i_2}\eta_{j_2}\ldots{}^{i_n}\eta_j$$

and $\phi \neq t^m$ is a power of an irreducible polynomial over $\mathbb{Z}/2$. The corresponding band module $W = W(\omega, \phi)$ is defined as follows:

$$W_1 = \bigoplus_r mC_{i_r}, \quad W_2 = mC_j \oplus (\bigoplus_r mC_{j_r}),$$

$$\xi(mC_{i_r}) \subset mC_{j_{r-1}},\ \eta(mC_{j_r}) \subset mC_{i_r},\ \xi(mC_{i_1}) \subset mC_j,\ \eta(nC_j) \subset mC_{i_n},$$

where $m = \deg \phi$, and the induced mappings coincide with $\gamma \mathrm{Id}$, except for $mC_j \to mC_{i_n}$ that is given by the matrix $\gamma\Phi$, where Φ is the Frobenius cell with the characteristic polynomial ϕ. In the case $p = 3$, $D_3 = 0$ and we are in the situation analogous to that of quadratic or 2-divisbile cubic modules. This time the values i_r, j_r are taken from the set $\{1, 2, 3, 4\}$ with $3 - 4$ and $2 \sim 3$. Gluing $\mathbf{C}_2$- and $\mathbf{C}_3$-modules gives the following

4.4. Theorem. *Indecomposable weakly alternative cubic modules correspond to the* $\mathbf{C}$*-modules of the following types: (1) Torsion modules: (a) 2-torsion:* $W(\omega)$ *and* $W(\omega, \phi)$ *such that* ξ^∞ *does not occur in* ω; *(b) 3-torsion: all band* $\mathbf{C}_3$*-modules and string modules of type* (iii); *(c) p-torsion for* $p > 3$, *which are* P/p^kP, *where* P *is an irreducible torsion free* $\mathbf{C}_p$*-module.*

(2) Torsion free modules, which are just irreducible modules and the projective module $\mathbf{C}(0, 1, e_{11})$.

(3) "Mixed" modules M, *which are also of three possible shapes given by their localization at* $p = 2$ *and* $p = 3$: *(a)* $M_2 = W(\omega)$, *where* ω *contains* ξ^∞, $M_3 = M^{\mathrm{D}}$, *where* D *is a string of type* (i) *or* (ii) *with* $i_{2n-1} = 2$ *or* $i_2 = 2$; *if both occur, it gives two non-isomorphic modules; (b)* $M_2 = W(\omega) \oplus W(\omega')$, *where both* ω *and* ω' *contains* ξ^∞, $M_3 = M^{\mathrm{D}}$, *where* D *is of type* (ii) *with* $j_{2n-1} = j_2 = 2$; *(c)* $M_3 = M^{\mathrm{D}}$, *where* D *is of type* (i) *or* (ii), M_2 *is torsion free (hence uniquely determined).*

C. Torsion free cubic modules. They are such modules that all groups M_i $(i = 1, 2, 3)$ are torsion free. As usually, we study them locally. The only non-trivial case is $p = 2$. Then the calculations of subsection 4A imply that the corresponding (localized) ring is isomorphic to the subring in $\mathbb{Z}_2^3 \times \mathrm{Mat}(2, \mathbb{Z}_2) \times \mathrm{Mat}(4, \mathbb{Z}_2)^2$ consisting of all sextuples satisfying the following congruences modulo 2:

$$(a_1, a_2, a_3, b, c, d) \quad \text{with} \quad a_1 \equiv b_{11} \equiv c_{11}\,,\ a_2 \equiv b_{22} \equiv c_{22} \equiv c_{33}\,,$$
$$a_3 \equiv c_{44}\,,\ b_{12} \equiv 0 \ \text{ and } \ c_{ij} \equiv 0 \ \text{ if } \ i < j\,.$$

It is a *Backström order*, i.e. its radical coincides with the radical of a hereditary order. Therefore we can apply the method of [9] that reduces the description of torsion free modules to some diagrams of vector spaces. The precise shape of our ring implies that in this case the corresponding diagram is a disjoint union of 4 diagrams of types A_2, A_3, D_4 and $\tilde{D}_4$. Hence the classification of such modules is again a tame (and rather easy) problem (cf. [3]). Moreover, the specific form of this order implies the following important corollary for *all* cubic modules, extending the claim (1) of Theorem 3.3.

4.5. Corollary. *Two cubic modules are isomorphic if and only if all their localizations are isomorphic.*

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ПРО ПОЛІНОМІАЛЬНІ ФУНКТОРИ

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HYPERSPACE FUNCTOR IN THE COARSE CATEGORY

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We consider the hyperspace monad in the category of topological coarse spaces and equivalence classes of coarse maps. It is proved that the G-symmetric power functor acting on the category of topological spaces can be naturally defined also for the category of topological coarse spaces and, on this category, it can be extended to the Kleisli category of the hyperspace monad.

Key words: coarse space, coarse map.

1. The coarse category was first introduced by Higson, Pedersen, and Roe [1]. Methods of coarse topology (geometry) found numerous applications in different areas of topology and analysis (see, e. g. [1–5]). The present paper is devoted to the hyperspace functor and hyperspace monad in the coarse category.

The paper is organized as follows. Section 2 contains necessary definitions. In Section 3 the hyperspace functor acting in the coarse category is defined and we prove in Section 4 that the hyperspace functor determines a monad in the coarse category. In Section 5 we consider the problem of extension of functors onto the Kleisli category of the hyperspace monad.

2.1. PRELIMINARIES. Coarse structures. Let X be a set and $M, M \subset X \times X$. The *composition* of M and N is the set $MN = \{(x,y) \in X \times X \mid \text{ there exists } z \in X \text{ such that } (x,z) \in M, (z,y) \in N\}$, the *inverse* of M is the set $M^{-1} = \{(x,y) \in X \times X \mid (y,x) \in M\}$.

A *coarse structure* on a set X is a family $\mathcal{E}$ of subsets, which are called the *entourages*, in the product $X \times X$ that satisfies the following properties:

1) any finite union of entourages is contained in an entourage;
2) for every entourage M, its inverse M^{-1} is contained in an entourage;
3) for every entourages M, N, their composition MN is contained in an entourage;
4) $\cup\mathcal{E} = X \times X$.

A coarse structure on X is called *unital* if the diagonal Δ_X is contained in an entourage. A coarse structure on X is called *anti-discrete* if $X \times X$ is an entourage.

If $\mathcal{E}_1$, $\mathcal{E}_2$ are coarse structures on X, then $\mathcal{E}_1 \leqslant \mathcal{E}_2$ means that for every $M \in \mathcal{E}_1$ there is $N \in \mathcal{E}_2$ such that $M \subset N$.

Two coarse structures, $\mathcal{E}_1$ and $\mathcal{E}_2$, are said to be *equivalent* if $\mathcal{E}_1 \leqslant \mathcal{E}_2$ and $\mathcal{E}_2 \leqslant \mathcal{E}_1$. We usually identify coarse spaces with equivalent coarse structures.

If $\mathcal{E}$ is a coarse structure on a set X, then, obviously, the coarse structure $\mathcal{E}_1 = \{M \cup M^{-1} \mid M \in \mathcal{E}\}$ is equivalent to $\mathcal{E}$ and is *symmetric* in the sense that $N^{-1} \in \mathcal{E}_1$ for every $N \in \mathcal{E}_1$.

Given $M \in \mathcal{E}$ and $A \subset X$, we define the *M-neighborhood* $M(A)$ of A as follows: $M(A) = \{x \in X \mid (a, x) \in M \text{ for some } a \in A\}$. We use the notation $M(a)$ instead of $M(\{a\})$. A set $A \subset X$ is *bounded* if there exists $x \in X$ such that $A \subset M(x)$.

Let $(X_i, \mathcal{E}_i)$, $i = 1, 2$, be coarse spaces. A map $f: X_1 \to X_2$ is called *coarse* if the following two conditions hold:

1) for every $M \in \mathcal{E}_1$ there exists $N \in \mathcal{E}_2$ such that $(f \times f)(M) \subset N$;
2) for any bounded subset A of X_2 the set $f^{-1}(A)$ is bounded.

It is easy to see that the coarse spaces and coarse maps form a category. We denote it by CS.

Definition 2.1. A subset A of X is called *coarsely dense* in X if there exists $M \in \mathcal{E}$ such that $M(A) = X$.

Lemma 2.2. *A subset A in X is coarsely dense in X iff the class $[i]$ of the inclusion map $i: A \to X$ is an isomorphism in $\mathcal{E}$.*

Proof. Suppose that A is coarsely dense in X, then there is $M \in \mathcal{E}$ such that $X = M(A)$.

Define a map $g: X \to A$ as follows: for any $x \in X$, $g(x)$ is an arbitrary point af A with $x \in M(g(x))$. Obviously, g is coarse.

Then,

$$(gi(x), x) = (g(x), x) \in M,$$

$$(ig(x), x) = (g(x), x) \in M,$$

i.e. $gi \sim 1_A, ig \sim 1_X$, which means that $[g][i] = [1_A], [i][g] = [1_X]$. □

If $[i]$ is an isomorphism, then there exists a coarse map $g: X \to A$ such that $[i][g] = [ig] = [g] = [1_X]$. That means that $g \sim 1_X$, i.e. there is $M \in \mathcal{E}$ such that $(g(x), x) \in M$, for every $x \in X$.

Proposition 2.3. *Let $f, g: (X, \mathcal{E}) \to (X', \mathcal{E}')$ be a coarse maps. If $f|_A \sim g|_A$ on some coarsely dense subset A of X, then $f \sim g$.*

Proof. Let $i: A \to X$ denote the inclusion map. Then $fi \sim gi$ and therefore $[f][i] = [fi] = [gi] = [g][i]$. Since $[i]$ is an isomorphism (by previous lemma), we obtain that $[f] = [g]$. □

2.2. PRELIMINARIES. Topological coarse structures. Now suppose that X is a Hausdorff topological space. A coarse structure $\mathcal{E}$ on X is called *topological* if the following conditions are satisfied:

1) every entourage is open in $X \times X$;
2) every bounded set is precompact.

Note that if a space X can be endowed with coarse structure, then X is necessarily locally compact.

Proposition 2.4. *In a coarse topological space, every dense subset is coarsely dense.*

2.3. PRELIMINARIES. Coarse categories. We denote it by CTS (respectively CTS) the category of coarse topological spaces and coarse maps (respectively, of coarse topological spaces and proper continuous maps).

We will need one more category related to the coarse structures. In order to define it, we introduce the following notion.

Let $f, g\colon (X,\mathcal{E}) \to (X',\mathcal{E}')$ be coarse maps. We say that f and g are *equivalent* (and write $f \sim g$) if there exists $M \in \mathcal{E}'$ such that $(f(x), g(x)) \in M$ for every $x \in X$. It is easy to verify that $\sim$ is an equivalence relation and we denote by $[f]$ the equivalence class of f.

Lemma 2.5. *Let $f_1, f_2\colon (X,\mathcal{E}) \to (X',\mathcal{E}')$, $g_1, g_2\colon (X',\mathcal{E}') \to (X'',\mathcal{E}'')$ be coarse maps. If $f_1 \sim f_2$ and $g_1 \sim g_2$, then $g_1f_1 \sim g_2f_2$.*

Proof. Since $f_1 \sim f_2$, there is $M' \in \mathcal{E}'$ such that $(f_1(x), f_2(x)) \in M'$ for every $x \in X$. Since g_1 is coarse, there is $M'' \in \mathcal{E}''$ such that $(g_1 \times g_1)(M') \subset M''$. We see that $(g_1f_1(x), g_1f_2(x)) \in M''$, for every $x \in X$, i. e. $g_1f_1 \sim g_1f_2$. Obviously, $g_1f_2 \sim g_2f_2$ and the result follows from the transitivity of $\sim$. □

Lemma 2.5 allows us to define a composition of the equivalence classes as $[gf] = [g][f]$. We define the category $CTS/\sim$ as the category whose objects are as in CTS and the morphisms are the equivalence classes of the morphisms in CTS with respect to the equivalence relation $\sim$.

3. Hyperspaces of coarse spaces. Given a Hausdorff topological space endowed with a topological coarse structure $\mathcal{E}$, denote by $\exp X$ the set of all nonempty compact subsets in X. A base for the *Vietoris topology* on $\exp X$ is formed by the sets

$$\langle U_1,\dots,U_k\rangle = \{A \in \exp X \mid A \subset \bigcup_{i=1}^{k} U_i,\ A \cap U_i \neq \emptyset \text{ for all } i = 1,\dots,k\},$$

where $U_1,\dots,U_k$ run over the topology of X. For every $M \in \mathcal{E}$ let $M_H = \{(A,B) \in \exp X \times \exp X \mid$ for every $a \in A$ there exists $b \in B$ with $(a,b) \in M$ and for every $b \in B$ there exists $a \in A$ with $(a,b) \in M\}$.

Proposition 3.1. *The family $\mathcal{E}_H = \{M_H \mid M \in \mathcal{E}\}$ is a topological coarse structure on $\exp X$. $\mathcal{E}_H$ is unital if so is $\mathcal{E}$.*

Proof. Obviously, if $M \subset N$, then $M_H \subset N_H$. Show that for every $M, N \in \mathcal{E}$ we have

$$M_HN_H = (MN)_H. \qquad (3.1)$$

Indeed, suppose that $(A,B) \in M_HN_H$. Then there exists $C \in \exp X$ such that $(A,C) \in M_H$, $(C,B) \in N_H$.

Given $a \in A$, there is $c \in C$ with $(a,c) \in M$ and there is $b \in B$ with $(c,b) \in N$. Therefore, $(a,b) \in MN$.

Similarly, we show that for every $b \in B$ there is $a \in A$ with $(a,b) \in MN$. This shows that $(A,B) \in (MN)_H$.

Using (3.1) we conclude that the product of entourages in $\mathcal{E}_H$ is contained in an entourage. Besides, if $M, N \in \mathcal{E}$, then $M_H \subset (M \cup N)_H$, $N_H \subset (M \cup N)_H$, i.e. $M_H \cup N_H \subset (M \cup N)_H$, which implies that the union of two entourages is contained in an entourage.

Finally, show that $\cup\mathcal{E}_H = \exp X \times \exp X$. Given $(A,B) \in \exp X \times \exp X$, find, for each $a \in A$, $b \in B$, an entourage $M_{ab} \in \mathcal{E}$ such that $(a,b) \in M_{ab}$. The cover $\{M_{ab} \mid a \in A,\ b \in B\}$ contains a finite subcover $\{M_{a_ib_i} \mid i = 1,\dots,k\}$ of $A \times B$.

There exists $M \in \mathcal{E}$ such that $\cup_{i=1}^{k} M_{a_i b_i} \subset M$. Then $A \times B \subset M$ and, obviously, $(A, B) \in M_H$.

Now suppose that $\mathcal{E}$ is a unital coarse structure on X. There exists $M \in \mathcal{E}$ with $\Delta_X \subset M$. Then, obviously, $\Delta_{\exp X} \subset M_H$.

Show that $\mathcal{E}_H$ is a topological coarse structure on $\exp X$.

First, show that every set M_H is open in $\exp X \times \exp X$, for every $M \in \mathcal{E}_1$. Indeed, suppose the opposite and let $(A, B) \in M_H$ be a non-interior point of M_H. Then there is a net $(A_\gamma, B_\gamma)_{\gamma \in \Gamma}$ converging to (A, B) such that $(A_\gamma, B_\gamma) \notin M_H$ for every $\gamma \in \Gamma$. Without loss of generality, we may assume that, for every $\gamma \in \Gamma$, there exists $b_\gamma \in B_\gamma \setminus M(A_\gamma)$.

There exists a subnet (b_{γ_i}) of (b_γ) converging to $b \in B$ (see the definition of the limit in the Vietoris topology [6]). Show that $b \notin M(A)$. Indeed, otherwise we would have a net (a_{γ_i}) converging to $a \in A$ such that $a_{\gamma_i} \in A_{\gamma_i}$. Since the net $(a_{\gamma_i}, b_{\gamma_i})$ converges to $(a, b) \in M$, there exists $i(0)$ such that $(a_{\gamma_{i(0)}}, b_{\gamma_{i(0)}}) \in M$, i. e. $b_{\gamma_{i(0)}} \in M(A_{\gamma_{i(0)}})$, a contradiction.

Now show that every subset of the form $M_H(A)$ is relatively compact. Note that the set $M(A)$ is bounded and, therefore, $\overline{M(A)}$ is compact. Obviously, $M_H(A) \subset \exp(\overline{M(A)})$, and therefore the closure of $M_H(A)$ is compact. $\square$

The coarse structure $\mathcal{E}_H$ is called the *Vietoris* coarse structure on $\exp X$. In the sequel, we always endow the hyperspace of a coarse topological space with the Vietoris coarse structure.

Let $f: (X, \mathcal{E}) \to (X', \mathcal{E}')$ be a coarse map between coarse topological spaces. Define the map $\exp f: \exp X \to \exp X'$ by the formula $\exp f(A) = \overline{f(A)}$. Note that $\exp f$ is well-defined as the set $f(A)$ is obviously bounded for every compact subset $A \subset X$ and therefore the set $\overline{f(A)}$ is compact.

Proposition 3.2. *The map* $\exp f: (\exp X, \mathcal{E}_H) \to (\exp X', \mathcal{E}'_H)$ *is coarse.*

Proof. Indeed, suppose that $M \in \mathcal{E}$. Then there is $M' \in \mathcal{E}'$ such that $(f \times f)(M) \subset M'$. Then it is easy to see that $(\exp f \times \exp f)(M_H) \subset M'_H$, this shows that $\exp f$ is coarsely uniform.

Show that $\exp f$ is coarsely proper. It suffices to show that the preimage under the map $\exp f$ of every set of the form $M'_H(\{x'\})$ is bounded. Since f is coarsely proper, there exist $M \in \mathcal{E}$ and $x \in X$ such that $f^{-1}(M'(x')) \subset M(x)$. It is easy to see that then $(\exp f)^{-1}(M'_H(\{x'\}) \subset M_H(\{x\})$. $\square$

It is not difficult to construct two coarse maps f, g such that $\exp(gf) \neq \exp g \exp f$. Indeed, consider the real line $\mathbb{R}$ with the *bounded* coarse structure, i. e. the coarse structure

$$\mathcal{E} = \{\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x - y| \leqslant C\} \mid C > 0\}.$$

Define $f, g: \mathbb{R} \to \mathbb{R}$ as follows: $f(x) = x$ whenever $x \leqslant 0$ and $f(x) = x+1$ otherwise, $g(x) = x$ whenever $x \leqslant 1$ and $g(x) = x + 1$ otherwise. Let $A = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$, then $\exp f(A) = \{0\} \cup \{1\} \cup \{1 + 1/n \mid n \in \mathbb{N}\}$ and

$$\exp g \exp f(A) = \{0\} \cup \{1\} \cup \{2\} \cup \{2 + 1/n \mid n \in \mathbb{N}\},$$

while

$$\exp(gf)(A) = \{0\} \cup \{2\} \cup \{2 + 1/n \mid n \in \mathbb{N}\}.$$

This example can be regarded as a motivation of introducing the category $CTS/\sim$.

Lemma 3.3. *If A is a subset of a coarse topological space $(X, \mathcal{E})$, then for every $M \in \mathcal{E}$ we have $\bar{A} \subset M(A)$.*

Proof. Suppose $x \in \bar{A}$, then there is $a \in A \cap M^{-1}(x)$. This means that $x \in M(A)$. □

Proposition 3.4. *Let $f_1, f_2: (X, \mathcal{E}) \to (X', \mathcal{E}')$ be coarse maps. If $f_1 \sim f_2$, then* $\exp f_1 \sim \exp f_2$.

Proof. There exists $M' \in \mathcal{E}'$ such that $(f_1(x), f_2(x)) \in M'$ for every $x \in X$. Without loss of generality, we may assume that $M' = (M')^{-1}$. If $A \in \exp X$, then $f_1(A) \subset M'(f_2(A))$ and $f_2(A) \subset M'(f_1(A))$. By Lemma 3:3,

$$\overline{f_1(A)} \subset M'(f_1(A)),\ \overline{f_2(A)} \subset M'(f_2(A))$$

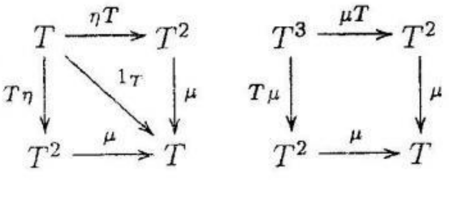
and we obtain

$$\overline{f_1(A)} \subset M'M'(f_2(A)),\ \overline{f_2(A)} \subset M'M'(f_1(A)).$$

The latter means that $(\exp f_1(A), \exp f_2(A)) \in M_H$. □

Proposition 3.4 allows us to define the hyperspace functor exp in the category $CTS/\sim$ as follows. Given a morphism $f: X \to Y$ in CTS, we define $\exp[f]: X \to Y$ in $CTS/\sim$ as $\exp[f] = [\exp f]$.

4. Hyperspace monad in the coarse category. Recall that a monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$ consisting of an endofunctor $T: \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta: 1_{\mathcal{C}} \to T$ (unit), $\mu: T^2 \to T$ (multiplication) making the diagrams



commutative (see [7] for details).

Theorem 4.1. *The triple $\mathbb{H} = (\exp, s, u)$ is a monad on the category $CTS/\sim$.*

Proof. First show that the map $u_x: (\exp^2 X, \mathcal{E}_{HH}) \to (expX, \mathcal{E}_H)$ is coarse. Let $(\mathcal{A}, \mathcal{B}) \in M_{HH}$, for some $M \in \mathcal{E}$. Show that $(\cup\mathcal{A}, \cup\mathcal{B}) \in M_H$. Indeed, if $a \in \cup\mathcal{A}$, then there is $A \in \mathcal{A}$ with $a \in A$. By the definition of M_{HH}, there is $B \in \mathcal{B}$ with $(A, B) \in M_H$. Then there is $b \in \cup\mathcal{B}$ with $(a, b) \in M$.

We can similarly prove that for every $b \in \cup\mathcal{B}$ there is $a \in \cup\mathcal{A}$ with $(a, b) \in M$. Together this means that $(\cup\mathcal{A}, \cup\mathcal{B}) \in M_H$.

To this end, we have to show that $u_x^{-1}(M_H(A))$ is bounded for every $A \in \exp X$ and every $M \in \mathcal{E}$. If $\mathcal{B} \in u_x^{-1}(M_H(A))$, then $(\cup\mathcal{A}, \cup\mathcal{B}) \in M_H$.

There exists an entourage $N \in \mathcal{E}$ such that $\overline{M(A)} \subset N(x)$, for some $x \in X$.

Suppose that $\mathcal{B} \in u_x^{-1}(M_H(A))$ and $B \in \mathcal{B}$. Then $B \subset M(A)$.

Given $b \in B$ we see that $b \in N(x)$, whence $x \in N^{-1}(b)$ and $A \subset NN^{-1}(b) \subset NN^{-1}(B)$. Let $L \in \mathcal{E}$ an entourage containing $M \cup (NN^{-1})$. Then for any $\mathcal{B} \in u_x^{-1}(M_H(A))$ and any $B \in \mathcal{B}$ we have $B \subset M(A) \subset L(A)$ and $A \subset NN^{-1}(B) \subset L(B)$. This means that $\mathcal{B} \in L_H(A)$.

Now we are going to show that $([u_X])$ is a natural transformation of $\exp^2$ into exp.

Given a coarse map $f\colon X \to Y$ we have to show that the diagram

$$\begin{array}{ccc} \exp^2 X & \xrightarrow{\exp^2[f]} & \exp^2 Y \\ {\scriptstyle [u_X]}\big\downarrow & & \big\downarrow{\scriptstyle [u_Y]} \\ \exp X & \xrightarrow[\exp[f]]{} & \exp Y \end{array}$$

is commutative.

We first start with finite sets.

Let $\mathcal{A} \in \exp^2 X$, then

$$\exp^2[f](\mathcal{A}) = \overline{\{\exp[f](A) | A \in \mathcal{A}\}} = \overline{\{\overline{f(A)} | A \in \mathcal{A}\}},$$

$$[u_Y](\exp^2[f](\mathcal{A})) = \cup\overline{\{\overline{f(A)} | A \in \mathcal{A}\}}.$$

On the other hand,

$$\exp[f]([u_X](\mathcal{A})) = \overline{f(\cup\mathcal{A})}.$$

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$. Then

$$[u_Y](\exp^2[f](\mathcal{A})) = \cup\overline{\{\overline{f(A_i)} | i = 1, \ldots, n\}} =$$

$$= \cup\{\overline{f(A_i)} | i = 1, \ldots, n\} = \overline{\cup\{f(A_i) | i = 1, \ldots, n\}} = \overline{f(\cup\mathcal{A})}.$$

The set $\{\mathcal{A} \in \exp^2 X \| \mathcal{A}| < \infty\}$ is dense in $\exp^2 X$ and therefore this set is coarsely dense in $\exp^2 X$. According to the Proposition 2.3 we conclude that the diagram is commutative for each $\mathcal{A} \in \exp^2 X$.

Show that the diagram

$$\begin{array}{ccc} \exp^3 X & \xrightarrow{\exp[u_X]} & \exp^2 X \\ {\scriptstyle [u_{\exp X}]}\big\downarrow & & \big\downarrow{\scriptstyle [u_X]} \\ \exp^2 X & \xrightarrow[[u_X]]{} & \exp X \end{array}$$

is commutative.

Similarly as above, we consider the set

$$F = \{\mathfrak{A} \in \exp^3 X \mid |u_X(u_{\exp X}(\mathfrak{A}))| < \infty\}.$$

It is well-known that F is dense in $\exp^3 X$ and the restriction of the above diagram on F is commutative. The result follows from Proposition 2.3. $\square$

5. Coarse structures on symmetric powers. Let $\mathbb{T}$ be a monad on a category $\mathcal{C}$. The *Kleisli category of the monad* $\mathbb{T}$ is the category $\mathcal{C}_{\mathbb{T}}$ defined as follows: $|\mathcal{C}_{\mathbb{T}}| = |\mathcal{C}|$, $\mathcal{C}_{\mathbb{T}}(X,Y) = \mathcal{C}(X,TY)$, and the composition $g * f$ of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X,Y)$, $g \in \mathcal{C}_{\mathbb{T}}(Y,Z)$ is given by $g * f = \mu Z \circ Tg \circ f$ (see[7]).

Note that the category $\mathcal{C}_{\mathbb{T}}$ can be embedded into $\mathcal{C}^{\mathbb{T}}$ as a full subcategory by means of the functor Φ

$$\Phi X = (TX, \mu X), \quad \Phi f = \mu Y \circ Tf, \; f \in \mathcal{C}_{\mathbb{T}}(X,Y).$$

A functor $\overline{F}\colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ called an *extension of the functor* $F\colon \mathcal{C} \to \mathcal{C}$ *on the Kleisli category* $\mathcal{C}_{\mathbb{T}}$ if $IF = \overline{F}I$.

The following theorem is a criterion of extension of functors onto the Kleisli category; see [8–12] for the proof.

Theorem 5.1. *There exists a bijective correspondence between extensions of functor* F *onto the Kleisli category* $\mathcal{C}_{\mathbb{T}}$ *of monad* $\mathbb{T}$ *and natural transformations* $\xi\colon FT \to TF$ *satisfying*

1) $\xi \circ F\eta = \eta F$;
2) $\mu F \circ T\xi \circ \xi T = \xi \circ F\mu$.

For any X, as usual, X^n denotes its nth cartesian power. Given a coarse structure $\mathcal{E}$ on X, define the coarse structure $\mathcal{E}^n$ on $\mathcal{X}^n$ as $\mathcal{E}^n = \{M^n \mid M \in \mathcal{E}\}$.

Let G be a subgroup of the symmetric group S_n (the group of bijections of the set $\{1,\dots,n\}$. Recall that the G-symmetric power functor is defined as follows. Define an equivalence relation $\sim_G$ on X^n by the condition: $(x_1,\dots,x_n) \sim_G (y_1,\dots,y_n)$ if and only if there exists $\sigma \in G$ such that $x_i = y_{\sigma(i)}$ for all $i = 1,\dots,n$. We denote by $[x_1,\dots,x_n]_G$ the equivalence class that contains $(x_1,\dots,x_n)$. By the definition, the G-symmetric power of X is $SP^n_G X = X^n/\sim_G$.

Given a map $f\colon X \to Y$, we define a map $SP^n_G f\colon SP^n_G X \to SP^n_G Y$ by the formula

$$SP^n_G f([x_1,\dots,x_n]_G) = [f(x_1),\dots,f(x_n)]_G.$$

Now suppose that $(X,\mathcal{E})$ is a coarse space. For any $M \in \mathcal{E}$ let

$$\hat{M} = \{([x_1,\dots,x_n]_G,[y_1,\dots,y_n]_G) \in SP^n_G X \times SP^n_G X$$
$$\mid \text{there is } \sigma \in G \text{ such that } (x_i, y_{\sigma(i)}) \in M \text{ for every } i = 1,\dots,n\}.$$

If X is a topological space, then $SP^n X$ is endowed with the quotient topology of X^n. A base of this topology is formed by the sets of the form

$$[U_1,\dots,U_n]_G = \{[x_1,\dots,x_n]_G \mid x_i \in U_i,\ i = 1,\dots,n\}.$$

Proposition 5.2. *The family* $\hat{\mathcal{E}} = \{\hat{M} \mid M \in \mathcal{E}\}$ *is a coarse structure on* $SP^n_G X$. *If* $\mathcal{E}$ *is topological (unital), then so is* $\hat{\mathcal{E}}$.

Proof. The fact that $\hat{\mathcal{E}}$ is a coarse structure easily follows from the equalities $(MN)\hat{} = \hat{M}\hat{N}$ and $(M^{-1})\hat{} = (\hat{M})^{-1}$.

Suppose now that $\mathcal{E}$ is topological and $([a_1,\dots,a_n]_G,[b_1,\dots,b_n]_G) \in \hat{M}$ for some $M \in \mathcal{E}$. Then there exists $\sigma \in G$ such that $(a_i, b_{\sigma(i)}) \in M$, for all $i = 1,\dots,n$. There exist open sets U_i and $V_{\sigma(i)}$ in X such that $(a_i, b_{\sigma(i)}) \in U_i \times V_{\sigma(i)} \subset M$. Then obviously

$$([a_1,\dots,a_n]_G,[b_1,\dots,b_n]_G) \in ([U_1,\dots,U_n]_G,[V_1,\dots,V_n]_G) \in \hat{M}.$$

□

Proposition 5.3. SP^n_G *is an endofunctor in the category* CS *(respectively* CTS*).*

Proof. We only consider the case of the category CTS. It is sufficient to verify that the map $SP^n_G f$ is coarse, for every coarse map $f\colon (X,\mathcal{E}) \to (X',\mathcal{E}')$. Given $M \in \mathcal{E}$ we can find $M' \in \mathcal{E}'$ such that $(f \times f)(M) \subset M'$. Then it can be immediately verified that $(SP^n_G f \times SP^n_G f)(\hat{M}) \subset \hat{M}'$.

Besides, we have to prove that for every $[a_1, \dots, a_n]_G \in SP^n_G X'$ and every $M' \in \mathcal{E}'$ the set $K = (SP^n_G f)^{-1}(\hat{M}'([a_1, \dots, a_n]_G))$ is relatively compact. It is easy to see that K is contained in the closure of the set $\cup_{i=1}^n f^{-1}(M'(a_i))$; the latter is relatively compact, because f is coarse. □

Theorem 5.4. *There exists an extension of the functor SP^n_G onto the Kleisli category $(CTS/\sim)_H$.*

Proof. We exploit an idea from [13]. For every coarse topological space X define a map $d_X : SP^n_G \exp X \to \exp SP^n_G$ by the formula

$$d_X([A_1, \dots, A_n]_G) = \{[a_1, \dots, a_n]_G \mid a_i \in A_i,\ i = 1, \dots, n\}.$$

It is easy to verify and we leave it to the reader that d_X is a coarse map for every X.

That $d = (d_X)$ is a natural transformation of the functor $SP^n_G \exp$ into the functor $\exp SP^n_G$ follows from the facts that

$$d_Y SP^n \exp f([A_1, \dots, A_n]_G) = \exp SP^n f d_X([A_1, \dots, A_n]_G)$$

for every finite $A_1, \dots, A_n$ (see [13]), that the set $\{[A_1, \dots, A_n]_G \mid A_i \text{ is finite}, i = 1, \dots, n\}$ is dense in $SP^n_G \exp X$, and Proposition 2.3.

Similarly, one can prove the equalities $d_X \circ SP^n_G s_X = s_{SP^n_G X}$ and $u_{SP^n_G X} \circ \exp d_X \circ d_{\exp X} = d_X \circ SP^n_G u_X$ which are known to be true for finite X (see [13]). Again, by Proposition 2.3, this shows that the conditions of Theorem 5.1. hold. Applying Theorem 5.1 we complete the proof. □

6. Remarks. The importance of the hyperspace monad in the category of compact Hausdorff spaces is closely related to the fact that the category of algebras for this monad (see [7] for the definition) can be described as the category of compact continuous semilattices [14]. A natural question arises whether a counterpart of this result exists in the coarse category.

Besides, in [13] the symmetric power functors are characterized as the normal functors of finite degree that have extensions to the Kleisli category of the hyperspace monad. In the forthcoming publication we are going to extend this result (at least partially) to the coarse category.

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ФУНКТОР ГІПЕРПРОСТОРУ В ГРУБІЙ КАТЕГОРІЇ

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Розглянуто монаду гіперпростору в категорії грубих топологічних просторів і класів еквівалентності грубих відображень. Доведено, що функтор G-симетричного степеня, що діє в категорії топологічних просторів, може бути природно означений і для категорії грубих топологічних просторів, і на цій останній категорії він може бути продовжений на категорію Клейслі монади гіперпростору.

Ключові слова: грубий простір, грубе відображення.

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REDUCTION OF A PAIR OF MATRICES OVER AN ADEQUATE DUO-RING TO A SPECIFIC TRIANGULAR FORM BY IDENTICAL UNILATERAL TRANSFORMATIONS

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It is proved that a pair of matrices over an adequate duo-ring can be reduced to a specific triangular form by means of identical unilateral transformations.

Key words: adequate ring, duo-ring, elementary divisor ring.

Throughout this paper, all rings are associative adequate duo-rings with identity. A ring is said to be a *duo-ring* if every its left or right ideal is two-sided. A ring is a *Bezout ring* if every its finitely generated right and left ideal is principal.

Matrices A and B over ring R are *equivalent* ($A \sim B$), if there exist invertible matrices P and Q over R such that $A = PBQ$.

An $m \times n$ matrix A *admits diagonal reduction* if A is equivalent to a diagonal matrix $[\epsilon_{ij}]$ (i.e. $\epsilon_{ij} = 0$ whenever $i \neq j$) with the property $R\epsilon_{i+1,i+1}R \subseteq R\epsilon_{i,i} \cap \epsilon_{i,i}R$ (in the case of a duo-ring we can write: $\epsilon_{i+1,i+1}R \subseteq \epsilon_{i,i}R$). If every matrix over R admits diagonal reduction, then R is an *elementary divisor ring*. The elements $\epsilon_{11}, \epsilon_{22}, \ldots, \epsilon_{rr}$ are called the *invariant factors* of the matrix A.

A ring R is called *right adequate* if R is a Bezout ring without zero divisors and for $a, b \in R$ with $a \neq 0$, there exist $r, s \in R$ such that $a = rs, rR + bR = R$, and $s'R + bR \neq R$ for any nonunit $s' : sR \subset s'R$.

Using left principal ideals by analogy we can define left adequate rings. In the class of duo-rings these notions are equivalent and we will use the term adequate ring.

Commutative adequate rings were considered in [1-3].

V.Petrychkovych investigated the reducibility of pairs of matrices by means of the generalized equivalent transformations to the diagonal form [4].

Let R be an adequate duo-ring.

Lemma 1. *Let $a, b, c \in R$ and $a \neq 0, c \neq 0$. Then there exists an element $r \in R$ such that $(a+rb)R+rcR = aR+bR+cR$ and if $aR+bR+cR = R$ then $rR+aR+bR+cR = R$.*

Proof. Let $aR+bR+cR = R$. Assume that $(a+rb)R+rcR = hR$, and $h \notin U(R)$. Then we obtain:

1) $rc = rrs, rR + hR = h'R$ and $(a+rb)R \subseteq h'R$.

It follows that $a \in h'R$ and $aR \subset h'R$. We obtain a contradiction with

$$R = rR + aR \subset h'R + aR \neq R.$$

2) If $rR + hR = R$, then

$$r^2R + hR = R, r^2u + hv = 1$$
$$r^2us + hvs = s, r^2su' + hsv' = s$$
$$u, v, u', v' \in R.$$

We have

$$sR \subset hR \text{ and } hR + aR = h'R,$$

where $h' \notin U(R)$. Thus,

$$(a + rb)R \subset h'R, rbR \subset h'R, R = rR + aR =' rR + h'R,$$
$$aR \subset h'R, bR \subset h'R, cR \subset h'R.'$$

This yields

$$R = aR + bR + cR \subset h'R,$$

and we have $h' \in U(R)$.

If $aR + bR + cR = dR, a = da_0, b = db_0, c = dc_0$ we provide the proof similarly for elements a_0, b_0, c_0.

Lemma 2. *Let $A_i, i = 1, 2$ be $2 \times k_i$ matrices over a ring R, and at least one of them is not a right zero divisor. Then there exist invertible matrices P and $Q_i, i = 1, 2$ over R such that*

$$PA_iQ_i = \begin{pmatrix} \varepsilon_1^{(i)} & 0 & 0 & \dots & 0 \\ * & \varepsilon_2^{(i)} & 0 & \dots & 0 \end{pmatrix},$$

where $\epsilon_j^{(i)}$ are invariant factors of matrices $A_i, i = 1, 2$.

Proof. We may assume that A_2 is not a right zero divisor, so that $k_1 \geqslant 1, k_2 \geqslant 2$. Since R is an elementary divisor ring [5], there exist invertible matrices S, M_1, M_2 over R such that

$$SA_1M_1 = \begin{pmatrix} \varepsilon_1^1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon_2^1 & 0 & \dots & 0 \end{pmatrix}, SA_2M_2 = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ b & c & 0 & \dots & 0 \end{pmatrix},$$

$\epsilon_2^1 R \subseteq \epsilon_1^1 R$ and $a \neq 0, c \neq 0$.

By Lemma 1 for elements $a, b, c \in R$ there exists an element $r \in R$ such that $(a + rb)R + rcR = aR + bR + cR$. Consider the matrix

$$T = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

It is easy to verify that matrices TSA_iM_i can be reduced to the form

$$\begin{pmatrix} \varepsilon_1^{(i)} & 0 & 0 & \dots & 0 \\ * & \varepsilon_2^{(i)} & 0 & \dots & 0 \end{pmatrix}.$$

using right-side multiplication by invertible matrices.

The proof is complete.

Theorem 1. *Let $A_i, i = 1,2$ be $m \times k_i$ matrices over a ring R, and at least one of them is not a right zero divisor.*

Then there exist invertible matrices P and $Q_i, i = 1,2$ over R such that

$$\begin{pmatrix} \epsilon_1^{(i)} & 0 & \dots & 0 & \dots & 0 \\ & \epsilon_2^{(i)} & \dots & 0 & \dots & 0 \\ & * & \ddots & & & \\ & & & \epsilon_m^{(i)} & \dots & 0 \end{pmatrix}$$

where $\epsilon_j^{(i)}$ are invariant factors of the matrices $A_i, i = 1,2$.

Proof. Assume that A_2 is not a right zero divisor. We shall prove the theorem by induction on number m of rows of the matrices. If $m = 2$, the Theorem is true by Lemma 2. Suppose that the theorem is true for matrices with the number of rows $m - 1$. Thus R is an adequate duo-ring and there exist invertible matrices $S, Q_i, i = 1,2$, such that

$$SA_1Q_1 = \begin{pmatrix} \epsilon_1^{(1)} & 0 & \dots & 0 & \dots & 0 \\ 0 & \epsilon_2^{(1)} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \\ 0 & 0 & \dots & \epsilon_m^{(1)} & \dots & 0 \end{pmatrix} = B_1,$$

$$SA_2Q_2 = \begin{pmatrix} a_{11} & 0 & \dots & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \\ a_{m1} & a_{m2} & \dots & a_m & \dots & 0 \end{pmatrix} = B_2,$$

where $\epsilon_i^{(1)}$ are invariant factors of the matrix A_1. Consider submatrices $B_i', i = 1,2$, of the matrices B_i obtained by crossing off the last rows of the matrices B_i. For them by the induction hypothesis there exist invertible matrices M, N_i such that

$$MB_1'N_1 = \begin{pmatrix} \epsilon_1^{(1)} & 0 & \dots & 0 & \dots & 0 \\ & \epsilon_2^{(1)} & \dots & 0 & \dots & 0 \\ & * & \ddots & & & \\ & & & \epsilon_{m-1}^{(1)} & \dots & 0 \end{pmatrix},$$

$$MB_2'N_2 = \begin{pmatrix} \varphi_1^{(2)} & 0 & \dots & 0 & \dots & 0 \\ & \varphi_2^{(2)} & \dots & 0 & \dots & 0 \\ & * & \ddots & & & \\ & & & \varphi_{m-1}^{(2)} & \dots & 0 \end{pmatrix},$$

where $\varphi_j^{(2)}$ are invariant factors of the matrix B_2'. Then

$$C_1 = \begin{pmatrix} & & & 0 \\ & M & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} B_1 \begin{pmatrix} & & & 0 \\ & N_1 & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \epsilon_1^{(1)} & 0 & \dots & 0 & \dots & 0 \\ & \epsilon_2^{(1)} & \dots & 0 & \dots & 0 \\ & * & \ddots & & & \\ 0 & \dots & 0 & \epsilon_m^{(1)} & \dots & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} & M & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} B_2 \begin{pmatrix} & N_2 & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \varphi_1^{(2)} & 0 & \dots & 0 & \dots & 0 \\ * & \ddots & & & & \\ & & \varphi_{m-1}^{(2)} & 0 & \dots & 0 \\ a'_{m1} & \dots & \dots & a'_{mm} & \dots & 0 \end{pmatrix}.$$

Let $\varphi_1^{(2)}R + a'_{m1}R + a'_{mm}R = R$. By Lemma 1, there exist $r \in R$ such that for the elements $\varphi_1^{(2)}, a'_{m1}, a'_{mm}$ we obtain

$$(\varphi_1^{(2)} + ra'_{m1})R + ra'_{mm}R = \varphi_1^{(2)}R + a'_{m1}R + a'_{mm}R,$$

$$rR + \varphi_1^{(2)}R + a'_{m1}R + a'_{mm}R = R. \tag{1}$$

Consider an $m \times m$ matrix of the form

$$T = \begin{pmatrix} 1 & 0 & \dots & r \\ 0 & 1 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Multiply the matrices $C_i, i = 1, 2$, on the left by this invertible matrix T:

$$TC_1 = \begin{pmatrix} \varepsilon_1^{(1)} & 0 & \dots & r\varepsilon_m^{(1)} & 0 & \dots & 0 \\ & & & * & & & \end{pmatrix},$$

$$TC_2 = \begin{pmatrix} \varphi_1^{(2)} + ra'_{m1}, & ra'_{m2}, & \dots & ra'_{mm}, & 0 & \dots & 0 \\ & & & * & & & \end{pmatrix}.$$

Using condition (1) we obtain that the greatest common right divisor of the elements of the first row of the matrix TC_2 is the greatest common right divisor of all elements of the matrix C. Since $\epsilon_m^{(m)}R \subset \epsilon_1^{(1)}R$, we have a similar situation for the elements of the first row of the matrix TC_1. Thus by multiplication on the right by the matrices $L_i, i = 1, 2$ the matrices TC_i can be reduced to the form

$$TC_iL_i = \begin{pmatrix} \epsilon_1^{(i)} & 0 & \dots & 0 & \dots & 0 \\ & b_{22}^{(i)} & \dots & 0 & \dots & 0 \\ & * & \ddots & & & \\ & & & b_{mm}^{(i)} & \dots & 0 \end{pmatrix},$$

where $\epsilon_1^{(i)}$ is the first invariant factor of the matrix A_i.

Consider the submatrices of the matrices TC_iL_i obtained by crossing off the first rows and columns. They have the number of rows $m-1$, satisfy the condition of the theorem and for them by induction hypothesis the theorem is true.

If $\varphi_1^{(2)}R + a'_{m1}R + a'_{mm}R = dR$ we can represent the matrix $C_2 = DC'_2$, where $D = diag[d, d, \dots, d]$ is a diagonal matrix and repeat the same arguments for the matrix C'_2. The proof is complete.

Theorem 2. *Let $C = AB$, where A, B are matrices over R which are not right and left zero divisors. Then the elementary divisors of the matrix C are divizible on coresponding elementary divisors of matrices A and B.*

The same result was obtained for other classes of rings in [2], [3], [6].

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ЗВЕДЕННЯ ПАРИ МАТРИЦЬ НАД АДЕКВАТНИМ ДУО-КІЛЬЦЕМ ДО СПЕЦІАЛЬНОГО ТРИКУТНОГО ВИГЛЯДУ ШЛЯХОМ ІДЕНТИЧНИХ ОДНОБІЧНИХ ПЕРЕТВОРЕНЬ

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Доведено, що пара матриць над адекватним дуо-кільцем зводиться до спеціального трикутного вигляду шляхом ідентичних однобічних перетворень.

Ключові слова: адекватне кільце, дуо-кільце, кільце елементарних дільників.

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COMBINATORIAL SIZE OF SUBSETS OF SEMIGROUPS AND ORGRAPHS

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A triple $\mathbf{B} = (X, P, B)$ is called a balls structure if X, P are nonempty sets and, for all $x \in X$, $\alpha \in P$, $B(x, \alpha) \ni x$ is a subset of X, called a ball of radius α around x. We classify subsets of X by their sizes with respect to the ball's structure $\mathbf{B}$ and apply this classification to semigroups and oriented graphs.

Key words: ball's structure, large and small subsets.

1. Ball's structures. Let X, P be nonempty sets and let, for any $x \in X$, $\alpha \in P$, $B(x, \alpha) \ni x$ be a subset of X, which is called *the ball of radius α around x*. Following [1], a triple $\mathbf{B} = (X, P, B)$ is called a *ball's structure.*

For any $x \in X$, $\alpha \in P$, put $B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}$. A ball's structure $\mathbf{B}^* = (X, P, B^*)$ is called dual to $\mathbf{B}$. Observe that $B^{**}(x, \alpha) = B(x, \alpha)$ for all x, α and thus $\mathbf{B}^{**} = \mathbf{B}$.

Define a preordering $\leqslant$ on the set P by the rule: $\alpha \leqslant \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset P' of P is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ with $\alpha \leqslant \beta$. A ball's structure $\mathbf{B}$ is called *symmetric* if there exists a cofinal subset $P' \subseteq P$ such that $B(x, \beta) = B^*(x, \beta)$ for all $x \in X$, $\beta \in P'$.

Given any subset $A \subseteq X$ and $\alpha \in P$, put

$$B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha), \quad Int(A, \alpha) = \{x \in X : B^*(x, \alpha) \subseteq A\}.$$

A ball's structure $\mathbf{B} = (X, P, B)$ is called *multiplicative* if, for any $\alpha, \beta \in P$ there exists $\gamma(\alpha, \beta) \in P$ such that $B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$ for every $x \in X$. Since $B^*(B^*(x, \alpha), \beta) \subseteq B^*(x, \gamma(\beta, \alpha))$, $\mathbf{B}$ is multiplicative if and only if $\mathbf{B}^*$ is multiplicative.

Example 1. Let $Gr = (V, E)$ be an oriented graph where V is the set of vertices of Gr and $E \subset V \times V$ is the set of its edges. For every $x \in V$, put $d(x, x) = 0$. If for distinct $x, y \in V$ there exists an oriented path from x to y, then let $d(x, y)$ be the length of the shortest oriented path from x to y. Otherwise, put $d(x, y) = \infty$. Given any $x \in V$ and $n \in \omega$, put $B(x, n) = \{y \in V : d(x, y) \leqslant n\}$. The ball's structure (V, ω, B) will be denoted by $\mathbf{B}(Gr)$. Taking into account that $B(B(x, n), m) \subseteq B(x, n + m)$ we conclude that $\mathbf{B}(Gr)$ is multiplicative. Note also that $\mathbf{B}^*(Gr)$ coincides with

$\mathbf{B}(Gr^*)$, where $Gr^* = (V, E^{-1})$, $E^{-1} = \{(y,x) : (x,y) \in E\}$. If $E = E^{-1}$, then $\mathbf{B}(Gr) = \mathbf{B}^*(Gr)$.

Example 2. Let S be a semigroup with the identity e and let Fin be the family of all finite subsets of S containing e. Given any $s \in S$ and $F \in Fin$, put

$$B_l(x, F) = Fx \text{ and } B_r(x, F) = xF.$$

The balls's structures (S, Fin, B_l) and (S, Fin, B_r) will be denoted by $\mathbf{B}_l(S)$ and $\mathbf{B}_r(S)$. If $x \in S$ and F, $F' \in Fin$, then

$$B_l(B_l(x,F),F') \subseteq B_l(x,F'F) \text{ and } B_r(B_r(x,F),F') \subseteq B_r(x,FF').$$

Hence, $\mathbf{B}_l(S)$ and $\mathbf{B}_r(S)$ are multiplicative. If S is a group, then $\mathbf{B}_l(S)$ and $\mathbf{B}_r(S)$ are symmetric [1, Example 2].

2. Classification of subsets by their sizes. Fix a ball's structure $\mathbf{B} = (X, P, B)$. A subset $A \subseteq X$ is called

- *large* if there exists $\alpha \in P$ such that $X = B(A, \alpha)$;
- *small* if $X \backslash B(A, \alpha)$ is large for every $\alpha \in P$;
- *extralarge* if $Int(A, \alpha)$ is large for every $\alpha \in P$;
- *piecewise large* if there exists $\beta \in P$ such that $Int(B(A, \beta), \alpha) \neq \emptyset$ for every $\alpha \in P$.

Observe that for a multiplicative ball's structure $\mathbf{B} = (X, P, B)$ a subset $A \subset X$ is large if and only if $B(A, \alpha)$ is large for some $\alpha \in P$.

Lemma 1. *Let $\mathbf{B} = (X, P, B)$ be a ball's structure, $A \subseteq X$, $\alpha \in P$. Then $Int(X \backslash A, \alpha) = X \backslash B(A, \alpha)$.*

Proof. Let $x \in Int(X\backslash A, \alpha)$. Then $B^*(x, \alpha) \cap A = \emptyset$, so $x \notin B(a, \alpha)$ for every $a \in A$. Hence, $x \in X \backslash B(A, \alpha)$.

Let $x \in X \backslash B(A, \alpha)$. Then $x \notin B(a, \alpha)$ for every $a \in A$. Hence, $a \notin B^*(x, \alpha)$ for every $a \in A$, so $B^*(x, \alpha) \subseteq X \backslash A$ and $x \in Int(X \backslash A, \alpha)$. □

The following statement is a refinement of Theorem 1 from [1].

Theorem 1. *Let $\mathbf{B} = (X, P, B)$ be a ball's structure and let $S \subseteq X$. Then the following statements are equivalent:*

1) S is small;

2) S is not piecewise large;

3) $X \backslash S$ is extralarge.

If, moreover, $\mathbf{B}$ is multiplicative, then the statements 1)–3) are equivalent to

4) $(X \backslash S) \cap L$ is large for every large subset L of X.

Proof. 1) $\Rightarrow$ 2). For every $\alpha \in P$, pick $\beta(\alpha) \in P$ such that $B(X \backslash B(S, \alpha), \beta(\alpha)) = X$. Take any $x \in X$ and choose $y \in X \backslash B(S, \alpha)$ with $x \in B(y, \beta(\alpha))$. Then $y \in B^*(x, \beta(\alpha))$ and $B^*(x, \beta(\alpha)) \cap (X \backslash B(S, \alpha)) \neq \emptyset$. Hence, $Int(B(S, \alpha), \beta(\alpha)) = \emptyset$ and S is not piecewise large.

2) $\Rightarrow$ 3). For every $\alpha \in P$, pick $\beta(\alpha) \in P$ such that $Int(B(S, \alpha), \beta(\alpha)) = \emptyset$. Then $B^*(x, \beta(\alpha)) \cap (X \backslash B(S, \alpha)) \neq \emptyset$ for every $x \in X$. By Lemma 1,

$$B^*(x, \beta(\alpha)) \cap (Int(X \backslash S, \alpha)) \neq \emptyset$$

for every $x \in X$. Hence, $X = B(Int(X \backslash S, \alpha), \beta(\alpha))$ and $X \backslash S$ is extralarge.

3) $\Rightarrow$ 1). For every $\alpha \in P$, pick $\beta(\alpha) \in P$ such that $B(Int(X\backslash S, \alpha), \beta(\alpha)) = X$. By Lemma 1, $B(X\backslash B(S, \alpha), \beta(\alpha)) = X$. Hence, S is small.

3) $\Rightarrow$ 4). Put $Y = X\backslash S$ and take any large subset L. Choose $\alpha \in P$ such that $X = B(L, \alpha)$. For every $x \in Int(Y, \alpha)$, choose $y(x) \in L$ with $x \in B(y(x), \alpha)$, equivalently, $y(x) \in B^*(x, \alpha)$. Put $Y' = \{y(x) : x \in Int(Y, \alpha)\}$ and note that $Y' \subseteq Y \cap L$. Since $Int(Y, \alpha) \subseteq B(Y', \alpha)$ and $Int(Y, \alpha)$ is large, by the multiplicativity of $\mathbf{B}$, Y' is large. Since $Y' \subseteq Y \cap L$, we get that $Y \cap L$ is large.

4) $\Rightarrow$ 3). Put $Y = X\backslash S$. Since $Y \cap X = Y$ and X is large, Y is large too. Fix any $\alpha \in P$ and show that $Int(Y, \alpha)$ is large. For every $x \in Y\backslash Int(Y, \alpha)$, pick $y(x) \in B^*(x, \alpha)\backslash Y$. Put $Y' = \{y(x) : x \in Y\backslash Int(Y, \alpha)\}$, $L = Y' \cup Int(Y, \alpha)$. Note that $Y \subseteq B(L, \alpha)$. Since Y is large, $B(L, \alpha)$ is large. By the multiplicativity of $\mathbf{B}$, L is large. By the assumption, $Y \cap L$ is large. Since $Y \cap L = Int(Y, \alpha)$, $Int(Y, \alpha)$ is large. □

Theorem 2. *Let $\mathbf{B} = (X, P, B)$ be a multiplicative ball's structure. If subsets $X_1, X_2, \dots, X_n$ of X are extralarge, then $X_1 \cap X_2 \cap \dots \cap X_n$ is extralarge. If subsets $S_1, S_2, \dots, S_n$ of X are small, then $S_1 \cup S_2 \cup \dots \cup S_n$ is small. If a piecewise large subset A of X finitely partitioned $A = A_1 \cup A_2 \cup \dots \cup A_n$, then at least one cell A_i of the partition is piecewise large. In particular, X can not be partitioned into finitely many small subsets.*

Proof. Take any large subset L of X. By equivalence $3 \Leftrightarrow 4$ Theorem 1, $X_n \cap L$ is large. Since $(X_1 \cap X_2 \cap \dots \cap X_n) \cap L = (X_1 \cap X_2 \cap \dots \cap X_{n-1}) \cap (X_n \cap L)$, by induction, $(X_1 \cap X_2 \cap \dots \cap X_n) \cap L$ is large. By equivalence $3 \Leftrightarrow 4$ of Theorem 1, $X_1 \cap X_2 \cap \dots \cap X_n$ is extralarge. The second statement follows from the first one and the equivalence $1 \Leftrightarrow 3$ of Theorem 1. The third statement follows from the second statement and the equivalence $1 \Leftrightarrow 2$ of Theorem 1. □

By Theorem 2, the family $\varphi(\mathbf{B})$ of all extralarge subsets of X is a filter on X.

Theorem 3. *Let $\mathbf{B} = (X, P, B)$ be a multiplicative ball's structure and let ψ be an ultrafilter on X. Then $\varphi(\mathbf{B}) \subseteq \psi$ if and only if every subset $A \in \psi$ is piecewise large.*

Proof. Suppose that $\varphi(\mathbf{B}) \subseteq \psi$ and take any subset $A \in \psi$. Assume that A is not piecewise large. By equivalence $1 \Leftrightarrow 2$ of Theorem 1, A is small. By equivalence $1 \Leftrightarrow 3$ of Theorem 1, $X\backslash A$ is extralarge. Hence, $X\backslash A \in \varphi(\mathbf{B})$, a contradiction with $A, X\backslash A \in \psi$.

Suppose that every subset $A \in \psi$ is piecewise large, but $\varphi(\mathbf{B}) \not\subseteq \psi$. Choose any subset $Y \in \varphi(\mathbf{B}), Y \notin \psi$. Since ψ is an ultrafilter, then $X\backslash Y \in \psi$. By equivalence $1 \Leftrightarrow 3$ of Theorem 1, $X\backslash Y$ is small, a contradiction with equivalence $1 \Leftrightarrow 2$ of Theorem 1. □

3. Resolvability of ball's structures. Let $\mathbf{B} = (X, P, B)$ be a ball's structure and let $\mathcal{L}$ be the family of all large subsets of X. A subset $A \subseteq X$ is called *$\mathcal{L}$-dense* if $A \cap L \neq \emptyset$ for every large subset L of X. A ball's structure $\mathbf{B}$ is called *ω-resolvable* if X can be partitioned into countably many $\mathcal{L}$-dense subsets.

Lemma 2. *Let $\mathbf{B} = (X, P, B)$ be a ball's structure. Suppose that there exists a cofinal linearly ordered sequence $\langle \alpha_n \rangle_{n \in \omega}$ of elements of P and a sequence $\langle x_n \rangle_{n \in \omega}$*

of elements of X such that the family $\{B^*(x_n, \alpha_n) : n \in \omega\}$ is disjoint. Then $\mathbf{B}$ is ω-resolvable.

Proof. Let $\omega = \bigcup_{k\in\omega} W_k$ be a partition of ω into countably many infinite subsets. It suffices to show that, for every $k \in \omega$, the subset $A_k = \bigcup_{n\in W_k} B^*(x_n, \alpha_n)$ is $\mathcal{L}$-dense. Take any large subset L of X and pick $\alpha \in P$ such that $X = B(L, \alpha)$. Choose $n \in W_k$ such that $\alpha_n > \alpha$. Since $X = B(L, \alpha_n)$, we get $x_n \in B(L, \alpha_n)$ and $B^*(x_n, \alpha_n) \cap L \neq \emptyset$. Hence, $L \cap A_k \neq \emptyset$. □

The following statement is a generalization of Theorem 5.31 from [2] concerning a resolvability of the ball's structures of groups.

Theorem 4. *Let $\mathbf{B} = (X, P, B)$ be a ball's structure such that the balls $B(x, \alpha)$, $B^*(x, \alpha)$ are finite for all $x \in X$, $\alpha \in P$. If there exists a cofinal linearly ordered sequence $\langle\alpha_n\rangle_{n\in\omega}$ of elements of P, then $\mathbf{B}$ is ω-resolvable.*

Proof. Using the assumptions, construct inductively a sequence $\langle x_n\rangle_{n\in\omega}$ of elements of X such that the family $\{B^*(x_n, \alpha_n)\}$ is disjoint. Then apply Lemma 2. □

4. Applications to semigroups. Let S be a semigroup with the identity e and let Fin be the family of all finite subsets of S containing e. Given any subsets $A, B \subseteq S$, put

$$A^{-1}B = \{s \in S : As \cap B \neq \emptyset\}, \quad AB^{-1} = \{s \in S : sB \cap A \neq \emptyset\}.$$

For every element $s \in S$ and every subset $A \subseteq S$, we write $A^{-1}s$ and sA^{-1} instead of $A^{-1}\{s\}$ and $\{s\}A^{-1}$.

A subset $A \subseteq S$ is called

- *left (right) large* if there exists $F \in Fin$ such that $S = FA$ $(S = AF)$;
- *left (right) small* if the subset $S\backslash FA$ $(S\backslash AF)$ is left(right) large for every subset $F \in Fin$;
- *left (right) extralarge* if $S\backslash A$ is left(right) small;
- *left (right) piecewise large* if there exists $F \in Fin$ such that, for every subset $H \in Fin$, there exists $x \in S$ with $H^{-1}x \subseteq FA$ $(xH^{-1} \subseteq AF)$.

Note that a left (right) size of subset A of semigroup S is exactly a size of A in the ball's structure $\mathbf{B}_l(S)$ ($\mathbf{B}_r(S)$).

A subset $A \subseteq S$ is called

- *left* (right*) large* if there exists $F \in Fin$ such that $S = F^{-1}A$ $(S = AF^{-1})$;
- *left* (right*) small* if $S\backslash F^{-1}A$ $(S\backslash AF^{-1})$ is left* (right*) large for every subset $F \in Fin$;
- *left* (right*) extralarge* if $S\backslash A$ is left* (right*) small;
- *left* (right*) piecewise large* if there exists $F \in Fin$ such that, for every subset $H \in Fin$, there exists $x \in S$ with $Hx \subseteq F^{-1}A$ $(xH \subseteq AF^{-1})$.

In topological dynamics [3], left* (right*) large subsets are called *left* (right*) syndetic* while left* (right*) piecewise large subsets are called *left (right) syndetic.*

Note that a left* (right*) size of subset $A \subseteq S$ is exactly a size of A in the ball's structure $\mathbf{B}_l^*(S)(\mathbf{B}_r^*(S))$.

Theorem 5. *For every finite partition of semigroup S, among the cells of the partition there exist a left piecewise large subset, a right piecewise large subset, a left* piecewise large subset, and a right* piecewise large subset.*

Proof. Apply Theorem 2 to the ball's structures $\mathbf{B}_l(S)$, $\mathbf{B}_r(S)$, $\mathbf{B}_l^*(S)$, $\mathbf{B}_r^*(S)$ respectively. □

Theorem 6. *Let S be a countable semigroup such that the subsets $F^{-1}x$ and xF^{-1} are finite for every subset $F \in Fin$. Then the ball's structures $\mathbf{B}_l(S)$, $\mathbf{B}_r(S)$, $\mathbf{B}_l^*(S)$, $\mathbf{B}_r^*(S)$ are ω-resolvable.*

Proof. Apply Theorem 4. □

Remark 1. By Theorem 6, every countable group G can be partitioned $G = \bigcup_{n\in\omega} A_n$ so that each subset $G\setminus A_n$ is not right large. In particular, there exist a partition $G = B_1 \cup B_2$ such that B_1, B_2 are not right large. Let X be an infinite set of cardinality γ and let $S = S(X)$ be the semigroup of all mappings $X \to X$. A.Ravsky [4] proved that, for every partition $S = \bigcup_{\alpha<\gamma} S_\alpha$, there exist $\alpha < \gamma$ and $s \in S$ such that $S = S_\alpha s$, i.e. at least one cell of the partition is right large. A countable counterpart of this statement was proved in [5]. There exist a countable semigroup S such that, for every finite partition $S = A_1 \cup A_2 \cup \ldots \cup A_n$, there exist $i \leqslant n$ and $s \in S$ such that $S = A_i s$. Obviously, the ball's structure $\mathbf{B}_r(S)$ is not resolvable, i.e. S can not be partitioned into two $\mathcal{L}$-dense subset, where $\mathcal{L}$ is a family of all right large subsets of S.

Remark 2. By [1], every infinite group can be partitioned into countably many subsets such that each of them is left and right small. Ravsky's results concerning $S(X)$ shows that this statement is not valid for all semigroups.

Question [5]. Does there exist an infinite semigroup S such that, for every partition $S = A_1 \cup A_2$, one of the cells A_1, A_2 is left and right large.

5. Application to orgraphs. Let $Gr = (V, E)$ be an oriented graph. By Theorem 2, for every finite partition of V, at least one cell of the partition is piecewise large with respect to the ball's structure $\mathbf{B}(Gr)$. In particular, if V is finite, then there exists a vertex $v \in V$ such that the subset $\{v\}$ is piecewise large. Let us illustrate the last observation.

Let $Gr = (V, E)$ be an arbitrary oriented graph. For every $v \in V$, denote by $St(v)$ (resp. $St^*(v)$) the set of all $x \in V$ such that there exists an oriented path from v to x (resp. from x to v). Define a preordering $\leqslant$ on V by the rule: $v_1 \leqslant v_2$ if and only if $St(v_1) \subseteq St(v_2)$.

Theorem 7. *Let $Gr = (V, E)$ be a finite orgraph and let $v \in V$. Then v is $\leqslant$-maximal if and only if $\{v\}$ is a piecewise large in the ball's structure $\mathbf{B}(Gr)$.*

Proof. Suppose that v is $\leqslant$-maximal. Since V is finite, it suffices to show that $St^*(v) \subseteq St(v)$. Take any $x \in St^*(v)$. Then $v \in St(x)$. By maximality of v, $x \in St(v)$. Hence, $St^*(v) \subseteq St(v)$.

Assume that $\{v\}$ is piecewise large. Since V is finite, then there exists $x \in V$ such that $St^*(x) \subseteq St(v)$. Take any element y with $v \in St(y)$. Then $y \in St^*(x)$, so $y \in St(v)$. Hence, v is $\leqslant$-maximal. □

An orgraph $Gr = (V, E)$ is called *locally finite* if the set $\{y \in V : (x, y) \in E\} \cup \{y \in V : (y, x) \in E\}$ is finite for every $x \in V$

Theorem 8. *Let $Gr = (V, E)$ be an infinite locally finite orgraph. Then the ball's structure $\mathbf{B}(Gr)$ is ω-resolvable.*

Proof. Apply Theorem 4. □

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КОМБІНАТОРНИЙ РОЗМІР ПІДМНОЖИН У НАПІВГРУПАХ НА ОРІЄНТОВАНИХ ГРАФАХ

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Трійка $\mathbf{B} = (X, P, B)$ називається кульовою структурою, якщо X, P – непорожні множини і для довільних $x \in X$ та $\alpha \in P$ в X зафіксовано підмножину $B(x, \alpha) \ni x$, яка називається кулею радіуса α навколо x. Класифікуємо підмножини X за їх розміром щодо кульової структури $\mathbf{B}$, застосовуємо отримані результати до проблеми розкладності напівгруп та орієнтованих графів.

Ключові слова: кульова структура, великі та малі підмножини.

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TOPOLOGICAL BRANDT λ-EXTENSIONS OF ABSOLUTELY H-CLOSED TOPOLOGICAL INVERSE SEMIGROUPS

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Some properties of homomorphic images of Brandt λ-extensions of algebraic semigroups are established. It is proved that for every cardinal $\lambda \geqslant 2$ any topological Brandt λ-extension of an absolutely H-closed topological inverse semigroup is absolutely H-closed in the class of topological inverse semigroups.

Key words: topological inverse semigroup, Brandt λ-extension, topological Brandt λ-extension, H-closed topological semigroup, absolutely H-closed topological semigroup, algebraically h-closed semigroup, topological semilattice, topological semigroup.

In this paper all spaces are Hausdorff.

A *topological* (*inverse*) *semigroup* is a topological space together with a continuous multiplication (and an inversion, respectively).

We follow the terminology of [2, 3, 7].

If S is a semigroup, then by $E(S)$ we denote the band (the subset of idempotents) of S, and by S^1 we denote the semigroup S with the adjoined unit (see: [3]). By ω we denote the first infinite ordinal. Further, we identify all cardinals with their corresponding initial ordinals. If Y is a subspace of a topological space X, and $A \subseteq Y$, then by $\mathrm{cl}_Y(A)$ we denote the topological closure of A in Y.

Let S be a semigroup and I_λ be a set of cardinality $\lambda \geqslant 2$. On the set $B_\lambda(S) = I_\lambda \times S^1 \times I_\lambda \bigcup\{0\}$ we define the semigroup operation " $\cdot$ " as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma, \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$ for $\alpha, \beta, \gamma, \delta \in I_\lambda$, $a, b \in S^1$. The semigroup $B_\lambda(S)$ is called the *Brandt-Howie semigroup of the weight λ over S* [8] or the *Brandt λ-extension of the semigroup S* [9]. Obviously $B_\lambda(S)$ is the Rees matrix semigroup $M^0(S^1; I_\lambda, I_\lambda, \mathcal{M})$, where $\mathcal{M}$ is the $I_\lambda \times I_\lambda$-identity matrix. Further, if $A \subseteq S^1$ then we shall denote $A_{\alpha\beta} = \{(\alpha, s, \beta) \mid s \in A\}$ for $\alpha, \beta \in I_\lambda$. If a semigroup S is trivial (i.e. S contains only one element), then $B_\lambda(S)$ is *the semigroup of $I_\lambda \times I_\lambda$ matrix units* [3] and we shall denote it by B_λ.

Further, by $\mathcal{S}$ we denote some class of topological semigroups.

Definition 1 [9]. Let λ be a cardinal $\geqslant 2$, and $(S, \tau) \in \mathcal{S}$. Let τ_B be a topology on $B_\lambda(S)$ such that

a) $(B_\lambda(S), \tau_B) \in \mathcal{S}$;

b) $\tau_B|_{(\alpha, S^1, \alpha)} = \tau$ for some $\alpha \in I_\lambda$.

Then $(B_\lambda(S), \tau_B)$ is called a *topological Brandt λ-extension of (S, τ) in $\mathcal{S}$*. If $\mathcal{S}$ coincides with the class of all topological semigroups, then $(B_\lambda(S), \tau_B)$ is called a *topological Brandt λ-extension of (S, τ)*.

A semigroup $S \in \mathcal{S}$ is called *H-closed in $\mathcal{S}$*, if S is a closed subsemigroup of any topological semigroup $T \in \mathcal{S}$ which contains S as a subsemigroup. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *H-closed*. H-closed topological semigroups were introduced by J. W. Stepp in [12], and there they were called *maximal semigroups*.

Definition 2 [10, 13]. A topological semigroup $S \in \mathcal{S}$ is called *absolutely H-closed in the class $\mathcal{S}$*, if any continuous homomorphic image of S into $T \in \mathcal{S}$ is H-closed in $\mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *absolutely H-closed*.

An algebraic semigroup S is called *algebraically h-closed in $\mathcal{S}$*, if S with discrete topology $\mathfrak{d}$ is absolutely H-closed in $\mathcal{S}$ and $(S, \mathfrak{d}) \in \mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *algebraically h-closed*.

Absolutely H-closed topological semigroups and algebraically h-closed semigroups were introduced by J. W. Stepp in [13], and there they were called *absolutely maximal* and *algebraic maximal*, respectively.

Obviously, any algebraically h-closed semigroup (in a class $\mathcal{S}$) is absolutely H-closed (in a class $\mathcal{S}$), and every absolutely H-closed topological semigroup (in a class $\mathcal{S}$) is H-closed (in a class $\mathcal{S}$). Further we shall show that the converse statements do not hold.

Recall [1], a topological group G is called *absolutely closed* if G is a closed subgroup of any topological group which contains G as a subgroup. In our terminology such topological groups are called H-closed in the class of topological groups. In [11] D. A. Raikov proved that a topological group G is absolutely closed if and only if it is Raikov complete, i.e. G is complete with respect to the two-sided uniformity.

A topological group G is called *h-complete* if for every continuous homomorphism $h: G \to H$ the subgroup $f(G)$ of H is closed [5]. The h-completeness is preserved under taking products and closed central subgroups [5].

For any $\lambda \geqslant 2$ the semigroup of $I_\lambda \times I_\lambda$ matrix units is a Brandt λ-extension of the trivial semigroup. The semigroup of $I_\lambda \times I_\lambda$ matrix units is algebraically h-closed in the class of topological inverse semigroups for each $\lambda \geqslant 2$ [10]. In [9] it is proved that for every $\lambda \geqslant 2$ any topological Brandt λ-extension of an H-closed topological inverse semigroup is H-closed in the class of topological inverse semigroups. In this paper we show that a similar statements hold for absolutely H-closed topological inverse semigroups and that any Brandt λ-extension of an algebraically h-closed inverse semigroup is algebraically h-closed in the class of topological inverse semigroups.

Proposition 3. *Let $h: B_\lambda(S) \to T$ be a homomorphism, such that $h((\alpha, x, \beta)) = h(0)$ for some $x \in S^1$, $\alpha, \beta \in I_\lambda$. Then $h((\gamma, y, \delta)) = h(0)$ for all $y \in S^1xS^1$, $\gamma, \delta \in I_\lambda$.*

Proof. Assume that $y \in S^1xS^1$. Then $y = axb$ for some $a, b \in S^1$. Therefore

$$h((\gamma,y,\delta)) = h((\gamma,a,\alpha)\cdot(\alpha,x,\beta)\cdot(\beta,b,\delta)) = h((\gamma,a,\alpha))\cdot h((\alpha,x,\beta))\cdot h((\beta,b,\delta)) =$$
$$h((\gamma,a,\alpha))\cdot h(0)\cdot h((\beta,b,\delta)) = h((\gamma,a,\alpha)\cdot 0\cdot(\beta,b,\delta)) = h(0).$$

A semigroup homomorphism $h: S \to T$ is called ***annihilating*** if there exists $c \in T$ such that $h(a) = c$ for all $a \in S$.

Corollary 4. *A homomorphism $h: B_\lambda(S) \to T$ is annihilating if and only if the homomorphism $h|_{B_\lambda}: B_\lambda = B_\lambda(1) \to T$ is annihilating.*

Proposition 5. *Let $h: B_\lambda(S) \to T$ be a homomorphism and $h((\alpha_1,a,\beta_1)) = h((\alpha_2,b,\beta_2))$ for some $a, b \in S^1$, $\alpha_1,\alpha_2,\beta_1,\beta_2 \in I_\lambda$. If $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$ then $h((\alpha_1,a,\beta_1)) = h(0)$.*

Proof. Assume that $\alpha_1 \neq \alpha_2$. Then

$$h((\alpha_1,a,\beta_1)) = h((\alpha_1,1,\alpha_1)(\alpha_1,a,\beta_1)) = h((\alpha_1,1,\alpha_1))\cdot h((\alpha_1,a,\beta_1)) =$$
$$h((\alpha_1,1,\alpha_1))\cdot h((\alpha_2,b,\beta_2)) = h((\alpha_1,1,\alpha_1)\cdot(\alpha_2,b,\beta_2)) = h(0).$$

The proof of the case $\beta_1 \neq \beta_2$ is similar.

Lemma 6. *Let $\lambda \geqslant 2$ and $B_\lambda(S)$ be a topological λ-extension of a topological semigroup S. Let T be a topological semigroup and $h: B_\lambda(S) \to T$ be a continuous homomorphism. Then the sets $h(A_{\alpha\beta})$ and $h(A_{\gamma\delta})$ are homeomorphic in T for all $\alpha,\beta,\gamma,\delta \in I_\lambda$, and $A \subseteq S^1$.*

Proof. If h is an annihilating homomorphism, then the statement of the lemma is trivial.

In the other case we fix $\alpha,\beta,\gamma,\delta \in I_\lambda$. Define the maps $\varphi^{\gamma\delta}_{\alpha\beta}: T \to T$ and $\varphi^{\alpha\beta}_{\gamma\delta}: T \to T$ by the formulae $\varphi^{\gamma\delta}_{\alpha\beta}(s) = h((\gamma,1,\alpha))\cdot s\cdot h((\beta,1,\delta))$ and $\varphi^{\alpha\beta}_{\gamma\delta}(s) = h((\alpha,1,\gamma))\cdot s\cdot h((\delta,1,\beta))$, $s \in T$. Obviously $\varphi^{\alpha\beta}_{\gamma\delta}\left(\varphi^{\gamma\delta}_{\alpha\beta}(h((\alpha,x,\beta)))\right) = h((\alpha,x,\beta))$, $\varphi^{\gamma\delta}_{\alpha\beta}\left(\varphi^{\alpha\beta}_{\gamma\delta}(h((\gamma,x,\delta)))\right) = h((\gamma,x,\delta))$, for all $\alpha,\beta,\gamma,\delta \in I_\lambda$, $x \in S^1$, and hence $\varphi^{\gamma\delta}_{\alpha\beta}|_{A_{\alpha\beta}} = (\varphi^{\alpha\beta}_{\gamma\delta})^{-1}|_{A_{\alpha\beta}}$. Since the maps $\varphi^{\gamma\delta}_{\alpha\beta}$ and $\varphi^{\alpha\beta}_{\gamma\delta}$ are continuous on T, then $\varphi^{\gamma\delta}_{\alpha\beta}|_{h(A_{\alpha\beta})}: h(A_{\alpha\beta}) \to h(A_{\gamma\delta})$ is a homeomorphism.

Proposition 7. *Let $\lambda \geqslant 2$ and $B_\lambda(S)$ be a topological λ-extension of a topological semigroup S. Let T be a topological semigroup and $h: B_\lambda(S) \to T$ be a continuous homomorphism, $A \subseteq h(B_\lambda(S))$, and the set A intersects at least two subsets of the type $h(S_{\alpha\beta})$. Then $h(0) \in A\cdot A$.*

Proof. The case $h(0) \in A$ is trivial. Assume that $h(0) \notin A$, $A \cap h(S_{\alpha_1\alpha_2}) \neq \varnothing$ and $A \cap h(S_{\beta_1\beta_2}) \neq \varnothing$ for some $\alpha_1,\alpha_2,\beta_1,\beta_2 \in I_\lambda$, i.e. there exist $x, y \in S^1$ such that $h((\alpha_1,x,\alpha_2)) \in A$ and $h((\beta_1,y,\beta_2)) \in A$. If $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$, then $h(0) = h((\alpha_1,x,\alpha_2))\cdot h((\alpha_1,x,\alpha_2)) \in A\cdot A$ or $h(0) = h((\beta_1,y,\beta_2))\cdot h((\beta_1,y,\beta_2)) \in A\cdot A$. If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then $\alpha_2 \neq \beta_1$, and hence $h(0) = h((\alpha_1,x,\alpha_2))\cdot h((\beta_1,y,\beta_2)) \in A\cdot A$.

Lemma 8. *Let $\lambda \geqslant 2$, $B_\lambda(S)$ and T be topological semigroups and $h\colon B_\lambda(S) \to T$ be a continuous homomorphism. Let $h(B_\lambda(S))$ be a dense subsemigroup of T and $h(S_{\alpha\beta})$ be a closed subset in T for some $\alpha, \beta \in I_\lambda$. Then $a \cdot a = h(0)$ for all $a \in T\backslash h(B_\lambda(S))$, and $h(0)$ is the zero of T.*

Proof. Since $h(B_\lambda(S))$ is a dense subsemigroup of T, then by Proposition 2 [9], $h(0)$ is the zero of T.

Assume that $a \cdot a = b \neq h(0)$ for some $a \in T\backslash h(B_\lambda(S))$. Then for any open neighbourhood $U(b) \not\ni h(0)$ there exists an open neighbourhood $V(a) \not\ni h(0)$ such that $V(a) \cdot V(a) \subseteq U(b)$. By Lemma 6 the set $h(S_{\gamma\delta})$ is closed for each $\gamma, \delta \in I_\lambda$. Therefore the neighbourhood $V(a)$ intersects infinitely many sets of the type $h(S_{\alpha\beta})$ $(\alpha, \beta \in I_\lambda)$. Then by Proposition 7 we have $h(0) \in V(a) \cdot V(a) \subseteq U(b)$, a contradiction with the choice of $U(b)$.

Proposition 9. *Let $\lambda \geqslant 2$, S and T be algebraic semigroups. Let $h\colon B_\lambda(S) \to T$ be a homomorphism, A and B be disjunctive subsets of $h(B_\lambda(S))$. If the sets A and B intersect at least two subsets of the type $h(S_{\alpha\beta})$ $(\alpha, \beta \in I_\lambda)$, then $h(0) \in A \cdot B$ or $h(0) \in B \cdot A$.*

Proof. The cases $h(0) \in A$ or $h(0) \in B$ are trivial. In the other case for $i = 1, 2, 3, 4$ we fix $\alpha_i, \beta_i \in I_\lambda$ such that $A \bigcap h(S_{\alpha_1\beta_1}) \neq \varnothing$, $A \bigcap h(S_{\alpha_2\beta_2}) \neq \varnothing$, $B \bigcap h(S_{\alpha_3\beta_3}) \neq \varnothing$ and $B \bigcap h(S_{\alpha_4\beta_4}) \neq \varnothing$. By Proposition 5 the sets $h(S_{\alpha_1\beta_1})\backslash h(0)$ and $h(S_{\alpha_2\beta_2})\backslash h(0)$ are disjunctive in $h(B_\lambda(S))$, hence $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. Let x_1, x_2, x_3, x_4 be elements of the semigroup S^1 such that $h((\alpha_1, x_1, \beta_1))$, $h((\alpha_2, x_2, \beta_2)) \in A$ and $h((\alpha_3, x_3, \beta_3)), h((\alpha_4, x_4, \beta_4)) \in B$. If $\alpha_1 \neq \alpha_2$, then $\alpha_1 \neq \beta_3$ or $\alpha_2 \neq \beta_3$, and hence

$$h(0) = h((\alpha_3, x_3, \beta_3) \cdot (\alpha_1, x_1, \beta_1)) = h((\alpha_3, x_3, \beta_3)) \cdot h((\alpha_1, x_1, \beta_1)) \in B \cdot A,$$

or

$$h(0) = h((\alpha_3, x_3, \beta_3) \cdot (\alpha_2, x_2, \beta_2)) = h((\alpha_3, x_3, \beta_3)) \cdot h((\alpha_2, x_2, \beta_2)) \in B \cdot A.$$

If $\beta_1 \neq \beta_2$ then $\beta_1 \neq \alpha_3$ or $\beta_2 \neq \alpha_3$, and hence

$$h(0) = h((\alpha_1, x_1, \beta_1) \cdot (\alpha_3, x_3, \beta_3)) = h((\alpha_1, x_1, \beta_1)) \cdot h((\alpha_3, x_3, \beta_3)) \in A \cdot B,$$

or

$$h(0) = h((\alpha_2, x_2, \beta_2) \cdot (\alpha_3, x_3, \beta_3)) = h((\alpha_1, x_1, \beta_1)) \cdot h((\alpha_3, x_3, \beta_3)) \in A \cdot B.$$

Theorem 10. *Let $\lambda \geqslant 2$, $B_\lambda(S)$ and T be topological inverse semigroups, $h\colon B_\lambda(S) \to T$ be a continuous homomorphism such that the set $h(S_{\alpha\beta})$ be a closed in T for some $\alpha, \beta \in I_\lambda$. Then $h(B_\lambda(S))$ is a closed subsemigroup of T.*

Proof. In the case $2 \leqslant \lambda < \omega$ the statement of the lemma follows from Lemma 6.

Let $\lambda \geqslant \omega$. We denote $G = \mathrm{cl}_T(h(B_\lambda(S)))$. By Proposition II.2 [6], G is a topological inverse semigroup. Let $b \in G\backslash h(B_\lambda(S))$. Then by Lemma 8, $b, b^{-1} \in G\backslash E(G)$. We remark that $b \cdot b^{-1} \neq h(0)$ and $b^{-1} \cdot b \neq h(0)$. Suppose contrary: $b \cdot b^{-1} = h(0)$ or $b^{-1} \cdot b = h(0)$. Since $h(0)$ is the zero of G, then $b = b \cdot b^{-1} \cdot b = h(0) \cdot b = h(0)$ or $b^{-1} = b^{-1} \cdot b \cdot b^{-1} = h(0) \cdot b^{-1} = h(0)$, a contradiction with $b \in G\backslash h(B_\lambda(S))$.

Therefore there exist $e, f \in E(G) = E(h(B_\lambda(S)))$ such that $b \cdot b^{-1} = e$ and $b^{-1} \cdot b = f$. At first we consider the case $e \neq f$. Let $W(e) \not\ni h(0)$ and $W(f) \not\ni h(0)$ be disjunctive open neighbourhoods of e and f in T, respectively. Then there exist disjunctive open neighbourhoods $U(b) \not\ni h(0)$ and $U(b^{-1}) \not\ni h(0)$ in T such that $U(b) \cdot U(b^{-1}) \subseteq W(e)$ and $U(b^{-1}) \cdot U(b) \subseteq W(f)$. By Lemma 6 the set $h(S_{\alpha\beta})$ is closed in T for each $\alpha, \beta \in I_\lambda$, and hence the sets $U(b)$ and $U(b^{-1})$ intersect infinitely many sets of the type $h(S_{\gamma\delta})\backslash h(0)$ $(\gamma, \delta \in I_\lambda)$. thus by Proposition 9 we get $h(0) \in U(b) \cdot U(b^{-1}) \subseteq W(e)$ or $h(0) \in U(b^{-1}) \cdot U(b) \subseteq W(f)$, a contradiction with the choice of the neighbourhoods $W(e)$ and $W(f)$.

In the case $e = f$ we similarly obtain a contradiction.

The obtained contradictions imply the statement of the theorem.

The proof of the following proposition is trivial.

Proposition 11. *If S as absolutely H-closed topological semigroup (in the class of topological semigroups $\mathcal{S}$), then so is S^1 (if $S^1 \in \mathcal{S}$).*

Propositions 6 and 11, and Theorem 10 imply

Theorem 12. *For any cardinal $\lambda \geqslant 2$, every topological Brandt λ-extension $B_\lambda(S)$ of an absolutely H-closed topological inverse semigroup S in the class of topological inverse semigroups, is absolutely H-closed in the class of topological inverse semigroups.*

Corollary 13. *For any cardinal $\lambda \geqslant 2$, every topological Brandt λ-extension $B_\lambda(S)$ of a compact topological inverse semigroup S in the class of topological inverse semigroups, is absolutely H-closed in the class of topological inverse semigroups.*

Theorem 14. *Let S be a topological inverse semigroup. Then the following conditions are equivalent:*

- *(i) S is an absolutely H-closed semigroup in the class of topological inverse semigroups;*
- *(ii) there exists a cardinal $\lambda \geqslant 2$ such that any topological Brandt λ-extension $B_\lambda(S)$ of the semigroup S is absolutely H-closed in the class of topological inverse semigroups;*
- *(iii) for each cardinal $\lambda \geqslant 2$ any topological Brandt λ-extension $B_\lambda(S)$ of the semigroup S is absolutely H-closed in the class of topological inverse semigroups.*

Proof. The implication (iii)⇒(ii) is trivial, and Theorem 12 implies the implications (i)⇒(ii) and (i)⇒(iii).

We shall show that the implication (ii)⇒(i) holds. Suppose contrary: there exists non absolutely H-closed topological inverse semigroup S in the class of topological inverse semigroups, and for some cardinal $\lambda_0 \geqslant 2$ every topological Brandt λ_0-extension $B_{\lambda_0}(S)$ is absolutely H-closed in the class of topological inverse semigroups. Then there exist a topological inverse semigroup T and a continuous homomorphism "into" $h: S \to T$ such that $h(S)$ is not closed subsemigroup of T.

Let τ_S and τ_T be direct sum topologies on $B_{\lambda_0}(S)$ and $B_{\lambda_0}(T)$, respectively (see: [8, p. 129]). Then $(B_{\lambda_0}(S), \tau_S)$ and $(B_{\lambda_0}(T), \tau_T)$ are topological inverse semigroups, S^1 and T^1 are homeomorphic to $S_{\alpha\beta}$ and $T_{\alpha\beta}$, for all $\alpha, \beta \in I_\lambda$ (see: [8, p. 129]). We define the map $\widetilde{h}: B_{\lambda_0}(S) \to B_{\lambda_0}(T)$ as follows: $\widetilde{h}(0) = 0$ and $\widetilde{h}((\alpha, s, \beta)) = (\alpha, h(s), \beta)$ for all $\alpha, \beta \in I_\lambda$, $s \in S^1$. Obviously, the homomorphism $\widetilde{h}: (B_{\lambda_0}(S), \tau_S) \to (B_{\lambda_0}(T), \tau_T)$ is continuous and $\widetilde{h}(B_{\lambda_0}(S))$ is not a closed subgroup of $(B_{\lambda_0}(T), \tau_T)$. Therefore, there exists a topological Brandt λ_0-extension $B_{\lambda_0}(S), \tau_S)$, which is not absolutely H-closed in the class of topological inverse semigroups.

The obtained contradiction implies the statement of the theorem,

The following example shows that there exists an absolutely H-closed topological Brandt λ-extension $B_\lambda(S)$ in the class of topological inverse semigroups of a topological inverse semigroup S, such that S is not absolutely H-closed in the class of topological inverse semigtoups.

Example 15. Obviously, $S = (\mathbb{N}, \max)$ with the discrete topology is a topological semigroup. We define a topology τ_B on $B_2(S)$ as follows:

a) (α, x, β) is an isolated point in $B_2(S)$ for all $\alpha, \beta = 1, 2$, $x \in S$;

b) the family $\mathcal{B}(0) = \{\{\{0\} \bigcup \{(\alpha, x, \beta) \mid \alpha, \beta = 1, 2, x \geqslant k\}\} \mid k \in \mathbb{N}\}$ is a base of the topology τ_B at the point $0 \in B_2(S)$.

It is easy to see that $(B_2(S), \tau_B)$ is a compact topological inverse semigroup, and hence it is absolutely H-closed. But S is not H-closed in the class of topological inverse semigroups.

Theorem 12 implies

Theorem 16. *For each cardinal $\lambda \geqslant 2$, every topological Brandt λ-extension $B_\lambda(S)$ of an algebraically h-closed inverse semigroup S in the class of topological inverse semigroups, is algebraically h-closed in the class of topological inverse semigroups.*

Theorem 14 implies

Theorem 17. *For an inverse semigroup S the following conditions are equivalent:*

(i) S is an algebraically h-closed semigroup in the class of topological inverse semigroups;

(ii) $B_\lambda(S)$ is algebraically h-closed in the class of topological inverse semigroups for some cardinal $\lambda \geqslant 2$;

(iii) $B_\lambda(S)$ is algebraically h-closed in the class of topological inverse semigroups for any cardinal $\lambda \geqslant 2$.

Since the band of a topological semigroup is a closed subset of it, then we have

Proposition 18. *If L is a subsemigroup of the band of a topological semigroup S, the so is $\mathrm{cl}_S(L)$.*

The closure of an Abelian subsemigroup of a topological semigroup is an Abelian semigroup [2, Vol. 1, pp. 9–10], then Proposition 18 implies

Corollary 19. *The closure of a topological semilattice in a topological semigroup is a semilattice.*

Therefore we get

Proposition 20. *A topological semilattice is H-closed if and only if it is H-closed in the class of topological semilattices.*

Since a homomorphic image of a semilattice is a semilattice, then Corollary 19 implies

Proposition 21. *A topological semilattice is absolutely H-closed if and only if it is absolutely H-closed in the class of topological semilattices.*

In [13] J. W. Stepp proved that a semilattice is algebraically h-closed if and only if any chain of it is finite.

Since a maximal subgroup of a topological inverse semigroup is a closed subset, then we have

Proposition 22. *A topological group is [absolutely] H-closed in the class of topological inverse semigroups if and only if it is [absolutely] H-closed in the class of topological groups.*

Absolutely H-closed topological groups in the theory of topological group are called *h-complete* [4]. Complete minimal topologically simple groups and locally compact totally minimal groups are h-complete [4]. There exist non-compact non-Abelian h-complete topological groups, but an h-complete Abelian topological group is compact [4, Example 3.8]. Every locally compact topological group is H-closed in the class of topological groups. Therefore in the class of topological groups the notions a compact group, an absolutely H-closed topological group, and an H-closed topological group, and hence in the class of topological inverse semigroups, are different. We also remark that there exists absolutely H-closed non-compact Abelian Clifford topological inverse semigroup, such as algebraically H-closed infinite semilattices [13].

The following example shows that there exists a Clifford topological inverse semigroup S with a compact band and finite maximal subgroups, such that S is not H-closed in the class of Clifford topological inverse semigroups.

Example 23. Let be $\mathcal{J} = \{0\} \bigcup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ with the usual topology, and operation "max". Then $(\mathcal{J}, \max)$ is a compact semilattice. Let $G = \{e, a\}$ be the two-elements group. Then $S = \mathcal{J} \times G$ with the product topology is a Clifford compact topological inverse semigroup. Obviously $T = S \setminus \{(0, e)\}$ is a subsemigroup of S, and T is not closed subset of S.

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ТОПОЛОГІЧНІ λ-РОЗШИРЕННЯ БРАНДТА АБСОЛЮТНО H-ЗАМКНЕНИХ ТОПОЛОГІЧНИХ ІНВЕРСНИХ НАПІВГРУП

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Вивчено властивості гомоморфних образів λ-розширень Брандта алгебричних напівгруп. Доведено, що для кожного кардинала $\lambda \geqslant 2$ довільне топологічне λ-розширення Брандта абсолютно H-замкненої топологічно інверсної напівгрупи є абсолютно H-замкненим у класі топологічних інверсних напівгруп.

Ключові слова: топологічна напівгрупа, топологічна інверсна напівгрупа, λ-розширення Брандта, H-замкнена топологічна група, абсолютно H-замкнена топологічна напівгрупа, алгебрично h-замкнена напівгрупа, топологічна напівгратка.

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ON ASSOCIATED GROUPS OF RINGS SATISFYING FINITENESS CONDITIONS

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We consider the construction of associated group of a ring with identity element. The characterization of rings with periodic, FC-group, nilpotent associated group are given. It is shown that some finiteness conditions or commutativity of a ring R follow from the finiteness conditions of the associated group $G(R)$.

Key words: associated group of a ring, adjoint group, FC-group, periodic group.

1. Let R be an associative ring with an identity element. The set of all elements of R forms a semigroup with the identity element $0 \in R$ under the operation $a \circ b = a+b+ab$ for all a and b of R. The group of all invertible elements of this semigroup is called the *adjoint group* of R and is denoted by R°. Clearly, if R has the identity 1, then $1 + R^\circ$ coincides with the group of units $U(R)$ of the ring R and the map $a \to 1 + a$ with $a \in R$ is an isomorphism from R° onto $U(R)$.

Many authors have studied the rings with prescribed adjoint groups (or equivalently, groups of units) (see, for example, [1-16]).

This paper is concerned with the question of how properties of associated group influence some characteristic of rings structure. The idea of associated group was introduced in [1] for radical ring. We extend this construction to arbitary associative rings with identity element.

In Sections 3,4,5 we obtain some results on rings determined by their associated groups which are periodic, FC-groups, nilpotent groups. It is proved that finiteness conditions of the associated group $G(R)$ imply some finiteness conditions or commutativity of a ring R.

2. Preliminaries. Let R be an associative ring (not necessarily with identity element) and R° its adjoint group. In the same way as in [1] we consider the set of pair $G(R) = \{(x,y) \mid x \in R, y \in R^\circ\}$ and define an operation by the rule

$$(x,y)(u,v) = (y \cdot u + u + x, y \circ v). \tag{2.1}$$

Definition 2.1. *Let R be an associative ring. Then $G(R) = A \rtimes B$ is a group with the neutral element $(0,0)$ with respect to the operation (2.1), where $A = \{(x,0) \mid x \in R\} \cong R^+$, $B = \{(0,y) \mid y \in R^\circ\} \cong R^\circ$.*

Following [1], the group $G(R)$ will be called the *associated group* of the ring R.

Lemma 2.2. *Let R be an associative ring with associated group $G(R)$. If S is a subring of R with associated group $G(S) = X \rtimes Y$ then following statements are true:*

(i) $G(S) \leqslant G(R)$, $X \leqslant A$, $Y \leqslant B$;

(ii) if S is a left ideal of the ring R, then $X \triangleleft G(R)$;

(iii) if $X \triangleleft G(R)$, then $rS \leqslant S$ for all $r \in R$;

(iv) if S is a right ideal of the ring R, then $G(S) \triangleleft A \rtimes Y$;

(v) if $G(S) \triangleleft A \rtimes Y$, then $S^{\circ} R \leqslant S$;

(vi) if S a two-side ideal of the ring R, then $G(S) \triangleleft G(R)$, $X \triangleleft G(R)$;

(vii) $C_A(B) = \{(a, 0) \mid a \in Ann_r(R^{\circ})\}$, $C_B(A) = \{(0, b) \mid b \in R^{\circ}$ *and* $b \in Ann_l(R)\}$; *in particular, if R is a ring with identity, then $C_B(A) = \langle (0,0) \rangle$ and if R is a domain, then $C_B(A) = C_A(B) = \langle (0, 0) \rangle$.*

Proof. (i) is immediate from Definition 2.1.

(ii) Let S be a left ideal of ring R and $rs \in S$ for all elements $r \in R$ and for all elements $s \in S$. Then for an arbitrary element $(a, b) \in G(R)$ and arbitrary element $(x, 0) \in X$ we have

$$(a, b)^{-1}(x, 0)(a.b) = (b^{(-1)}x + x, 0) \in X, \tag{2.2}$$

hence X is a normal subgroup in $G(R)$.

(iii) If $X \triangleleft G(R)$, then (2.2) implies that $b^{(-1)}x \in S$ for all $b \in R^{\circ}$ and all $x \in S$.

(iv) Let S be a right ideal of the ring R and $sr \in S$ for all $s \in S$, $r \in R$. Then for all elements $(x, y) \in X \rtimes Y$ and all $(a, c) \in A \rtimes Y$ we have

$$\begin{gathered}(a, c)^{-1}(x, y)(a, c) = (-c^{(-1)}a - a, c^{(-1)})(x, y)(a, c) = \\ (ya + c^{(-1)}ya + c^{(-1)}x + x,\ y + c^{(-1)}y + yc + c^{(-1)}yc) \in X \rtimes Y,\end{gathered} \tag{2.3}$$

because $c, y \in S$. Therefore $G(R) \triangleleft A \rtimes Y$.

(v) If $c = 0$, then (2.3) yields $S^{\circ} R \leqslant S$.

(vi) Since S is a two-side ideal of the ring R, for arbitrary elements $(x, y) \in X \rtimes Y$ and $(u, v) \in G(R)$ we have

$$\begin{gathered}(u, v)^{-1}(x, y)(u, v) = \\ (yu + v^{(-1)}yu + v^{(-1)}x + x,\ y + v^{(-1)}y + yv + v^{(-1)}yv) \in G(S).\end{gathered} \tag{2.4}$$

In particular, if $y = 0$ then $(u, v)^{-1}(x, 0)(u, v) = (v^{(-1)}x + x, 0) \in X$, hence $X \triangleleft G(R)$.

(vii) Let $(a, 0) \in C_A(B)$. Then for arbitrary elements $(0, b) \in B$ we have

$$(0, b) = (a, 0)^{-1}(0, b)(a, 0) = (ba, b) \tag{2.5}$$

and consequently $ba = 0$ for all $b \in R^{\circ}$. Therefore $a \in Ann_r(R^{\circ})$. The converse statement is also true.

Let $(0, b) \in C_B(A)$. Then for all elements $(a, 0) \in A$ we have

$$(a, 0) = (0, b)^{-1}(a, 0)(0, b) = (b^{(-1)}a + a, 0) \tag{2.6}$$

and hence $b^{(-1)}a = 0$ for all $a \in R$. It follows that

$$0 = 0 \cdot a = (b + b^{(-1)} + bb^{(-1)})a = ba, \tag{2.7}$$

hence $b \in Ann_l(R)$.

Lemma 2.3. *Let R be a ring and I be an ideal of R such that $I \leqslant J(R)$. Then*

$$G(R/I) \cong G(R)/G(I). \tag{2.8}$$

Proof. Let $G(R) = A \rtimes B$ (respectively $G(I) = X \rtimes Y$, $G(R/I) = C \rtimes D$) be an associated group of the ring R (respectively of the ideal I, of the quotient-ring R/I). Then

$$\begin{aligned} G(R)/G(I) = AB/G(I) \cong AG(I)/G(I) \cdot BG(I)/G(I) = \\ (AXY/XY) \rtimes (BXY/XY) = (AY/XY) \rtimes (XB/XY). \end{aligned} \tag{2.9}$$

Moreover,

$$\begin{aligned} D \cong (R/I)^{\circ} \cong R^{\circ}/I^{\circ} \cong B/Y \cong XB/XY, \\ C \cong (R/I)^{+} \cong R^{+}/I^{+} \cong A/X \cong AY/XY. \end{aligned} \tag{2.10}$$

(2.8) is immediate from the above equations.

The next corollary follows from Lemma 4.2 [3].

Corollary 2.4. *Let S be unital subring of ring R such that $|R^{+} : S^{+}| < \infty$. Then $|G(R) : G(S)| < \infty$.*

3. Rings with Periodic Associated Group. By analogy with Lemma 1.1 [3] the following lemma can be proved.

Lemma 3.1. *Let R be a ring and $J = J(R)$ its Jacobson radical. Then $G(R)$ is a periodic group if and only if J is a nil ideal with periodic additive group J^{+} and the group $G(R/J)$ is periodic.*

Remark 3.2. *It is clear that for any ring R with identity the following statements are equivalent:*

1) the group $G(R)$ is periodic if and only if so is the group of units $U(R)$;

2) $charR$ is finite.

Let us recall that a field T is *absolute* if T is a field of prime characteristic p and T is an algebraic extension of its simple subfields. Hence the multiplicative group T^{*} of an absolute field T is a periodic p'-group.

Lemma 3.3. *Let R be a comutative ring with identity. Suppose that R has no zero divisors and $Q(R)$ its field of quotients. Then $G(Q(R))$ is a periodic group if and only if R is an absolute field.*

Proof. ($\Leftarrow$) Sufficiency of the lemma is clear.

($\Rightarrow$) Suppose that $G(Q(R))$ is a periodic group. Then for all elements $r \in R$ there exists $n = n(r) \in \mathbb{N}$ such that $r^{n} = 1$. Therefore the element r is invertible in R. The lemma is proved.

Theorem 3.4. *Let R be a ring with identity and suppose that R has no zero divisors. Then $G(R)$ is periodic group if and only if the following statements are equivalent:*

1) $P[x]$ is a field, where P is simple subfield of R;

2) the element $x \in R$ is algebraic over P;

3) $x \in U(R)$.

Proof. Necessity. Suppose that the group $G(R)$ is periodic. Then $charR = p$, where p is prime.

(1) $\Rightarrow$ (2). If $P[x]$ is a field, then the element x is invertible. It follows that $x^n = 1$ for some $n \in \mathbb{N}$, hence x is algebraic over P.

(2) $\Rightarrow$ (1). If x is algebraic over P, then the domain $P[x]$ is finite and therefore it is a field.

Implications (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.

Sufficiency. Suppose that the items (1), (2) and (3) are equivalent for the ring R. Assume the contrary, that a is an element of infinite order in the adjoint group R°. Then $1 + a \in U(R)$, hence $P[1 + a]$ is a field and the condition (2) imply that element a is algebraic over P. This contradiction completes the proof.

Corallary 3.5. *Let R be a ring with identity, P be a prime subring of R. If R has no zero divisors, then $R^{\circ} = \{0\}$ if and only if the following statements are true:*

1) $P \cong GF(2)$;

2) any element $x \in R - P$ is transcendental over P;

3) $P[x]$ is not a field for arbitrary element $x \in R - P$.

Proof. Suppose $R^{\circ} = \{0\}$, then $2 = -2$ and therefore $char R = 2$. Assume that there exists an element $a \in R - P$ algebraic over P. Then $P[a]$ is a finite ring without zero divisors. It means that $P[a]$ is a field and $a \in U(R)$, giving a contradiction. So condition (2) is true. Condition (3) is obvious. The converse is trivial.

The rings R with torsion free additive group R^+ and periodic group of units $U(R)$ were studied in paper [5].

Remark 3.6. *If $K[G]$ is a group ring, of a non-trivial group G over a skew field K of zero characteristic, then the group of units $U(K[G])$ is not periodic.*

Indeed, if $char K = 0$, then the prime subfield P of skew field K is isomorphic to $\mathbb{Q}$, but $\mathbb{Q}^*$ is not a periodic group.

Corollary 3.7. *Let $K[H]$ be a group algebra of a group H over a skew field K. Then the following statements are equivalent:*

1) $G(K[H])$ is a periodic group;

2) $U(K[H])$ is a periodic group;

3) K is an absolute field, H is a locally finite group.

Proof. (1) $\Leftrightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Since the groups H and K^* can be embedded in $U(K[H])$, it follows from Lemma 2.1 [15] that K is an absolute field and H is a periodic group.

Let $y_1, \dots, y_n$ be arbitrary elements of the group H. Since $K = \bigcup_{i=1}^{\infty} K_i$, where K_i are finite fields and $K_i[y_1, \dots, y_n]$ are finite domains (hence fields), the subgroup $\langle y_1, \dots, y_n \rangle \leqslant H$ is finite.

(3) $\Rightarrow$ (2). Clearly, for any element $x \in K[H]$ there exists a finite subfield F of the field K such that $x \in F[C]$ for certain finite subgroup C of the group H. Since subring $F[C]$ is finite, the group $U(K[H])$ is periodic.

4. Associated Groups with Finite Conjugacy Classes. A group G is called an *FC-group* if every conjugacy class is finite, i.e., if $|G : C_G(x)| < \infty$ for all element $x \in G$.

Lemma 4.1. *Let R be a ring with identity. Then $G(R)$ is an FC-group if and only if $G(R)$ is a locally normal group.*

Proof. Let $G(R) = A \rtimes B$, where $A \cong R^+$ and $B \cong R^\circ$. If the group R° is not periodic, then by Corollary 3.10 [20] $C_B(A) \neq 1$. But it contradicts Lemma 2.2 (vii). Therefore, the subgroup R° is periodic. Let $(a, 0)$ be an arbitrary element of A. Since $(a,0)^n \in Z(G(R))$ for some $n = n(a) \in \mathbb{N}$, we obtain

$$(na, b) = (na, 0)(0, b) = (0, b)(na, 0) = (bna + na, b). \tag{4.1}$$

Hence,

$$bna = 0 \tag{4.2}$$

for arbitrary non-zero element $a \in R$.

If $char R = 0$, then $(-2)e \in R^\circ$, where e is the identity element of the ring R. From (4.2), if we put $a = e$ we get $nb = 0$ for arbitrary $b \in R^\circ$. It contradicts that the order $|-2e|_+$ is infinite. Therefore $char R = n$ is finite. Thus $G(R)$ is a locally normal group. The converse is trivial. The lemma is proved.

Corollary 4.2. *Let R be a ring with identity. Then $G(R)$ is a fibrewise finite group if and only if R is a finite ring.*

Corollary 4.3. *Let R be a ring with identity. Suppose R has no zero divisors, then $G = G(R)$ is an FC-group if and only if $R^\circ = \{0\}$ or R is a finite field.*

Indeed, if the adjoint group R° is not trivial, then it follows from Lemma 4.1 and fact, that quotient-group $G/C_G(x^G)$ (where $x^G = \langle g^{-1}xg \mid g \in G\rangle$) of FC-group G is finite for all $x \in G$.

Theorem 4.4. *Let R be a ring with identity. If $G = G(R)$ is an FC-group, then $G = A \rtimes B$ is a locally normal group with finite commutant, moreover, the subgroup B is finite, $|G : Z(G)| < \infty$ and $B \cap Z(G) = 1$.*

Proof. Let $G = G(R) = A \rtimes B$ be an FC-group. Then for all element $g \in G$ the quotient-group $G/C_G(g^G)$ is finite. Lemma 4.1 implies that subgroup B is finite. By Lemma 3.10 [20] $|G : Z(G)| < \infty$ and by theorem of Baer the commutant G' is finite.

Corollary 4.5. *Let $K[H]$ be a group algebra of a group H over a field K. Then $G(K[H])$ is an FC-group if and only if the algebra $K[H]$ is finite.*

Proof. Taking into account that the groups H and K^* can be embedded into the adjoint group $(K[H])^\circ$, we see that H and K^* are finite by Theorem 4.4. Therefore, the algebra $K[H]$ is finite as well. The converse is trivial.

5. Rings with nilpotent associated groups.

Lemma 5.1. *Let T be a skew field. Then $G(T)$ is a nilpotent group if and only if $T \cong GF(2)$.*

Proof. ($\Leftarrow$) is obvious.

($\Rightarrow$). If the associated group $G(T)$ is nilpotent, then T is a field of characteristic p for some prime p. Since the field $GF(p)$ embeds in T and by Lemma 2.2 we have $|GF(p)| = p - 1 = 1$, so $p = 2$. Let $p \cong GF(2)$ be a prime subfield of T, then Exercise 9 [19] implies that that $T \supseteq P$ is a finite algebraic extension and $T = P$.

Remark 5.2.

$$U(\mathbb{Z}_{2^n}) \cong \begin{cases} 1, & n=1; \\ \mathbb{Z}_2, & n=2; \\ \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}, & n \geqslant 3. \end{cases} \tag{5.1}$$

The equation above implies that $G(\mathbb{Z}_{2^n})$ is a nilpotent 2-group.

Remark 5.3. *If p is an odd prime and $n \in \mathbb{N}$, then*

$$U(\mathbb{Z}_{p^n}) \cong \mathbb{Z}_{p^{n-1}(p-1)}. \tag{5.2}$$

From Lemma 2.2 (vii) it follows that the group $G(\mathbb{Z}_{p^n})$ is not nilpotent.

Lemma 5.4. *Let R be a ring with identity e and suppose that R has no zero divisors. Then $G(R)$ is a nilpotent group if and only if $char R = 2$ and $R^\circ = \{0\}$.*

Proof. Let $G(R) = A \rtimes B$ be a nilpotent group. Then $C_A(B) \neq 1$ by Proposition 1.6 [20]. According to Lemma 2.2 (vii), B is an identity group and consequently $R^\circ = \{0\}$. Moreover, $char R = 2$. Conversely, if $R^\circ = \{0\}$, then $G(R) \cong R^\circ$ is an abelian group. The lemma is proved.

Below $\mathcal{N}(R)$ will denote the set of all nilpotent elements of a ring R.

Theorem 5.5. *Let R be a ring with identity e. If the associated group $G(R)$ is nilpotent, then $char R = 2^m$ ($m \in \mathbb{N}$). If, thereto, ring R is a commutative, then $R^\circ = \mathcal{N}(R)$.*

Proof. Let additive order $|e|_+ = m$ for some $m \in \mathbb{N} \cup \{0\}$, then the group $G(\mathbb{Z}_m)$ is embedded in $G(R)$ (where $\mathbb{Z}_0 = \mathbb{Z}$). According to Lemma 5.4 $m \neq 0$. If $m = 2^a p_1^{a_1} \dots p_l^{a_l}$ is a canonical decomposition of m, then by Theorem 3 [19]

$$U(\mathbb{Z}_m) \cong U(\mathbb{Z}_{2^a}) \times U(\mathbb{Z}_{p_1}^{a_1}) \times \dots \times U(\mathbb{Z}_{p_l}^{a_l}), \tag{5.3}$$

where $U(\mathbb{Z}_{p_i}^{a_i}) \cong \mathbb{Z}_{p_i^{a_i-1}(p_i-1)}$ and $U(\mathbb{Z}_{2^a})$ is described in Remark 5.2. Remark 5.3 implies $a_1 = \dots = a_l = 0$ and $m = 2^a$.

Let $\bar{R} = R/2R$. If a torsion part $T(\bar{R}^\circ) \neq \{0\}$ then by Lemma 2.2 (vii), $T(\bar{R}^\circ)$ is a 2-group and therefore $T(\bar{R}^\circ) \subset \mathcal{N}(\bar{R})$. Conversely, let $\bar{x} \in \mathcal{N}(\bar{R})$, then $\bar{x}^n = \bar{0}$ for some $n \in \mathbb{N}$. It follows, that the adjoint power $\bar{x}^{(2^s)} = \bar{0}$, where $s \in \mathbb{N}$ is such that $n \leqslant 2^s$. Hence $T(\bar{R}^\circ) = \mathcal{N}(\bar{R})$.

Suppose R is a commutative ring. Then, clearly, $\mathcal{N}(\bar{R})$ is an ideal of R. Let $G(D) = \bar{A} \rtimes \bar{B}$ is a group associated with a ring $D = R/\mathcal{N}(R)$, then $\bar{B}$ is torsion free and $C_{\bar{B}}(\bar{A}) = \bar{1}$. This means, that $\bar{B}$ is embedded in the group $Aut(\bar{A})$ of the subgroup $\bar{A}$.

If $\bar{B}$ is not identity subgroup, then $[\bar{A}, \bar{B}] = \bar{A}$. It contradict to the nilpotency of the group $G(D)$. Hence $\bar{B}$ is an identity subgroup and $R^\circ = \mathcal{N}(R)$. The theorem is proved.

Remark 5.6. *Let $R = \mathbb{Q}[a]$, where $a^2 = 0$. Then R is a local Artinian ring. From the results in [21] we have $R = B + J(R)$, where the field $B = \mathbb{Q}$. It follows that $R^\circ = B^\circ \times J(R)^\circ$ is a mixed abelian group. Assume $(a, 0)$ is non zero element of $G(R)$, then*

$$(a,0)^{-1}(0,-2)(a,0) = (-a,0)(0,-2)(a,0) = (-2a,-2) \notin T(G(R)). \tag{5.4}$$

Since $(0,-2) \in T(G(R))$, *then* $G(R)$ *is a nilpotent group.*

Remark 5.7. *Let* $F = GF(p^n), n \geqslant 2$ *and* σ *is the Frobenius automorphism of the field* F. *Suppose* $F[x,\sigma]$ *is a skew polynomial algebra such that* $xa = \sigma(a)x$ *for all* $a \in F$. *Then* $R = F[x,\sigma]/(x^2)$ *is a local Artinian ring. Since* $R = J(R) + B$, *where field* $B \cong F$, *then* $U(R) \cong (1+J(R)) \rtimes B^*$, *where* $1+J(R)$ *is a p-group,* $|B^*| = p^n - 1$. *As a corollary of [11] we have that the group* $U(R)$ *is not nilpotent.*

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ПРО АСОЦІЙОВАНІ ГРУПИ КІЛЕЦЬ З УМОВАМИ СКІНЧЕННОСТІ

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Розглянуто конструкцію асоційованої групи кільця з одиницею. Охарактеризовано кільця з періодичною, FC-групою, нільпотентною асоційованими групами. Показано, що з умов скінченності для асоційованої групи $G(R)$ випливають певні умови скінченності чи комутативність кільця R.

Ключові слова: асоційована група кільця, приєднана група, FC-група, періодична група.

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LATTICES OF SUBGROUPS OF FINITE GROUPS AND FORMATIONS

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The paper is devoted to the study of formations $\mathfrak{F}$, for which set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of all subgroups in any finite group. The review of the main results on formations with the given property obtained in Gomel algebraic school.

Key words: finite group, lattice of subgroups, subgroups functor, formation.

1. All groups considered are finite. Following Wielandt [1] we say that a subset $\mathfrak{L}$ of subgroups of a group G is a lattice if $A \cap B \in \mathfrak{L}$ and $< A, B > \in \mathfrak{L}$ for any A and B in $\mathfrak{L}$. By classical Wielandt theorem, the set of subnormal subgroups of G is a lattice. There are two generalizations of subnormality in the theory of formations. A formation is a class $\mathfrak{F}$ of groups which is closed under homomorphic images and is such that each group G has unique smallest normal subgroup $G^{\mathfrak{F}}$ (the $\mathfrak{F}$-residual of G) with factor group in $\mathfrak{F}$. Let $\mathfrak{F}$ be a non-empty formation. A subgroup H of G is called:

1) $\mathfrak{F}$-subnormal in G (Carter-Hawkes formation subnormality [2]) if either $H = G$ or there exists a chain

$$G = H_0 \supset H_1 \supset \ldots \supset H_n = H$$

such that H_i is a $\mathfrak{F}$-normal maximal subgroup in H_{i-1} for any $i \in 1, \ldots, n$;

2) $K\mathfrak{F}$-subnormal in G (Kegel formation subnormality [3]) if there exists a chain

$$G = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_n = H$$

such that for every $i \in 1, \ldots, n$ a subgroup H_i is either normal in H_{i-1} or $H_{i-1}^{\mathfrak{F}} \subseteq H_i$.

In 1978 L. A. Shemetkov ([4], problem 12; [5], problem 9.75) and O. Kegel [3] posed a problem of finding conditions under which the set of $\mathfrak{F}$-subnormal ($K\mathfrak{F}$-subnormal) subgroups of G is a lattice.

We will say that a formation $\mathfrak{F}$ has the lattice property (briefly, lattice formation) if the set of all $\mathfrak{F}$-subnormal subgroups is a lattice in every group.

2. Lattice formations. Saturated case. In 1992 at the conference of Byelorussian mathematicians (Grodno) S. F. Kamornikov, V. N. Semenchuk and A. F. Vasil'ev reported on the solution of problems of Kegel and Shemetkov for saturated formations (see [6]). The detailed article was published in the Kiev book dedicated to the memory of the algebraist S. N. Chernicov [7]. First of all, in [7] it was shown that Kegel's problem and Shemetkov's problem are equivalent. Remind that the formation $\mathfrak{F}$ is said to be saturated if the group G belongs to $\mathfrak{F}$ whenever the factor group $G/\Phi(G) \in \mathfrak{F}$.

2.1. Theorem ([7]). *Let $\mathfrak{F}$ be an S-closed (soluble S_n-closed) saturated formation. Then the following statements are equivalent:*

1) the set of all $K\mathfrak{F}$-subnormal subgroups is a lattice in every (soluble) group;

2) the set of all $\mathfrak{F}$-subnormal subgroups is a lattice in every (soluble) group.

Let $\{\mathfrak{F}_i : i \in I\}$ be a set of groups classes. Then $D_0(\cup_{i\in I}\mathfrak{F}_i)$ denotes a class of all groups G which is presented in the form $G = G_{i_1} \times \ldots \times G_{i_t}$ where $i_k \in I$ and $G_{i_k} \in \mathfrak{F}_{i_k}, k = 1, \ldots, t$. In this terminology we will formulate the results which is devoted to the Kegel-Shemetkov problem for saturated formations in the class of all (soluble) groups.

2.2. Theorem ([7]). *Let $\mathfrak{F}$ be an S-closed (soluble S_n-closed) saturated formation. Then the following statements are equivalent:*

1) the set of all $K\mathfrak{F}$-subnormal subgroups is a lattice in every group;

2) $\mathfrak{F}$ can be presented in the form $\mathfrak{F} = D_0(\cup_{i\in I}\mathfrak{S}_{\pi_i})$, where $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$ in I.

Recall that a class $\mathfrak{F}$ of groups is a Fitting class if $\mathfrak{F}$ is closed under taking normal subgroups and in every group G there is a unique normal subgroup that is maximal with respect to being in $\mathfrak{F}$; that subgroup, the $\mathfrak{F}$-radical, will be denoted by $G_{\mathfrak{F}}$.

2.3. Theorem ([7]). *Let $\mathfrak{F}$ be an S-closed saturated formation. Then the following statements are equivalent:*

1) the set of all $\mathfrak{F}$-subnormal subgroups is a lattice in every group;

2) $\mathfrak{F}$ can be presented in the form $\mathfrak{F} = D_0(\mathfrak{M} \cup \mathfrak{H})$, where $\mathfrak{M}$ and $\mathfrak{H}$ are S-closed local formations satisfying the following conditions:

a) $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \emptyset$;

b) $\mathfrak{H} = D_0(\cup_{i\in I}\mathfrak{S}_{\pi_i})$, where $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$ in I;

c) $\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M}$ is a Fitting class which is normal in $\mathfrak{M}\mathfrak{M}$;

d) every non-cyclic minimal non-$\mathfrak{M}$-group G has the following property: $G/\Phi(G)$ is monolithic, $Soc(G/\Phi(G)) = (G/\Phi(G))^{\mathfrak{M}}$ is non-abelian and $G/G^{\mathfrak{M}}\Phi(G)$ is a cyclic group of prime power order.

For the results in this direction also see [8].

3. Lattice formations. Nonsaturated case. The condition of saturation for a formation played an essential role in the proof of the results given above. In [9,10] we give another approach, different from the approach given in [7, 8], which allows to give a constructive description of soluble S-closed formations $\mathfrak{F}$ such that the set of all $\mathfrak{F}$-subnormal ($K\mathfrak{F}$-subnormal) subgroups is a lattice for every group.

3.1. Theorem ([9, 10]). *Let $\mathfrak{F}$ be a soluble S-closed formation. Then the following statements are equivalent:*

1) the set of all $K\mathfrak{F}$-subnormal subgroups in every group is a lattice;

2) the set of all $\mathfrak{F}$-subnormal subgroups in every group is a lattice;

3) there is a partition $\{\pi_i \mid i \in I\}$ *of* $\pi(\mathfrak{F})$ *such that* $\mathfrak{F} = D_0(\cup_{i\in I}\mathfrak{F}_{\pi_i})$, *where* $\mathfrak{F}_{\pi_i} = \mathfrak{F} \cap \mathfrak{S}_{\pi_i}$ *and* $\mathfrak{F}_{\pi_i} = \mathfrak{S}_{\pi_i}$ *if* $|\pi_i| > 1$.

A class of groups which is both a Fitting class and a formation is called a Fitting formation. There are Fitting formations which are neither subgroup-closed nor saturated (see [11]).

3.2. Theorem. *Let* $\mathfrak{F}$ *be a soluble Fitting formation. Then the following conditions are pairwise equivalent:*

1) the set of all $K\mathfrak{F}$*-subnormal subgroups in every (soluble) group is a lattice;*

2) the set of all $\mathfrak{F}$*-subnormal subgroups in every (soluble) group is a lattice;*

3) $\mathfrak{F}$ *can be presented in the form* $\mathfrak{F} = D_0(\cup_{i\in I}\mathfrak{S}_{\pi_i})$, *where* $\pi_i \cap \pi_j = \emptyset$ *for all* $i \neq j$ *in* I.

In order to prove our theorems, we need the following results about the formations with Shemetkov property. We say formation $\mathfrak{F}$ has the Shemetkov property if every minimal non-$\mathfrak{F}$-group is either a Schmidt group or a cyclic group of prime order.

3.3. Proposition ([12]). *Let* $\mathfrak{F}$ *be a soluble* S*-closed formation with the Shemetkov property. Then* $\mathfrak{F}$ *is saturated.*

3.4. Proposition ([13]). *A soluble* S*-closed saturated formation* $\mathfrak{F}$ *is a formation with the Shemetkov property if and only if* $\mathfrak{F} = LF(f)$ *and* f *satisfies the following conditions:*

1) $f(p) = \mathfrak{S}_{\pi(f(p))}$ *for each* $p \in \pi(\mathfrak{F})$;

2) $f(p) = \emptyset$ *for each* $p \notin \pi(\mathfrak{F})$;

3) $f(p) = \mathfrak{N}_p f(p)$ *for each prime* p.

4. Subgroups lattice functors. In spite of the completed results of [7, 8] the issue of the existence in finite groups of other natural lattices similar to the lattice of all subnormal subgroups is still under discussion. In [14] we introduce another (functor) approach to the development of Wielandt results.

Axiomatizing the main properties of subnormal subgroups (invariantness under homomorphism, transitivity, heredity in subgroups), we introduce the notion of the natural transitive lattice functor and describe all the lattices induced by such functors in finite soluble groups.

Let A, B be groups, $\phi : A \to B$ is an epimorphism, and let Ω and Σ be some systems of subgroups from A and B respectively. Further $\Omega^\phi = \{H^\phi | H \in \Omega\}$, and $\Sigma^{\phi^{-1}} = \{H^{\phi^{-1}} | H \in \Sigma\}$ is the full inverse image of all subgroups from Σ in A.

Let Θ be the map, which associates with every group G some non-empty system $\Theta(G)$ of its subgroups. It is reported in [15], that Θ is a group functor, if the condition of abstractness is

$$(\Theta(G))^\phi = \Theta(G^\phi)$$

for every isomorphism ϕ of every group G.

If H is a subgroup of group G, then we write $H \cap \Theta(G) = \{H \cap R | R \in \Theta(G)\}$.

Subgroup functor Θ will be called:

1) natural, if $(\Theta(A))^\phi \subseteq \Theta(B^\phi)$ and $(\Theta(B))^{\phi^{-1}} \subseteq \Theta(A)$ for any epimorphism $\phi : A \to B$, and also $H \cap \Theta(G) \subseteq \Theta(H)$ for any subgroup H of group G;

2) transitive, if $\Theta(H) \subseteq \Theta(G)$ for any subgroup $H \in \Theta(G)$;

3) lattice, if always from $H, K \in \Theta(G)$ follows, that $H \cap K \subseteq \Theta(G)$ and $< H, K > \subseteq \Theta(G)$.

The examples of natural transitive lattice functors (further called NTL-functors) are the functors, which in every finite group G take the set $S(G)$ of its all subgroups; the set $\{G\}$; the set $sn(G)$ of its all subnormal subgroups. Other examples of NTL-functors are given in [7].

There are posed the problem of finding all NTL-functors, given on groups from the given class of groups $\mathfrak{X}$.

The following theorem solves the problem for the case, when $\mathfrak{X}$ is the class of all soluble finite groups.

4.1. Теорема ([14, 15]). *Let Θ be a subgroup NTL-functor. Then*

1) class $\mathfrak{X}_\Theta = \{G | \Theta(G) = S(G)\}$ is an S-closed saturated formation;

2) there exists a partition $\{\pi_i | i \in I\}$ of set $\pi(\mathfrak{X}_\Theta)$, such that $\mathfrak{X}_\Theta = D_0(\cup_{i \in I} \mathfrak{S}_{\pi_i})$;

3) $\Theta(G) = sn_{\mathfrak{X}_\Theta}(G)$ for any group G.

5. Some characterizations and applications of lattice formations. From theorems 2 and 3 it follows that the saturated subgroup-closed lattice formations are Fitting formations, moreover a totally-saturated (primitive in soluble case) Fitting formations. These formations are generalizations of the class of all nilpotent groups in the sense that the groups in the lattice formation are the direct product of all Hall subgroups corresponding to pairwise disjoint sets of primes.

As applicatoins we will show that some well-known properties of the class of all nilpotent groups characterize lattice formations.

It is well-known that nilpotent radical $F(G)$ of a group G can be obtained as the join of the subnormal nilpotent subgroups of G.

5.1. Theorem ([7]). *Let $\mathfrak{F}$ be an S-closed saturated formation. Then the following statements are pairwise equivalent:*

1) $\mathfrak{F}$ is a Fitting class such that the $\mathfrak{F}$-radical $G_\mathfrak{F}$ of a group G is presented in the form

$G_\mathfrak{F} =< H \in \mathfrak{F} | H$ is $K\mathfrak{F}$-subnormal in $G >$;

2) if H and K are two $K\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of a group G, then $< H, K > \in \mathfrak{F}$;

3) $\mathfrak{F}$ is a lattice formation.

In [16] B. Amberg, B. Höfling and L. S. Kazarin studied subgroup-closed formations $\mathfrak{F}$ which are closed under taking groups, which are products of pairwise permutable $\mathfrak{F}$-subgroups. Earlier [17] R. Bryce and J. Cossey described subgroup-closed formations of soluble groups which are closed under taking products of normal $\mathfrak{F}$-subgroups. In this direction the next two theorems are obtained. We consider only soluble groups.

5.2. Theorem ([18]). *Let $\mathfrak{F}$ be a soluble S-closed saturated formation. Then the following statements are pairwise equivalent:*

1) if $G = AB$, where A and B are abnormal $\mathfrak{F}$-subgroups of G, then $G \in \mathfrak{F}$;

2) if the group G have abnormal $\mathfrak{F}$-subgroups A and B such that $(|G : A|, |G : B|) = 1$ than $G \in \mathfrak{F}$;

3) $\mathfrak{F}$ is a lattice formation.

5.3. Theorem ([19]). *Let $\mathfrak{F}$ be a soluble Fitting formation. Then the following statements are pairwise equivalent:*

1) if $G = AB$, where A and B are abnormal $\mathfrak{F}$-subgroups of G, then $G \in \mathfrak{F}$;

2) $\mathfrak{F}$ is S-closed saturated lattice formation.

Following [15, 20], will say that a function $\omega_{\mathfrak{F}} : G \to G^{\mathfrak{F}}$ is a Wielandt-Kegel operator if $< H, K >^{\mathfrak{F}} = < H^{\mathfrak{F}}, K^{\mathfrak{F}} >$ for any two $K\mathfrak{F}$-subnormal subgroups H and K of an arbitrary group G. S. F. Kamornikov investigated this operators in [15, 20–21].

5.4. Theorem ([15, 20]). *Let $\mathfrak{F}$ be an S-closed saturated lattice formation. Then $\omega_{\mathfrak{F}}$ is a Wielandt-Kegel operator.*

5.5. Theorem ([15, 20]). *Let $\mathfrak{F}$ be a soluble formation. Then $\omega_{\mathfrak{F}}$ is a Wielandt-Kegel operator if and only if $\mathfrak{F}$ is S-closed saturated lattice formation.*

In [22] V. S. Monakhov proved that $F(A) \cap F(B) \subseteq F(G)$ for every finite group $G = AB$ which is the product of two subgroups A and B. The result of Johnson [23] says that for every finite soluble group $G = AB$ and for every set of primes π, the maximal normal π-subgroups satisfy $O_\pi(A) \cap O_\pi(B) \subseteq O_\pi(G)$. B. Amberg and L. S. Kazarin in [24] showed that the result of Johnson cannot be extended to arbitrary finite groups and indicated conditions under which this problem has an affirmative solution.

Since classes of all nilpotent groups and all π-groups are Fitting classes it is natural to look for Fitting classes $\mathfrak{F}$ such that $A_{\mathfrak{F}} \cap B_{\mathfrak{F}} \subseteq G_{\mathfrak{F}}$ for every group $G = AB$.

5.6. Theorem ([25]). *For the universe of all soluble groups any two of the following statements about a Fitting formation $\mathfrak{F}$ are equivalent:*

1) if $G = AB$ then $A_{\mathfrak{F}} \cap B_{\mathfrak{F}} \subseteq G_{\mathfrak{F}}$;

2) if $G = AB$ and $A, B \in \mathfrak{F}$ then $A \cap B \subseteq G_{\mathfrak{F}}$;

3) $\mathfrak{F}$ is S-closed saturated lattice formation.

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ГРАТКИ ПІДГРУП СКІНЧЕННИХ ГРУП І ФОРМАЦІЇ

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Вивчено формації $\mathfrak{F}$, для яких множина всіх $\mathfrak{F}$-субнормальних ($\mathfrak{F}$-досяжних) підгруп утворює підгратку гратки всіх підгруп довільної скінченної групи. Розглянуто основні результати про формації з заданою властивістю, одержані в Гомельській алгебричній школі.

Ключові слова: скінченна група, гратка підгруп, підгруповий функтор, формація.

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ABSORBING SETS RELATED TO HAUSDORFF DIMENSION

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It is proved that the hyperspace of compact sets in the n-dimensional cube $\mathbb{I}^n$ of the Hausdorff dimension $> \alpha$, $0 < \alpha < n$, forms an $\mathcal{F}_\sigma$-absorber in the hyperspace $\exp(\mathbb{I}^n)$ homeomorphic to the Hilbert cube. Moreover, for arbitrary sequence (α_i), $0 < \alpha_1 < \alpha_2 < \ldots < n$, the sequence of hyperspaces of compact sets in $\mathbb{I}^n$ of the Hausdorff dimension $> \alpha_i$ forms an $\mathcal{F}_\sigma$-absorbing sequence in $\exp(\mathbb{I}^n)$.

Key words: hyperspace, Hausdorff dimension, Hilbert cube, absorbing system.

The classical result of West, Curtis, and Schori asserts that the hyperspace of any nondegenerate Peano continuum is homeomorphic to the Hilbert cube. This allows us to apply methods of infinite-dimensional topology to investigation of classes of sets with prescribed geometric properties.

In particular, in a series of papers [1],[2],[3],[5], the topology of the hyperspace of sets of given Lebesgue dimension (see also [1] for the case of cohomological dimension) is described. In this note we consider the case of the Hausdorff dimension.

PRELIMINARIES. A typical metric will be denoted by d. By $\mathrm{diam}(A)$ we denote the diameter of a subset A in a metric space. Given a cover $\mathcal{U}$ of a metric space, we define $\mathrm{mesh}(\mathcal{U})$ as $\sup\{\mathrm{diam}(U)|U \in \mathcal{U}\}$. For $x \in X$ and $\varepsilon > 0$ the set $O_\varepsilon(x) = \{y \in X|d(x,y) < \varepsilon\}$ is an *open ε-ball* centered at x.

By Q we denote the Hilbert cube, $Q = \prod_{i=1}^{\infty}[-1,1]_i$. The class of absolute neighborhood retracts is denoted by ANR. A closed subset A of $X \in \mathrm{ANR}$ is called a *Z-set* in X if for every continuous function $\varepsilon: X \longrightarrow (0,\infty)$ there exists a map $f: X \longrightarrow X\setminus A$ which is ε-close to the identity in the sense that $d(x, f(x)) < \varepsilon(x)$, for every $x \in X$. An embedding $g: Y \longrightarrow X$ is called a *Z-embedding* if its image $g(Y)$ is a Z-set in X. By $B(Q) = Q\setminus \prod_{i=1}^{\infty}(-1,1)_i$ we denote the *pseudoboundary* of Q.

HYPERSPACES. Let X be a metric space. The hyperspace of X is the space $\exp X$ of nonempty compact subsets of X endowed with the Vietoris topology. A base of this topology consists of the sets

$$\langle V_1,\ldots,V_n\rangle = \{A \in \exp X|A \subset \bigcup_{i=1}^{n} V_i \text{ and for every } i \in \{1,2,\ldots,n\}\ A\cap V_i \neq \emptyset\},$$

where $V_1, ..., V_n$ run over the topology of X. The Vietoris topology is generated by the Hausdorff metric d_H,

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\}.$$

For $n \in \mathbb{N}$, we denote by $\exp_n X$ the subspace of $\exp X$ consisting of sets of cardinality $\leqslant n$. Let $\exp_\omega X = \cup\{\exp_n X | n \in \mathbb{N}\}$.

HAUSDORFF DIMENSION. Let F be a subset of $\mathbb{R}^n$ for some n and s a non-negative number. For $\varepsilon > 0$ define

$$\mathcal{H}^s_\varepsilon(F) = \inf_{\mathcal{B}} \sum_{B \in \mathcal{B}} (\mathrm{diam} B)^s,$$

where the infimum is over all covers $\mathcal{B}$ of F with $\mathrm{mesh}(\mathcal{B}) < \varepsilon$.

Let $\mathcal{H}^s(F) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(F)$. There exists a unique number s_0, the *Hausdorff dimension* of F, such that $\mathcal{H}^s(F) = \infty$ whenever $0 \leqslant s < s_0$ and $\mathcal{H}^s(F) = 0$ whenever $s_0 < s < \infty$. We write $\dim_H F = s_0$.

Proposition. *For every $\alpha \geqslant 0$ the set $C_\alpha = \{A \in \exp \mathbb{R}^n \mid \dim_H(A) \leqslant \alpha\}$ is a G_δ-subset of $\exp \mathbb{R}^n$.*

Proof. For every $A \in C_\alpha$, by the definition of Hausdorff dimension, $\mathcal{H}^{\alpha+1/i}(A) = 0$, for every $i \in \mathbb{N}$. Therefore, for every $A \in C_\alpha$ there exist open sets $U_{m_1}, ..., U_{m_k}$, which are elements of a fixed countable base $\mathcal{U}$ of $\mathbb{R}^n$, such that

$$A \in \langle U_{m_1}, ..., U_{m_k}\rangle \text{ and } \sum_{j=1}^{k} (\mathrm{diam} U_{m_j})^{\alpha+1/i} < 1/i.$$

Let

$$V_i = \cup\{\langle U_{m_1}, ..., U_{m_k}\rangle \mid \sum_{j=1}^{k} (\mathrm{diam} U_{m_j})^{\alpha+1/i} < 1/i,\ U_{m_1}, ..., U_{m_k} \in \mathcal{U}\}.$$

We have just shown that $C_\alpha \subset \bigcap_{i=1}^{\infty} V_i$. Prove the inclusion $\bigcap_{i=1}^{\infty} V_i \subset C_\alpha$. Assuming the opposite, choose $B \in \exp \mathbb{R}^n$ such that $\dim_H B = s > \alpha$ and $B \in \bigcap_{i=1}^{\infty} V_i$. Then there is $i_0 \in \mathbb{N}$ such that $\alpha + 1/i < s$ for every $i \in \mathbb{N}$, $i \geqslant i_0$.

We therefore have $\mathcal{H}^{\alpha+1/i}(B) = \infty$ for all $i \in \mathbb{N}$, $i \geqslant i_0$. Taking into account that $\mathcal{H}^{\alpha+1/i}_\varepsilon(B) > 0$, we conclude that $l(i) = \inf\{\mathcal{H}^{\alpha+1/i}_\varepsilon(B) | 0 < \varepsilon \leqslant 1\} > 0$.

The function $l(i)$ is an increasing function of i. Thus, there is $i_1 \in \mathbb{N}$ with $l(i_1) > 1/i_1$. Then obviously $B \notin V_{i_1} \supset \bigcap_{i=1}^{\infty} V_i$ and we obtain a contradiction.

We have proven that $C_\alpha = \bigcap_{i=1}^{\infty} V_i$. Since V_i are open in $\exp \mathbb{R}^n$, this completes the proof. □

ABSORBING SYSTEMS. We briefly recall some definitions from the theory of absorbing systems; see [3], [4] for details.

Let Γ be an ordered set and $\mathcal{M}_\gamma$ a class of metric spaces for $\gamma \in \Gamma$. Put $\mathcal{M}_\Gamma = (\mathcal{M}_\gamma)_{\gamma\in\Gamma}$. An $\mathcal{M}_\Gamma$-*system* in a space X is an order preserving indexed collection $(A_\gamma)_{\gamma\in\Gamma}$ of subsets of X such that $A_\gamma \in \mathcal{M}_\gamma$ for every γ.

An $\mathcal{M}_\Gamma$-system $\mathcal{X}$ in $X \in$ ANR is called ***strongly*** $\mathcal{M}_\Gamma$-***universal*** in X if for every $\mathcal{M}_\Gamma$-system (A_γ) in Q, every map $f\colon Q \longrightarrow X$ that restricts to a Z-embedding on some compact subset K of Q can be approximated by a Z-embedding $g\colon Q \longrightarrow X$ such that $g|K = f|K$ and for every $\gamma \in \Gamma$ we have $g^{-1}(X_\gamma)\backslash K = A_\gamma\backslash K$.

An $\mathcal{M}_\Gamma$-system $\mathcal{X}$ is called $\mathcal{M}_\Gamma$-***absorbing*** in X if the set $\bigcup_{\gamma\in\Gamma} X_\gamma$ is contained in a σ-compact σ-Z-set in X and $\mathcal{X}$ is strongly $\mathcal{M}_\Gamma$-universal in X.

By $\mathcal{F}_\sigma$ we denote the class of σ-compact spaces.

MAIN RESULT.

Lemma 1. *Let $n \in \mathbb{N}$. For every continuous function $f\colon Q \longrightarrow \exp(\mathbb{I}^n)$ that restricts to a Z-embedding on some compact subset K of Q and for every $\varepsilon > 0$ there is a Z-embedding $h\colon Q \longrightarrow \exp(\mathbb{I}^n)$ such that $h|K = f|K$, for every $x \in Q\backslash K$ $d_H(f(x), h(x)) < \varepsilon$ and $\dim_H(h(x)) = 0$.*

Proof. Consider a sequence of compact subsets $\{B_i\}_{i=1}^\infty$ in $\mathbb{I}^n$ defined as follows:

$$B_1 = \frac{1}{2}\cdot\mathbb{I}^n;$$

$$B_2 = \frac{1}{2^2}\cdot\mathbb{I}^n + \frac{1}{2}\cdot y_0;$$

$$\dots$$

$$B_k = \frac{1}{2^k}\cdot\mathbb{I}^n + \left(\frac{1}{2} + \dots + \frac{1}{2^{k-1}}\right)\cdot y_0,$$

$$\dots$$

where $y_0 = (1, 1, \dots, 1)$.

Let α_i be an embedding $[-1, 1]$ into B_i. For every $x \in Q$, $x = (x_i)_{i=1}^\infty$ let $\hat{x} \in Q$ be defined as follows:

$$\hat{x} = (x_1, x_1, x_2, x_1, x_2, x_3, x_1, x_2, x_3, x_4, \dots).$$

Let the map ξ be given by the formula

$$\xi(x) = \bigcup_{i=1}^{\infty} \alpha_i(\hat{x}_i) \cup \{y_0\}.$$

It is clear that for every $x \in Q$, $\xi(x)$ is a compact subset in $\mathbb{I}^n$. On the other hand, $\xi(x)$ is a countable subset of $\mathbb{I}^n$, therefore, $\dim_H(\xi(x)) = 0$.

Choose two points $x, x' \in Q$, $x = (x_i)_{i=1}^\infty$, $x' = (x'_i)_{i=1}^\infty$. If $x \neq x'$, then there is $i \in \mathbb{N}$ such that $x_i \neq x'_i$. In this case for some $j \in \mathbb{N}$, $\alpha_j(\hat{x}_j) \neq \alpha_j(\hat{x}'_j)$. Therefore, $\xi(x) \neq \xi(x')$. This implies that ξ is an injective map.

Let $\varepsilon > 0$. Let $f\colon Q \longrightarrow \exp(\mathbb{I}^n)$ be a map that restricts to a Z-embedding on some compact subset K of Q. Without loss of generality we may assume that f is a Z-embedding because $\exp(\mathbb{I}^n)$ is homeomorphic to the Hilbert cube (see [4]). Define $\mu\colon Q \longrightarrow [0, 1]$ by $\mu(x) = \frac{1}{3}\cdot\min\{\varepsilon, d_H(f(x), f[K])\}$. The set $\exp(\mathbb{I}^n)\backslash \exp_\omega(\mathbb{I}^n)$

is locally homotopy negligible in $\exp(\mathbb{I}^n)$(see [4]). Therefore, there is a homotopy $H: \exp(\mathbb{I}^n) \times \mathbb{I} \longrightarrow \exp(\mathbb{I}^n)$ such that

1) $H_0 = 1_{\exp(\mathbb{I}^n)}$;
2) for every $t \in (0,1]$, $H_t(\exp(\mathbb{I}^n)) \subseteq \exp_\omega(\mathbb{I}^n)$.

It is clear that we may additionally assume that

3) for every $t \in [0,1]$, $\hat{d}_H(H_t, 1_{\exp(\mathbb{I}^n)}) \leqslant 2t$;
4) for every $t \in (0,1]$, $H_t(\exp(\mathbb{I}^n)) \subseteq \exp_\omega([0, 1-3t/4]^n)$.

For every $x \in Q$, let $F(x) = H(f(x), \mu(x))$. Then, if $\mu(x) > 0$, $F(x)$ is a finite approximation of $f(x)$.

Now define $h: Q \longrightarrow \exp(\mathbb{I}^n)$ as follows:

$$h(x) = F(x) \cup \bigcup_{y \in F(x)} [\mu(x)/4 \cdot \xi(x) + y].$$

CLAIM 1. The map h is well-defined, continuous and satisfies $h|K = f|K$. Moreover, for every $x \in Q$, $d_H(f(x), h(x)) \leqslant \frac{3}{4}\min\{\varepsilon, d(f(x), f[K])\}$ and for every $x \in Q \backslash K$, $\dim_H(h(x)) = 0$.

a) Let $x \in Q$. Then by (4), $F(x) \subseteq [0, 1-3\mu(x)/4]^n$. For every $y \in F(x)$, the diameter of the set $[\mu(x)/4 \cdot \xi(x) + y]$ does not exceed $\mu(x)/4$, which implies that $h(x) \subseteq [0, 1-\mu(x)/2]^n$.

b) If $\mu(x) > 0$, then $h(x)$ is compact and non-empty, being a finite union of compact non-empty sets. If $\mu(x) = 0$, then $h(x) = f(x)$ which is also compact and non-empty. Therefore for every $x \in Q$, $h(x) \in \exp(\mathbb{I}^n)$.

c) That h is continuous follows from the continuity of the involved maps.

d) If $\mu(x) > 0$, then $h(x)$ is a finite union of countable sets. Therefore for every $x \in Q \backslash K$ $\dim_H(h(x)) = 0$.

e) Fix $x \in Q$. It is clear that $d_H(f(x), h(x)) \leqslant 2 \cdot \mu(x) + \mu(x)/4 = 9 \cdot \mu(x)/4$, from which it follows that $d_H(f(x), h(x)) \leqslant 3/4 \cdot \min\{\varepsilon, d_H(f(x), f[K])\}$. So we are done because this inequality implies that $h|K = f|K$.

CLAIM 2. The map h is injective.

Let us first observe that from Claim 1 and the fact that f is an embedding it follows that

$$h[Q \backslash K] \cap h[K] = \emptyset. \qquad (*)$$

Now fix $x, x' \in Q$. If both x and x' belong to K, then since $h|K = f|K$ and since f is an embedding, it is trivial that $h(x) = h(x')$ implies $x = x'$. If $x \notin K$ and $x' \in K$, then from $(*)$ it follows that $h(x) \neq h(x')$. So without loss of generality we may assume that $x, x' \in Q \backslash K$.

Let $h(x) = h(x')$. Our task is to show that $x = x'$. We will first prove that $\mu(x) = \mu(x')$. Assume the contrary, e.g. assume $\mu(x) < \mu(x')$. For some $y \in F(x)$, consider in $\mathbb{I}^n$ the set $B_y = (\mu(x)/4) \cdot \mathbb{I}^n + y$. There exists a point $m \in h(x)$ such that $|m| \leqslant |p|$ for all $p \in h(x)$. Moreover, this point m is an element of $F(x) \cap F(x')$ (by construction of the map h and since $h(x) = h(x')$). For this m, we see that $B_m \cap h(x)$ is infinite while $B_m \cap h(x')$ is a finite set, being a finite union of finite sets. This contradiction establishes that $\mu(x) = \mu(x')$.

Again consider the point $\hat{m} = (m_1, \dots, m_n) \in h(x)$ such that $|p| \leqslant |\hat{m}|$ for every $p \in h(x)$. Since $\mu(x) = \mu(x')$, we have

$$m^* = (m_1 - \mu(x)/4, \dots, m_n - \mu(x)/4) \in F(x) \cap F(x').$$

Since $F(x)$ and $F(x')$ are finite, $\hat{m}$ is maximal, there are a neighborhood U of $\hat{m}$ and a $\delta \in (0,1]$ such that

$$U \cap h(x) = m^* + \mu(x)/4(\xi(x) \cap O_\delta(y_0)) = \\ = m^* + \mu(x')/4(\xi(x') \cap O_\delta(y_0)).$$

Since the coordinates of x appear infinitely often in the coordinates of $\hat{x}$, and the same is true for x', it now easily follows that $x = x'$.

CLAIM 3. The map h is a Z-embedding.

Since $h[K] = f[K]$ is a Z-set, it suffices to show that $h[Y]$ is a Z-set if $Y \subseteq Q \setminus K$ is compact. But this easily follows from the fact that the map $h' \colon Q \longrightarrow \exp(\mathbb{I}^n)$ defined by

$$h'(x) = \bigcup_{y \in F(x)} \overline{O_\delta(y)} \cup [\mu(x)/4 \cdot \xi(x) + y]$$

maps Q into the complement of $h[Y]$, for every positive δ, and is δ-close to the identity. This completes the proof of the Lemma 1. □

Theorem 1. *If $n \geqslant 1$ and $\alpha \in (0,n)$, then the set $D_{>\alpha}(\mathbb{I}^n) = \{A \in \exp(\mathbb{I}^n) | \dim_H A > \alpha\}$ is strongly $\mathcal{F}_\sigma$-universal in* $\exp(\mathbb{I}^n)$.

Proof. Let $\varepsilon > 0$. Choose a sequence $A_1 \subseteq A_2 \subseteq \dots$ of compact subset in the Hilbert cube Q and let $A = \bigcup_{n=1}^{\infty} A_n$. Let $f \colon Q \longrightarrow \exp(\mathbb{I}^n)$ be a map that restricts to a Z-embedding on some compact subset K of Q. Let $\mu \colon Q \longrightarrow [0,1]$, $H \colon \exp(\mathbb{I}^n) \times \mathbb{I} \longrightarrow \exp(\mathbb{I}^n)$, $F \colon Q \longrightarrow \exp(\mathbb{I}^n)$ be maps, as in the proof of Lemma 1.

For every $t \in [0,1]$ let $\phi \colon \mathbb{I} \longrightarrow \exp(\mathbb{I}^n)$ be defined as follows: $\phi(t) = H_t(\mathbb{I}^n)$. Then, it is clear that $\phi(0) = \mathbb{I}^n$ and $\phi((0,1]) \subseteq \exp_\omega(\mathbb{I}^n)$.

Let $\{B_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of $\mathbb{I}^n$, as in the proof of Lemma 1, let $\beta_i \colon \mathbb{I}^n \longrightarrow B_i$ be a homeomorphism. For some $\lambda \in (0,1]$ and $y \in \mathbb{I}^n$ define $(\beta_i)_y^\lambda = \lambda \beta_i + y + \lambda y_0$, where $y_0 = (1, 1, \dots, 1)$.

Let $h \colon Q \longrightarrow \exp(\mathbb{I}^n)$ be a map that satisfies the conditions of Lemma 1.

Now define $g \colon Q \longrightarrow \exp(\mathbb{I}^n)$ as follows

$$g(x) = h(x) \cup \bigcup_{y \in F(x)} \left[\bigcup_{i=1}^{\infty} (\beta_i)_y^{\mu(x)/4}(\phi(d(x, A_i))) \cup \{\mu(x)/2 \cdot y_0 + y\} \right] \\ \cup \{h(x) + \mu(x)/2 \cdot y_0\}.$$

We claim that g is a required map, i.e., g is an approximation of f with the properties stated in the definition of strong $\mathcal{F}_\sigma$-universality.

CLAIM 1. The map g is well-defined, continuous, and satisfies $g|K = f|K$. Moreover, for every $x \in Q$, $d_H(f(x), g(x)) \leqslant \frac{11}{12} \min\{\varepsilon, d(f(x), f[K])\}$.

a) Let $x \in Q$. Then by Lemma 1, $h(x) \subseteq [0, 1 - \mu(x)/2]^n$. For every $y \in F(x)$, the diameter of the set $\bigcup_{i=1}^{\infty} (\beta_i)_y^{\mu(x)/4}(\phi(d(x, A_i)))$ does not exceed $\mu(x)/4$, which implies that $g(x) \subseteq \mathbb{I}^n$.

b) If $\mu(x) > 0$, then $g(x)$ is compact and non-empty, being a finite union of compact non-empty sets. If $\mu(x) = 0$, then $g(x) = f(x)$ which is also compact and non-empty. Therefore for every $x \in Q$, $g(x) \in \exp(\mathbb{I}^n)$.

c) That g is continuous follows from the continuity of the involved maps.

d) Fix $x \in Q$. It is clear, by the proof of Lemma 1, that $d_H(f(x), g(x)) \leqslant 9/4 \cdot \mu(x) + \mu(x)/2 = 11 \cdot \mu(x)/4$, from which it follows that $d_H(f(x), g(x)) \leqslant 11/12 \cdot \min\{\varepsilon, d_H(f(x), f[K])\}$. So we are done because this inequality implies that $g|K = f|K$.

CLAIM 2. The map g is injective.

Injectivity of g follows from injectivity of h and construction of the map g.

CLAIM 3. We have $g^{-1}[D_{>\alpha}(\mathbb{I}^n)] \backslash K = A \backslash K$.

By analogy to the proof of Lemma 1, we first observe that from Claim 1 and the fact that f is an embedding it follows that

$$g[Q \backslash K] \cap g[K] = \emptyset. \qquad (*)$$

Choose $x \in Q \backslash K$. If $x \in A_k$ for certain k, then $d(x, A_k) = 0$. This implies that $\phi(d(x, A_k)) = \mathbb{I}^n$. In this case, we see that $g(x)$ contains the n-dimensional cube and this implies that $\dim_H(g(x)) \geqslant n$. Therefore, $g(x) \in D_{>\alpha}(\mathbb{I}^n)$.

If $x \notin A$, then $d(x, A_k) > 0$ for every $k \in \mathbb{N}$ and $\phi(d(x, A_k))$ is a finite set for all $k \in \mathbb{N}$. In this case, by construction, $g(x)$ is a countable set, being a countable union of finite sets. This implies that $\dim_H(g(x)) = 0$. Therefore, $g(x) \notin D_{>\alpha}(\mathbb{I}^n)$. Equality $(*)$ completes the proof of Claim 3.

CLAIM 4. The map g is a Z-embedding.

Follows from the same results for the map h.

This completes the proof of Theorem 1. □

Corollary. *In the assumptions of Theorem 1, the pair* $(\exp(\mathbb{I}^n), D_{>\alpha}(\mathbb{I}^n))$ *is homeomorphic to* $(Q, B(Q))$.

Proof follows from the standard results of the theory of absorbing sets in Q; see [4]. □

Theorem 2. *If* $n \geqslant 1$ *and* $\Gamma = \{\gamma_k\}_{k=1}^{\infty}$ *is a countable ordered set, where* $0 < \gamma_1 < \ldots < \gamma_k < \ldots < n$ *then the sequence* $\{D_{>\gamma_k}(\mathbb{I}^n)\}_{k=1}^{\infty}$ *is strongly* $\mathcal{F}_\sigma$*-universal in* $\exp(\mathbb{I}^n)$.

Proof. Let $\varepsilon > 0$. Choose a decreasing sequence of σ-compact subsets $\{\mathcal{A}_m\}_{m=1}^{\infty}$ in Q and a map $f: Q \longrightarrow \exp(\mathbb{I}^n)$ that restricts to a Z-embedding on some compact subset K of Q.

Write $\mathbb{N}$ as the disjoint union of infinitely many infinite sets, say, $N_1, N_2, \ldots$. It is clear that for every $\gamma \in (0, n]$ there is a set $C \in \exp(\mathbb{I}^n)$ such that $\dim_H(C) = \gamma$. For $p \geqslant 1$ and $i \in N_p$, let $C_i \in \exp(\mathbb{I}^n)$ be a set such that $\dim_H(C_i) = \gamma_{p+1}$.

Since the set $\exp(\mathbb{I}^n)\setminus\exp_\omega(\mathbb{I}^n)$ is locally homotopy negligible in $\exp(\mathbb{I}^n)$(see [4]), we can find a continuous function $\phi_i\colon \mathbb{I}\longrightarrow\exp(\mathbb{I}^n)$ such that $\phi_i(0)=C_i$ and $\phi_i((0,1])\subseteq\exp_\omega(\mathbb{I}^n)$.

Let $\mu\colon Q\longrightarrow[0,1]$, $F\colon Q\longrightarrow\exp(\mathbb{I}^n)$ be maps, as in the proof of Lemma 1.

For every $m\geqslant 1$ write $\mathcal{A}_m=\bigcup_{p=1}^{\infty}A_m^p$, where A_m^p are compact subsets of Q. Let $i(m,p)$ be the pth element of N_m.

Let $\{B_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of $\mathbb{I}^n$, as in the proof of Lemma 1, and $\beta_i\colon\mathbb{I}^n\longrightarrow B_i$ be a homeomorphism. For some $\lambda\in(0,1]$ and $y\in\mathbb{I}^n$ define $(\beta_i)_y^\lambda=\lambda\beta_i+y+\lambda y_0$, where $y_0=(1,1,\ldots,1)$.

Let $h\colon Q\longrightarrow\exp(\mathbb{I}^n)$ be a map that satisfies the conditions of Lemma 1.

Now define $g\colon Q\longrightarrow\exp(\mathbb{I}^n)$ as follows:

$$g(x)=h(x)\cup\bigcup_{y\in F(x)}\left[\bigcup_{m=1}^{\infty}\bigcup_{p=1}^{\infty}(\beta_{i(m,p)})_y^{\mu(x)/4}(\phi_{i(m,p)}(d(x,A_m^p)))\cup\{\mu(x)/2\cdot y_0+y\}\right]$$
$$\cup\{h(x)+\mu(x)/2\cdot y_0\}\,.$$

We claim that g is a required map, i.e., g is an approximation of f with the properties stated in the definition of strong $\mathcal{F}_\sigma$-universality.

CLAIM 1. The map g is well-defined, continuous and satisfies $g|K=f|K$. Moreover, for every $x\in Q$, $d_H(f(x),g(x))\leqslant\frac{11}{12}\min\{\varepsilon,d(f(x),f[K])\}$.

a) Let $x\in Q$. Then by Lemma 1, $h(x)\subseteq[0,1-\mu(x)/2]^n$. For every $y\in F(x)$, the diameter of the set $\bigcup_{m=1}^{\infty}\bigcup_{p=1}^{\infty}(\beta_{i(m,p)})_y^{\mu(x)/4}(\phi_{i(m,p)}(d(x,A_m^p)))$ does not exceed $\mu(x)/4$, which implies that $g(x)\subseteq\mathbb{I}^n$.

b) If $\mu(x)>0$, then $g(x)$ is compact and non-empty, being a finite union of compact non-empty sets. If $\mu(x)=0$, then $g(x)=f(x)$ which is also compact and non-empty. Therefore, for every $x\in Q$, $g(x)\in\exp(\mathbb{I}^n)$.

c) That g is continuous follows from the continuity of the involved maps.

d) Fix $x\in Q$. It is clear, by the proof of Lemma 1, that $d_H(f(x),g(x))\leqslant 9/4\cdot\mu(x)+\mu(x)/2=11\cdot\mu(x)/4$, from which it follows that $d_H(f(x),g(x))\leqslant 11/12\cdot\min\{\varepsilon,d_H(f(x),f[K])\}$. So we are done, because this inequality implies that $g|K=f|K$.

CLAIM 2. The map g is injective.

Injectivity of g follows from injectivity of h and construction of the map g.

CLAIM 3. For every $k\in\mathbb{N}$ we have $g^{-1}[D_{>\gamma_k}(\mathbb{I}^n)]\setminus K=\mathcal{A}_k\setminus K$.

By analogy to the proof of Lemma 1, we first observe that from Claim 1 and the fact that f is an embedding it follows that

$$g[Q\setminus K]\cap g[K]=\emptyset.\qquad(*)$$

Choose $x\in Q\setminus K$. If $x\in\mathcal{A}_k\setminus\mathcal{A}_{k+1}$ for certain k, then $x\in A_k^p$ for some p. This implies that $d(x,A_k^p)=0$ and $\phi_{i(k,p)}(d(x,A_k^p))=C_{i(k,p)}$, where $\dim_H(C_{i(k,p)})=\gamma_{k+1}$. Thus, $g(x)$ is a union of finitely many countable sets and countable union of the sets for which the Hausdorff dimension does not exceed γ_{k+1}. Therefore, $\dim_H(g(x)) = \gamma_{k+1}$. This implies that for $x \in \mathcal{A}_k \backslash \mathcal{A}_{k+1}$, $g(x) \in D_{>\gamma_k}(\mathbb{I}^n) \backslash D_{>\gamma_{k+1}}(\mathbb{I}^n)$.

If $x \notin \mathcal{A}_k$ for every $k \in \mathbb{N}$ then $g(x)$ is a countable set and therefore $\dim_H(g(x)) = 0$. Equality $(*)$ completes the proof of Claim 3.

CLAIM 4. The map g is a Z-embedding.

Follows from the same results for the map h.

This completes the proof of Theorem 2. □

Corollary. *The pair* $(\exp(\mathbb{I}^n), D_{=n}(\mathbb{I}^n))$, *where* $D_{=n}(\mathbb{I}^n) = \{A \in \exp(\mathbb{I}^n) | \dim_H A = n\}$, *is homeomorphic to* $(Q^\omega, B(Q)^\omega)$.

Proof. Since $B(Q)^\omega$ is $\mathcal{F}_{\sigma\delta}$-absorbing set in Q^ω (see [4]) and we can write $D_{=n}(\mathbb{I}^n) = \bigcap_{i=1}^{\infty} D_{>n-1/i}(\mathbb{I}^n)$, this follows from Theorem 2. □

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ПОГЛИНАЮЧІ МНОЖИНИ ПОВ'ЯЗАНІ З ВИМІРОМ ГАУСДОРФА

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Доведено, що гіперпростір компактних множин у n-вимірному кубі $\mathbb{I}^n$, вимір Гаусдорфа яких $> \alpha$, $0 < \alpha < n$, є $\mathcal{F}_\sigma$-поглинаючою множиною в гіперпросторі $\exp(\mathbb{I}^n)$. Крім того, для довільної послідовності (α_i), $0 < \alpha_1 < \alpha_2 < \ldots < n$, послідовність гіперпросторів компактних множин в $\mathbb{I}^n$, вимір Гаусдорфа яких $> \alpha_i$, є $\mathcal{F}_\sigma$-поглинаючою послідовністю в $\exp(\mathbb{I}^n)$.

Ключові слова: гіперпростір, вимір Гаусдорфа, Гільбертів куб, поглинаюча система.

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SOME PROPERTIES OF MINORS OF INVERTIBLE MATRICES

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The invariants of matrices which reduced a matrix over a commutative elementary divisor domain to canonical diagonal form is investigated. Some properties of minors of invertible matrices are considered.

Key words: commutative elementary divisor domain, transformable matrix, invertible matrix, minors, invariants of matrices.

Let R be a commutative elementary divisor domain [1]. Let A be a nonsingular $n\times n$ matrix with the canonical diagonal form $\Phi = \operatorname{diag}(\varphi_1, \dots, \varphi_n)$. Therefore, there exist invertible matrices P and Q such that $PAQ = \Phi$. The matrices P and Q are called the left and the right transformable matrices of the matrix A, respectively. By $\mathbf{P}_A$ we denote the set of left transformable matrices of the matrix A. It follows from the results of the papers [2, 3] that $\mathbf{P}_A = \mathbf{G}_\Phi P$, where $\mathbf{G}_\Phi$ is the group of invertible matrices of the form

$$\left\| \begin{matrix} h_{11} & h_{12} & \dots & h_{1n-1} & h_{1n} \\ \frac{\varphi_2}{\varphi_1}h_{21} & h_{22} & \dots & h_{2n-1} & h_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\varphi_n}{\varphi_1}h_{n1} & \frac{\varphi_n}{\varphi_2}h_{22} & \dots & \frac{\varphi_n}{\varphi_{n-1}}h_{nn-1} & h_{nn} \end{matrix} \right\|. \tag{1}$$

It means that the set $\mathbf{P}_A$ is a left conjugate class of the complete linear group $GL_n(R)$ with respect to the subgroup $\mathbf{G}_\Phi$. In the papers [2, 3, 4, 5] it is proved that the set $\mathbf{P}_A$ plays the main role in the description of the divisors of matrices. The papers [2, 3, 6, 7] are dedicated to investigations of invariants and description of properties of the set $\mathbf{P}_A$. Our paper is connected to this topic. We also study properties of minor determinants of invertible matrices.

Let U be an $m\times n$ matrix over R, $m \leqslant n$. The matrix U is called *primitive* if the greatest common divisor (g.c.d.) of the minors of order m of the matrix U is equal to 1. We denote by $U_{i_1,\dots,i_k}$ the matrix consisting of $i_1, \dots, i_k$ columns of the matrix U, where $k \leqslant m$, $1 \leqslant i_1 < \dots < i_k \leqslant n$.

Lemma. *The g.c.d. of the minors of order m of the matrix U, which contain the matrix $U_{i_1,\dots,i_k}$ is equal to the g.c.d. of the minors of order k of the matrix $U_{i_1,\dots,i_k}$.*

Proof. If $k = m$ the lemma clearly holds. Suppose that $k < m$. A suitable permutation of the columns of U gives $\| S \quad U_{i_1,\dots,i_k} \| = UL$, where L is a suitable permutation matrix. There exists an invertible matrix K such that

$$KU_{i_1,\dots,i_k} = \left\| \begin{array}{ccc} u_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & u_k \\ \hline & \mathbf{0} & \end{array} \right\| = \left\| \begin{array}{c} U'_{i_1,\dots,i_k} \\ \mathbf{0} \end{array} \right\|,$$

where $u_1 \cdots u_k$ is g.c.d. of the minors of order k of the matrix $U_{i_1,\dots,i_k}$. Then

$$KUL = K \| S \quad U'_{i_1,\dots,i_k} \| = \left\| \begin{array}{cc} * & U'_{i_1,\dots,i_k} \\ T & \mathbf{0} \end{array} \right\|.$$

Every minor of order m of the matrix KUL which contains the matrix $\left\| \begin{array}{c} U'_{i_1,\dots,i_k} \\ \mathbf{0} \end{array} \right\|$ has the form

$$\det \left\| \begin{array}{cc} * & U'_{i_1,\dots,i_k} \\ T_{m-k} & \mathbf{0} \end{array} \right\|,$$

where T_{m-k} is an $(m-k) \times (m-k)$ submatrix of the matrix T. Therefore, g.c.d. of such minors is equal to $u_1 \dots u_k \tau$, where τ is the g.c.d. of the minors of order $m-1$ of the matrix T. Since the matrix KUL is primitive, it follows that the matrix T is primitive. Thus $\tau = 1$. Since any minor of the order m of the matrix U differs from respective minor of the matrix KUL by a unit multiplier of the ring R, the proof of our statement is complete. □

Let $U \in \mathbf{P}_A$. We denote by U^m the matrix consisting of the last m rows of the matrix U, $1 \leqslant m < n$, by $U^m_{i_1,\dots,i_k}$ the matrix consisting of $i_1, \dots, i_k$ columns of the matrix U^m, $1 \leqslant k \leqslant n$, $1 \leqslant i_1 < \dots < i_k \leqslant n$, and by $\delta^m_{i_1,\dots,i_k}$ the g.c.d. of the minors of order $\min\{m,k\}$ of the matrix $U^m_{i_1,\dots,i_k}$.

Theorem. *The elements $\left(\delta^m_{i_1,\dots,i_k}, \dfrac{\varphi_{n-m+1}}{\varphi_{n-m}}\right)$ are invariant with respect to the choice of transformable matrices from $\mathbf{P}_A$, for all indices $i_1, \dots, i_k$, $1 \leqslant i_1 < \dots < i_k \leqslant n$, $k = 1, \dots, n$, $m = 1, \dots, n-1$.*

Proof. Let $V \in \mathbf{P}_A$. By $\Delta^m_{i_1,\dots,i_k}$ we denote the g.c.d. of the minors of order $\min\{m,k\}$ of the matrix $V^m_{i_1,\dots,i_k}$. At first we consider the case $k = m$. Since $V = HU$, where H is an invertible matrix of form (1), it follows that

$$V^m_{i_1,\dots,i_m} = \left\| \begin{array}{cccccc} \frac{\varphi_s}{\varphi_1} h_{s\,1} & \dots & \frac{\varphi_s}{\varphi_{s-1}} h_{s\,s-1} & h_{s\,s} & \dots & h_{s\,n-1} & h_{s\,n} \\ \frac{\varphi_{s+1}}{\varphi_1} h_{s+1\,1} & \dots & \frac{\varphi_{s+1}}{\varphi_{s-1}} h_{s+1\,s-1} & \frac{\varphi_{s+1}}{\varphi_s} h_{s+1\,s} & \dots & h_{s+1\,n-1} & h_{s+1\,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\varphi_n}{\varphi_1} h_{n\,1} & \dots & \frac{\varphi_n}{\varphi_{s-1}} h_{n\,s-1} & \frac{\varphi_n}{\varphi_s} h_{n\,s} & \dots & \frac{\varphi_n}{\varphi_{n-1}} h_{n\,n-1} & h_{n\,n} \end{array} \right\| \rangle$$

$$\times \left\| \begin{matrix} u_{1\,i_1} & \dots & u_{1\,i_m} \\ \dots & \dots & \dots \\ u_{n\,i_1} & \dots & u_{n\,i_m} \end{matrix} \right\|,$$

where $s = n - m + 1$. Since $\frac{\varphi_{s+i}}{\varphi_j}$ are divisible by $\frac{\varphi_s}{\varphi_{s-1}}$ for any $i = 1, \dots, n-s$, $j = 1, \dots, s-1$, $s + i > j$, all minors of order m which are built on last m rows of the matrix H, except the minor

$$\left| \begin{matrix} h_{s\,s} & \dots & h_{s\,n-1} & h_{s\,n} \\ \frac{\varphi_{s+1}}{\varphi_s} h_{s+1\,s} & \dots & h_{s+1\,n-1} & h_{s+1\,n} \\ \dots & \dots & \dots & \dots \\ \frac{\varphi_n}{\varphi_s} h_{n\,s} & \dots & \frac{\varphi_n}{\varphi_{n-1}} h_{n\,n-1} & h_{n\,n} \end{matrix} \right| = |H_s|$$

are divisible by $\frac{\varphi_s}{\varphi_{s-1}}$. Since H is an invertible matrix, we have $\left(|H_s|, \frac{\varphi_s}{\varphi_{s-1}}\right) = 1$. By the Cauchy-Binet formula and from what has been said it follows that

$$|V^m_{i_1,\dots,i_m}| = \frac{\varphi_s}{\varphi_{s-1}} d + |H_s| \left|U^m_{i_1,\dots,i_m}\right|,$$

where

$$|U^m_{i_1,\dots,i_m}| = \left| \begin{matrix} u_{1\,i_1} & \dots & u_{1\,i_m} \\ \dots & \dots & \dots \\ u_{n\,i_1} & \dots & u_{n\,i_m} \end{matrix} \right|.$$

Then

$$\left(\frac{\varphi_s}{\varphi_{s-1}}, |V^m_{i_1,\dots,i_m}|\right) = \left(\frac{\varphi_s}{\varphi_{s-1}}, |H_s| \left|U^m_{i_1,\dots,i_m}\right|\right) = \left(\frac{\varphi_s}{\varphi_{s-1}}, \left|U^m_{i_1,\dots,i_m}\right|\right).$$

We remark that $\delta^m_{i_1,\dots,i_m} = |U^m_{i_1,\dots,i_m}|$ and $\Delta^m_{i_1,\dots,i_m} = |V^m_{i_1,\dots,i_m}|$, hense the result holds for $k = m$.

Let $m < k \leqslant n$. We choose a minor μ of order m of the matrix $U^m_{i_1,\dots,i_k}$. The corresponding minor of the matrix $V^m_{i_1,\dots,i_k}$ is denoted by ν. The first case implies that

$$\left(\mu, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\nu, \frac{\varphi_s}{\varphi_{s-1}}\right).$$

Since $\delta^m_{i_1,\dots,i_k}$ is g.c.d. of the minors of order m of the matrix $U^m_{i_1,\dots,i_k}$, we have

$$\left(\delta^m_{i_1,\dots,i_k}, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\Delta^m_{i_1,\dots,i_k}, \frac{\varphi_s}{\varphi_{s-1}}\right).$$

Finally we consider the case $1 \leqslant k < m$. Let $\mu_1, \dots, \mu_t$ be minors of order m of the matrix U^m which contains the matrix $U^m_{i_1,\dots,i_k}$. By $\nu_1, \dots, \nu_t$ we denote the respective minors of the matrix V^m. By Lemma we get

$$(\mu_1, \dots, \mu_t) = \delta^m_{i_1,\dots,i_k}, \ (\nu_1, \dots, \nu_t) = \Delta^m_{i_1,\dots,i_k}.$$

Thus,

$$\left(\delta^m_{i_1,\dots,i_k}, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\mu_1, \dots, \mu_t, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\left(\mu_1, \frac{\varphi_s}{\varphi_{s-1}}\right), \dots, \left(\mu_t, \frac{\varphi_s}{\varphi_{s-1}}\right)\right) =$$

$$= \left(\left(\nu_1, \frac{\varphi_s}{\varphi_{s-1}}\right), \dots, \left(\nu_t, \frac{\varphi_s}{\varphi_{s-1}}\right)\right) = \left(\nu_1, \dots, \nu_t, \frac{\varphi_s}{\varphi_{s-1}}\right) = \left(\Delta^m_{i_1,\dots,i_k}, \frac{\varphi_s}{\varphi_{s-1}}\right).$$

The proof of the Theorem is complete. □

Proposition 1. *Let A be an $m \times s$ matrix with the canonical diagonal form* $\mathrm{diag}(\underbrace{1,\dots,1}_{k}, \alpha_{k+1}, \dots, \alpha_s)$, *where $\alpha_{k+1} \notin U(R)$, $m \geqslant s$. Then the matrix A is a submatrix of some invertible $n \times n$ matrix if and only if $n - m \geqslant s - k$.*

Proof. **Necessity.** Let A be a submatrix of an invertible $n \times n$ matrix U. We may assume, without loss of generality, that

$$U = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix},$$

where B, C, D are matrices of respective sizes. Let P and Q be transformable matrices of A, i.e.

$$PAQ = \begin{Vmatrix} \begin{matrix} E_k & 0 \\ 0 & S \end{matrix} \\ \hline 0 \end{Vmatrix} = \Phi,$$

where E_k is the identity $k \times k$ matrix, $S = \mathrm{diag}(\alpha_{k+1}, \dots, \alpha_s)$. Then

$$\begin{Vmatrix} P & 0 \\ 0 & E_{n-m} \end{Vmatrix} U \begin{Vmatrix} Q & 0 \\ 0 & E_{n-s} \end{Vmatrix} = \begin{Vmatrix} \Phi & PB \\ CQ & D \end{Vmatrix}.$$

Put F be an $(m-k) \times n$ matrix consisting of $k+1, k+2, \dots, m$ columns of the matrix $\begin{Vmatrix} \Phi & PB \\ CQ & D \end{Vmatrix}$. Therefore

$$F = \begin{Vmatrix} 0 \\ S \\ 0 \\ C_k \end{Vmatrix},$$

where C_k is a submatrix of C. Since $\alpha_{k+1} | \alpha_j$ for $j = k+1, \dots, s$, we see that α_{k+1} divides all minors of order $s - k$ of the matrix F, which contain at least one of first m rows of F. The matrix F is primitive, therefore there exists a minor of order $s - k$ which does not contain the first m rows of F. It means that the $(n-m) \times (s-k)$ matrix C_k contains at least $s - k$ rows, i.e. $n - m \geqslant s - k$.

Sufficiency. Let

$$U = \begin{Vmatrix} P^{-1} & 0 \\ 0 & E_{s-k} \end{Vmatrix} \begin{Vmatrix} E_k & 0 & 0 \\ 0 & S & E_{m-k} \\ 0 & 0 & \\ 0 & E_{s-k} & 0 \end{Vmatrix} \begin{Vmatrix} Q^{-1} & 0 \\ 0 & E_{m-k} \end{Vmatrix}.$$

The matrix U is an example of a desired invertible matrix of order $n = m + s - k$. □

Corollary. *Let U be an invertible $n \times n$ matrix, V be a square submatrix of U and* $\mathrm{diag}(\underbrace{1,\dots,1}_{k}, \alpha_{k+1}, \dots, \alpha_{n-t})$ *be the canonical diagonal form of V. Then $k \leqslant t$.* □

Proposition 2. *Let be $U = \|u_{ij}\|_1^n$ and $U^{-1} = V = \|v_{ij}\|_1^n$. Then g.c.d. of the minors of maximal order of matrices*

$$U' = \begin{Vmatrix} u_{i_1k_1} & u_{i_1k_2} & \dots & u_{i_1k_p} \\ u_{i_2k_1} & u_{i_2k_2} & \dots & u_{i_2k_p} \\ \dots & \dots & \dots & \dots \\ u_{i_sk_1} & u_{i_sk_2} & \dots & u_{i_sk_p} \end{Vmatrix} \text{ and } V' = \begin{Vmatrix} v_{k'_1i'_1} & v_{k'_1i'_2} & \dots & v_{k'_1i'_{n-s}} \\ v_{k'_2i'_1} & v_{k'_2i'_2} & \dots & v_{k'_2i'_{n-s}} \\ \dots & \dots & \dots & \dots \\ v_{k'_{n-p}i'_1} & v_{k'_{n-p}i'_2} & \dots & v_{k'_{n-p}i'_{n-s}} \end{Vmatrix},$$

coincides, where $i_1 < i_2 < \ldots < i_s$ together with $i'_1 < i'_2 < \ldots < i'_{n-s}$ and $k_1 < k_2 < \ldots < k_p$ together with $k'_1 < k'_2 < \ldots < k'_{n-p}$ form the complete system of indices $1, 2, \ldots, n$.

Proof. Suppose that $s \geqslant p$. It is well known that the minors $U\begin{pmatrix} i_1 \dots i_p \\ k_1 \dots k_p \end{pmatrix}$ and $V\begin{pmatrix} k'_1 \dots k'_{n-p} \\ i'_1 \dots i'_{n-p} \end{pmatrix}$ differ from each other by a unit multiplier of R (see: [6, Part 1, §4]). All the minors $V\begin{pmatrix} k'_1 \dots k'_{n-p} \\ i'_1 \dots i'_{n-p} \end{pmatrix}$ can be characterized as all minors that contain the submatrix V'. Then Lemma implies our statement. The case $s < p$ is similar. □

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ДЕЯКІ ВЛАСТИВОСТІ МІНОРІВ ОБОРОТНИХ МАТРИЦЬ

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Досліджено інваріанти тих матриць, які зводять матрицю над комутативною областю елементарних дільників до її канонічної діагональної форми. Зазначено деякі властивості мінорів оборотних матриць.

Ключові слова: комутативна область елементарних дільників, перетворювальні матриці, оборотні матриці, мінори, інваріанти матриць.

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INVARIANT HYPERCOMPLEX STRUCTURES

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G-invariant Kähler structures (J_1, Ω) on the cotangent bundles $T^*(G/K)$ (symplectic manifolds with the canonical 2-form Ω) of Hermitian symmetric spaces with the standard antiholomorphic involution are considered. For arbitrary such a structure (J_1, Ω) a hypercomplex manifold $(T^*(G/K), \{J_1, J_2\})$ is constructed.

Key words: invariant hypercomplex structures.

1. A hypercomplex manifold $(X, \{J_1, J_2\})$ is a pair consisting of a $4n$-dimensional manifold X together with two anticommuting complex structures J_1, J_2. It then follows that X has a family of complex structures $J_\lambda = \lambda_1 J_1 + \lambda_2 J_2 + \lambda_3 J_3$, $J_3 = J_1 J_2$, parametrized by points $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ in the unit sphere $S^2 \subset \mathbb{R}^3$.

This article concerns the construction of hypercomplex structures on the cotangent bundle of Hermitian symmetric spaces. Non-compact homogeneous manifolds carrying such a structure were considered by Barberis and Miatello in [1], the case of compact homogeneous manifolds was considered by Joyce in [2].

Let $M = G/K$ be a Hermitian symmetric space and $\sigma : TM \to TM$ the involution which maps any tangent vector Y at $m \in M$ onto $-Y$ at m. Let Ω be the canonical symplectic structure on TM (the standard G-invariant metric $\mathrm{g_M}$ on M identifies the cotangent bundle T^*M and the tangent bundle TM). The main purpose of this note is to construct the following rich family of examples: let $\mathfrak{P}$ be a set of all G-invariant Kähler structures on some tube $T^sM = \{v \in TM : \mathrm{g}(v,v) < s^2\}$ (with Ω as the Kähler form) such that the mapping σ is an antiholomorphic involution. We prove here that for any $J_1 \in \mathfrak{P}$ there is a complex structure J_2 on T^sM for which $(T^sM, \{J_1, J_2\})$ is a hypercomplex manifold. The proof is simple because it is based on a Lie algebraic method of description of the elements $J_1 \in \mathfrak{P}$ [4] (usually the tensors J_1 are described in terms of geometric structures associated with the metric $\mathrm{g_M}$ on M [5,6]). Remark that the set $\mathfrak{P}$ is non-empty because it contains the adapted complex structure [3]; for all rank-one symmetric spaces this set $\mathfrak{P}$ is described in [4]. Moreover, the obtained set of hypercomplex structures on TM contains the hyper-Kähler structure constructed in [5,6].

2. Anticommuting structures. We recall some facts on Kähler and hypercomplex structures (see for example [4]). Let X be a (real) manifold with a symplectic form Ω and J be an almost complex structure on X. J is a complex structure if the complex subbundle F of $(0,1)$-vectors of J is an involutive subbundle of the complexified tangent bundle $T^{\mathbb{C}}X$. By definition, for any $x \in X$ we have $F(x) = \{Y + iJ_x(Y), Y \in T_xX\}$ $(J_x^2 = -1)$. We say that a complex structure J is a Kähler structure with the Käler form Ω if 1) $\Omega_x(J_x(Y_1), J_x(Y_2)) = \Omega_x(Y_1, Y_2)$ for any $Y_1, Y_2 \in T_xX$; 2) the quadratic form $B_x(Y_1, Y_2) = \Omega_x(J_xY_1, Y_2)$ is symmetric and positive-definite. Such a Kähler structure J will be denoted by (J, F, Ω).

A pair (J_1, J_2) formed by two anticommuting complex structures J_1, J_2 is a hypercomplex structure on X. Then $J_3 = J_1J_2$ is also a complex structure on X (for a proof see [7]).

3. G-invariant complex structures. Let $M = G/K$ be a symmetric space with a real reductive connected Lie group G and a compact connected subgroup K. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the groups G and K respectively,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m} \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \tag{1}$$

Suppose that there is a nondegenerate $\mathrm{Ad}\, G$-invariant bilinear form $\langle , \rangle$ on $\mathfrak{g}$ such that its restriction $\langle , \rangle|\mathfrak{m}$ is a positive definite form and $\mathfrak{k} \perp \mathfrak{m}$. This form defines the G-invariant Riemannian metric $\mathbf{g_M}$ on $M = G/K$. The metric $\mathbf{g_M}$ identifies the cotangent bundle T^*M and the tangent bundle TM and thus we can also talk about the canonical symplectic 2-form Ω on TM. This form Ω is G-invariant with respect to the natural action of G on TM.

Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\mathrm{Ad}\, K$-invariant (orthogonal) splitting of $\mathfrak{g}$, we can consider the trivial vector bundle $G \times \mathfrak{m}$ with the two Lie group actions (which commute) on it: the left G-action, $l_h : (g, w) \mapsto (hg, w)$ and the right K-action $r_k : (g, w) \mapsto (gk, \mathrm{Ad}_{k^{-1}} w)$. Let $\pi : G \times \mathfrak{m} \to G \times_K \mathfrak{m}$ be the natural projection. It is well known that $G \times_K \mathfrak{m}$ and TM are isomorphic. Using the corresponding G-equivariant diffeomorphism $\phi : G \times_K \mathfrak{m} \to TM$, $[(g, w)] \mapsto \frac{d}{dt}\Big|_0 g \exp(tw)K$ and the projection π define the G-equivariant submersion $\Pi : G \times \mathfrak{m} \to TM$, $\Pi = \phi \circ \pi$. Let ξ^l be the left-invariant vector field on the Lie group G defined by a vector $\xi \in \mathfrak{g}$. Since Ω is a symplectic form, the kernel $\mathcal{K} \subset T(G \times \mathfrak{m})$ of the 2-form $\tilde{\Omega} = \Pi^*\Omega$ is the kernel of Π_*, i.e. is generated by the global (left) G-invariant vector fields ζ^L, $\zeta \in \mathfrak{k}$ on $G \times \mathfrak{m}$, $\zeta^L(g, w) = (\zeta^l(g), [w, \zeta])$.

For given s, $0 < s \leqslant \infty$ consider the tube $T^sM \stackrel{\text{def}}{=} \{v \in TM \text{ of length } < s\}$. Put $W^s \stackrel{\text{def}}{=} \{w \in \mathfrak{m} : |w| < s\}$, where $|w| \stackrel{\text{def}}{=} \sqrt{\langle w, w\rangle}$. We will say that a smooth mapping $P : W^s \to \mathrm{GL}(\mathfrak{m})$, $w \mapsto P_w$ is K*-equivariant* if

$$\mathrm{Ad}_k \circ P_w \circ \mathrm{Ad}_{k^{-1}} = P_{\mathrm{Ad}_k w} \quad \text{on } \mathfrak{m} \quad \text{for all} \quad w \in W^s, k \in K. \tag{2}$$

This mapping determines a complex (left) G-invariant subbundle $\mathcal{F}(P) \subset T^{\mathbb{C}}(G \times W^s)$ generated by nowhere vanishing on $G \times W^s$ (left) G-invariant vector fields ξ^L, $\xi \in \mathfrak{m}$ and $\zeta^L \in \Gamma\mathcal{K}$, $\zeta \in \mathfrak{k}$, where

$$\xi^L(g, w) = (\xi^l(g), iP_w(\xi)). \tag{3}$$

The subbundle $\mathcal{F}(P)$ is (right) K-invariant because the mapping P is K-equivariant. Therefore $F(P) \stackrel{\text{def}}{=} \Pi_*(\mathcal{F}(P))$ is a well-defined (smooth) complex subbundle of the complexified tangent bundle $T^{\mathbb{C}}(T^sM)$ $(\mathcal{K}^{\mathbb{C}} \subset \mathcal{F}(P))$.

Consider two (left) G-invariant and (right) K-invariant subbundles $\mathcal{T}_h$, $\mathcal{T}_v$ of the tangent bundle $T(G\times\mathfrak{m})$ given by

$$\mathcal{T}_h(g,w)=\{(\xi^l(g),0),\ \xi\in\mathfrak{m}\},\quad \mathcal{T}(g,w)=\{(0,u),\ u\in\mathfrak{m}=T_w\mathfrak{m}\}.$$

Put $\mathcal{T}=\mathcal{T}_h\oplus\mathcal{T}_v$. Since $T(G\times\mathfrak{m})=\mathcal{K}\oplus\mathcal{T}$, the map $\Pi_*|\mathcal{T}_{(g,w)}$ is an isomorphism of the spaces $\mathcal{T}_{(g,w)}$ and $T_{\Pi(g,w)}(TM)$, in particular, by (right) K-invariance of $\mathcal{T}_h$ and $\mathcal{T}_v$ the images $\Pi_*(\mathcal{T}_h)$ and $\Pi_*(\mathcal{T}_v)$ are well-defined subbundles. But the natural projection $p:G\to G/K$ is a locally trivial fiber bundle so that for any $g\in G$ there is a (regular) submanifold $D\subset G$ such that the restriction $p:D\to G/K$ is an embedding. Since $\Pi=\phi\circ\pi$, the restriction $\Pi:D\times W^s\to TM$ is also an embedding. Denote by U_D the image $\Pi(D\times W^s)$. Now using the splitting

$$TU_D=\Pi_*(\mathcal{T}_h|D\times W^s)\oplus\Pi_*(\mathcal{T}_v|D\times W^s),\tag{4}$$

we obtain that the subbundle $F(P)|U_D$ is a subbundle of $(0,1)$-vectors of the almost complex tensor $J(P)|U_D:TU_D\to TU_D$, where $J_{\Pi(g,w)}(P)=\begin{pmatrix}0 & -P_w^{-1}\\ P_w & 0\end{pmatrix}$ $(J^2(P)=-1)$. The tensor field $J(P)$ on $T(T^sM)$ is smooth because $F(P)$ is a well-defined subbundle. $J(P)$ defines a complex structure if the subbundle $F(P)$ is involutive.

Fix base $\{W_b\}$ in $\mathfrak{m}$. Let $\{w_b\}$ be the coordinates in $\mathfrak{m}$ with respect to the basis $\{W_b\}$. For any vector-function $\tau:W^s\to\mathfrak{m}$, $\tau(w)=\sum_b\tau_b(w)W_b$ by $\overrightarrow{\tau}$ we denote the vector field $\overrightarrow{\tau}\overset{\text{def}}{=}\sum_b\tau_b\frac{\partial}{\partial w_b}$. Let $P(\xi)$, where $\xi\in\mathfrak{m}$, denote the vector-function $P(\xi):w\mapsto P_w(\xi)$.

3.1. Proposition. [4] *Suppose that $M=G/K$ is a Riemannian symmetric space. Let $\mathcal{F}(P)$ be a complex subbundle of $T^{\mathbb{C}}(G\times W^s)$ defined by a K-equivariant mapping $P:W^s\to GL(\mathfrak{m})$. Then*
1) the subbundle $F(P)=\Pi_*(\mathcal{F}(P))$ is involutive on T^sM if and only if the Lie bracket identities $\left[\overrightarrow{P(\xi)},\overrightarrow{P(\eta)}\right](w)=-[w,[\xi,\eta]]$ hold on W^s for all (fixed) $\xi,\eta\in\mathfrak{m}$;
2) the complex structure $J(P)$ such that $\sigma_*(F(P))=\overline{F(P)}$ is a Kähler structure with the Kähler form Ω if and only if P_w is a symmetric positive-definite operator for each $w\in W^s$ (with respect to the bilinear form $\langle,\rangle$ on $\mathfrak{m}$).

For any G-invariant Kähler structure (J,F,Ω) on T^sM for which $\sigma_*(F)=\overline{F}$ there exists a unique K-equivariant mapping $P:W^s\to GL(\mathfrak{m})$ such that $J=J(P)$ and $F=F(P)$.

4. Examples: adapted complex structures. Let J^A be a (smooth) complex structure on some tube T^sM. The complex structure J^A on T^sM is called *adapted* [3,8] if for every geodesic γ in M a map $\hat\gamma:\mathbb{C}\to T(G/K)$, $(x+iy)\mapsto y\dot\gamma(x)$ is holomorphic on $\hat\gamma^{-1}(T^sM)$. Since the Riemannian manifold M is complete, an adapted complex structure on T^sM is unique (if it exists) [8]. Since the Riemannian manifold $(M,\mathrm{g_M})$ is real-analytic and is also a symmetric space, on some tube T^sM there exists a real-analytic adapted structure J^A [8]. If the Lie group G is compact, by Corollary 21.1 of [9] (see also [5]) $F^A=F(P^A)$, where

$$P^A:W^s\to\mathrm{GL}(\mathfrak{m}),\ w\mapsto P_w^A,\ P_w^A=\left.\frac{\mathrm{ad}_w\cos\mathrm{ad}_w}{\sin\mathrm{ad}_w}\right|_{\mathfrak{m}},\quad w\in\mathfrak{m}^s,\ s=\infty.\tag{5}$$

In this case the adapted structure F^A is defined on the whole tangent bundle TM and (J^A, F^A, Ω) is a Kähler structure.

If G is a noncompact semisimple Lie group and if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition of its Lie algebra $\mathfrak{g}$ then the (real) Lie algebra $\mathfrak{k} \oplus i\mathfrak{m} \subset \mathfrak{g}^{\mathbb{C}}$ is compact. Now it follows easily from Proposition 3.1 that formula (5) defines a Kähler structure $(J(P^A), F(P^A), \Omega)$ on some tube T^sM, $0 < s < \infty$. For this s eigenvalues of symmetric operators $\mathrm{ad}_w^2 |\mathfrak{m}$, $w \in W^s$ are positive.

5. Hypercomplex structures on the cotangent bundles of Hermitian symmetric spaces. We continue with the previous notations but in this subsection it is assumed in addition that G/K is an irreducible Hermitian symmetric space (of the compact or noncompact type).

We will review a few facts about Hermitian symmetric spaces (see Ch.VIII, §§4–7,[10]). Since G/K is an irreducible Hermitian symmetric space, then $\mathfrak{g}$ is a simple Lie algebra and the center $\mathfrak{c}$ of $\mathfrak{k}$ is one-dimensional. There exists a unique (up to sign) element $z_0 \in \mathfrak{c} \subset \mathfrak{k}$ such that for the operator $I = \mathrm{ad}_{z_0} |\mathfrak{m}$ on $\mathfrak{m}$ we have $I^2 = -1$. Moreover, taking into account relations (1) and the Jacobi identity, we obtain that

$$[I\xi, I\eta] = [\xi, \eta], \qquad I[\xi, \zeta] = [I\xi, \zeta] \qquad \text{for all } \xi, \eta \in \mathfrak{m},\ \zeta \in \mathfrak{k}. \tag{6}$$

Since the Lie group K is connected, the group $\mathrm{Ad}(K)$ commutes elementwise with I (on $\mathfrak{m}$). This endomorphism determines an G-invariant complex structure on M [10].

Now fix some K-equivariant mapping $P : W^s \to GL(\mathfrak{m})$. The mapping PI, $(PI)_w \stackrel{\text{def}}{=} P_w I$ is also K-equivariant because the group $\mathrm{Ad}(K)$ commutes elementwise with I. As an application of the proposition above we will prove

5.1. Lemma. *If $J(P)$ is a complex structure then so is $J(PI)$.*

Proof. Suppose that $J(P)$ is a complex structure, i.e. $F(P)$ is an involutive subbundle. Since I is independent of w, by the definition of the Lie bracket and from relations (6) it follows that

$$\begin{aligned}
\left[\overrightarrow{(PI)(\xi)}, \overrightarrow{(PI)(\eta)}\right](w) &= \frac{d}{dt}\Big|_{t=0}\Big((PI)_{w+t(PI)_w(\xi)}(\eta) - (PI)_{w+t(PI)_w(\eta)}(\xi)\Big)\\
&= \frac{d}{dt}\Big|_{t=0}\Big(P_{w+tP_w(I\xi)}(I\eta) - P_{w+tP_w(I\eta)}(I\xi)\Big)\\
&= \left[\overrightarrow{P(I\xi)}, \overrightarrow{P(I\eta)}\right](w)\\
&= -[w, [I\xi, I\eta]]\\
&= -[w, [\xi, \eta]]
\end{aligned}$$

on W^s for any $\xi, \eta \in \mathfrak{m}$. Thus by assertion 1) of Proposition 3.1 the subbundle $F(PI)$ is involutive, i.e. $J(PI)$ is a complex structure on $T(T^sM)$.

Remark that locally on the open subset $U_D \subset T(T^sM)$ with respect to the splitting (4) the maps $J(P)$, $J(PI)$ and its product $J(P)J(PI)$ are represented by

$$J_{\Pi(g,w)}(P) = \begin{pmatrix} 0 & -P_w^{-1} \\ P_w & 0 \end{pmatrix}, \quad J_{\Pi(g,w)}(PI) = \begin{pmatrix} 0 & IP_w^{-1} \\ P_w I & 0 \end{pmatrix},$$

$$J_{\Pi(g,w)}(P) J_{\Pi(g,w)}(PI) = \begin{pmatrix} -I & 0 \\ 0 & P_w I P_w^{-1} \end{pmatrix}.$$

Therefore $J(P)$ and $J(PI)$ is a pair of anticommuting complex structures on T^sM. Thus we have proved

5.2. Theorem. *Let $(J(P), F(P), \Omega)$ be an arbitrary G-invariant Kähler structure on T^sM such that $\sigma_*(F(P)) = \overline{F(P)}$. Then $(T^sM, \{J_1 = J(P), J_2 = J(PI)\})$ is a hypercomplex manifold.*

5.3. Remark. Using the results of [5,6] we obtain that the constructed above hypercomplex structure $(T^sM, \{J_1 = J(P), J_2 = J(PI)\})$ is a hyper-Kähler structure if and only if $PI = IP$, i.e. if $P_wI = IP_w$ for any $w \in W^s$.

5.4. Remark. If G/K is a rank-one symmetric space isomorphic to $U(n+1)/(U(1)\times U(n))$, $n \geqslant 2$ then each G-invariant Kähler structure $J(P)$ on TM is determined by the following operator-function $P : \mathfrak{m} \to GL(\mathfrak{m})$, $w \mapsto P_w$ [4]

$$P_w(\xi) = \psi(r)\xi + \big(\psi(2r) - \psi(r)\big)r^{-2} \cdot \langle Iw, \xi\rangle Iw + \big(\lambda(r) - \psi(r)\big)r^{-2} \cdot \langle w, \xi\rangle w,$$

where $w \in \mathfrak{m}$, $r = \|w\| = \sqrt{-\frac{1}{2}\operatorname{Tr} w^2}$, $\lambda, \psi : [0, +\infty) \to \mathbb{R}$ are arbitrary positive functions satisfying the relations $\psi(r) = r\dfrac{\cosh\alpha(r)}{\sinh\alpha(r)}$, $\alpha'(r) = \dfrac{1}{\lambda(r)}$.

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ІНВАРІАНТНІ ГІПЕРКОМПЛЕКСНІ СТРУКТУРИ

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Розглянуто G-інваріантні келерові структури (J_1, Ω) на кодотичних розшаруваннях $T^*(G/K)$ (симплектичних многовидах з канонічною 2-формою Ω) ермітових симетричних просторів зі стандартною антиголоморфною інволюцією. Для довільної такої структури (J_1, Ω) побудовано гіперкомплексний многовид $(T^*(G/K), \{J_1, J_2\})$.

Ключові слова: інваріантні гіперкомплексні структури.

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TRIPLEABILITY OF THE CATEGORY OF (STRONGLY) SEMICONVEX COMPACTA OVER THE CATEGORY OF COMPACTA

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The notion of (strongly) semiconvex compactum and semiconvex combination generalizes a notion of convex compactum and convex combination (a "segment" that connects a point with itself is allowed to be a non-trivial loop). It is proved that a quotient space of a (strongly) semiconvex compactum for an equivalence relation closed under semiconvex combination is a (strongly) semiconvex compactum as well. Also tripleability of the category of (strongly) semiconvex compacta over the category of compacta is established.

Key words: compactum, (strongly) semiconvex compactum, left adjoint functor, tripleability.

First recall some definitions and facts from [5]. We use the following terminology and denotations : $I=[0,1]$ is *a unit segment*, *a compactum* is a (not necessarily metrizable) (bi)compact Hausdorff topological space. *A semiconvex compactum* is a compactum X with a continuous ternary operation $c: X\times X\times I\to X$ (we usually call it semiconvex combination and write $\lambda(x,y)$ instead of $c(x,y,\lambda)$) satisfying the following axioms:

1) for all $x,y\in X$, $\lambda\in I$: $\lambda(x,y)=(1-\lambda)(y,x)$ (commutative law);

2) for all $x,y,z\in X$, $\lambda,\mu,\nu\in I$, $\lambda+\mu+\nu=1$, $\mu\neq 0$:

$$\lambda(x,\frac{\mu}{\mu+\lambda}(y,z))=(\lambda+\mu)(\frac{\lambda}{\lambda+\mu}(x,y),z)$$

(associative law);

3) for all $x,y\in X$: $1(x,y)=x$.

4) there exists a base β of a unique uniformity inducing the topology on X [2] such that $B\in\beta$, $(x,y),(z,t)\in B$, $\lambda\in I$ implies $(\lambda(x,z),\lambda(y,t))\in B$.

The last axiom is equivalent to the following :

4') the topology on X is generated by a saturated family of pseudometrics [2] $(d_\alpha)_{\alpha\in\mathcal{A}}$ such that $x,y,z,t\in X$, $\epsilon>0$, $\alpha\in\mathcal{A}$, $d_\alpha(x,y)<\epsilon$, $d_\alpha(z,t)<\epsilon$, $\lambda\in I$ implies $d_\alpha(\lambda(x,z),\lambda(y,t))<\epsilon$.

Extend the notion of semiconvex combination onto a finite number of elements of X. Let $\Delta_{n-1} = \{(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1} : \lambda_0, \dots, \lambda_n \geqslant 0,\ \lambda_0 + \dots + \lambda_n = 1\}$ denote the standard n-dimensional simplex. Given $(\lambda_0, \dots, \lambda_n) \in \Delta_n$ and points $x_0, \dots, x_n \in X$ let

$$(\lambda_0, \dots, \lambda_n)(x_0, \dots, x_n) = \begin{cases} x_0, & \text{if } \lambda_0 = 1; \\ \lambda_1(x_0, (\frac{\lambda_1}{1-\lambda_0}, \dots, \frac{\lambda_n}{1-\lambda_0})(x_1, \dots, x_n)), & \text{if } \lambda_0 \neq 1. \end{cases}$$

If *arguments* $x_0, \dots, x_n$ of semiconvex combination are permutted simultaneously with the respective *coefficients* $\lambda_0, \dots, \lambda_n$, the value of semiconvex combinations does not change.

For any subset $A \subset X$ the set

$$\mathrm{Cl}\{(\lambda_0, \dots, \lambda_n)(x_0, \dots, x_n) \mid n \in \mathbb{N},\ x_0, \dots, x_n \in A,\ (\lambda_0, \dots, \lambda_n) \in \Delta_n\}$$

is the least closed subset in X that contains A and is closed under semiconvex combination. It is called *the semiconvex hull* of A.

There exists the largest closed under semiconvex combination closed subset $A \subset X$ such that $\lambda : A^2 \to A$ is surjective for any $\lambda \in I$. It is called *the weak center* of X and denoted $WCtr(X)$. *The center* $Ctr(X)$ of the semiconvex compactum X is a closed subset consisting of all points $b \in X$ such that $\lambda(b, b) = b$ for any $\lambda \in I$. Always $Ctr(X) \subset WCtr(X)$, and

$$WCtr(X) = \bigcap\{(\lambda_0, \dots, \lambda_n)(x_0, \dots, x_n) \mid n \in \mathbb{N},\ (\lambda_0, \dots, \lambda_n) \in \Delta_n,\ x_0, \dots, x_n \in X\}$$
$$Ctr(X) = \bigcap\{(\lambda_1, \dots, \lambda_n)(x, \dots, x) \mid n \in \mathbb{N},\ (\lambda_0, \dots, \lambda_n) \in \Delta_n,\ x \in X\}.$$

The center of X is closed under semiconvex combination and with operation induced becomes a convex compactum. Always $Ctr(X) \subset WCtr(X)$. If $Ctr(X) = WCtr(X)$, then X is called *a strongly semiconvex compactum*. Here is an alternative definition : X is a strongly semiconvex compactum if and only if for any $x \in X$ the point $(\lambda_0, \dots, \lambda_n)(x, \dots, x)$ converges to a unique point $y \in X$, when $(\lambda_0, \dots, \lambda_n) \in \Delta_n$ and $\max(\lambda_1, \dots, \lambda_n) \to 0$. This implies that if $f : X \to Y$ is a surjective map of strongly semiconvex compacta that preserves semiconvex combination (i.e., $f(\lambda(x_1, x_2)) = \lambda(f(x_1), f(x_2))$ for any $x_1, x_2 \in X$, $\lambda \in I$), and X is strongly semiconvex, then Y is strongly semiconvex as well.

For proofs see [5].

Theorem 1. *Let X be a (strongly) semiconvex compactum and "$\sim$"$\subset X \times X$ be a closed equivalence relation that is closed under semiconvex combination. If by $[x]$ the equivalence class that contains $x \in X$, is denoted, then the formula $\lambda([x], [x']) = [\lambda(x, x')]$, $x, x' \in X$, $\lambda \in I$, correctly defines an operation $Y \times Y \times I \to Y$ on $Y = X/_\sim$ such that Y is a (strongly) semiconvex compactum.*

Proof. Since "$\sim$" is closed, $X/_\sim$ is a compactum [2]. Denote by $q : X \to X/_\sim$ the quotient map. Let $x_1 \sim x_1'$, $x_2 \sim x_2'$, $x_1, x_1', x_2, x_2' \in X$, $\lambda \in I$. Then by the assumption of the theorem $\lambda(x_1, x_2) \sim \lambda(x_1', x_2')$, and the operation is well defined. Axioms (1)–(3) for Y are easy consequences of (1)–(3) for semiconvex combination in

X. Since q is a surjective continuous map of compacta, the diagram

$$\begin{array}{ccc} X\times X\times I & \xrightarrow{\text{semiconvex combination}} & X \\ {\scriptstyle q\times q\times 1_I}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ (X/_\sim)\times(X/_\sim)\times I & \xrightarrow[\text{new operation}]{} & X/_\sim \end{array}$$

shows that new defined operation is continuous.

Denote by $\exp Z$ [3] the set of all nonempty closed subsets of an arbitrary compactum Z. Then the multivalued map $q^{-1}: Y \to \exp X$ is *upper semicontinuous*, i.e., for any open set $U \subset X$ the set $\{y \in Y \mid q^{-1}(y) \subset U\}$ is open. Thus for any closed $F \subset X$ the set $\{y \in Y \mid q^{-1}(y) \cap U \neq \emptyset\}$ is closed. It is easy to prove that the map $Q = (\times)\circ(q^{-1}\times q^{-1}): Y\times Y \to \exp(X\times X)$, $Q(y_1,y_2) = d^{-1}(y_1)\times d^{-1}(y_2)$, is upper semicontinuous as well.

Take a saturated family $(\rho_\alpha)_{\alpha\in\mathcal{A}}$ of pseudometrics that generates the topology on X and satisfies (4'). For any $\alpha \in \mathcal{A}$ the formula

$$\tilde{\rho}_\alpha((x_1,x_2),(x_3,x_4)) = \max\{\rho_\alpha(x_1,x_3), \rho_\alpha(x_2,x_4)\}$$

defines a continuous pseudometric on $X\times X$. For each $\epsilon > 0$ the set

$$F_\epsilon^\alpha = \{(x_1,x_2)\in X^2 \mid \tilde{\rho}_\alpha((x_1,x_2),\text{“}\sim\text{”}) \leqslant \epsilon\} = \\ \{(x_1,x_2)\in X^2 \mid \exists z_1,z_2\in X : z_1\sim z_2, \rho_\alpha(x_1,z_1)\leqslant\epsilon, \rho_\alpha(x_2,z_2)\leqslant\epsilon\}$$

is closed, as well as the set

$$V_\epsilon^\alpha = \{(y_1,y_2)\in Y^2 \mid Q(y_1,y_2)\cap F_\epsilon^\alpha \neq \emptyset\} = \{(y_1,y_2)\in Y^2 \mid \exists x_1\in q^{-1}(y_1), \\ \exists x_2 \in q^{-1}(y_2), \exists z_1,z_2\in X : z_1\sim z_2, \rho_\alpha(x_1,z_1)\leqslant\epsilon, \rho_\alpha(x_2,z_2)\leqslant\epsilon\}.$$

Since $(\rho_\alpha)_{\alpha\in\mathcal{A}}$ is saturated, the family $(F_\epsilon^\alpha)_{\alpha\in\mathcal{A},\epsilon>0}$ is a centered system of nonempty closed subsets of $X\times X$, and $\bigcap_{\alpha\in\mathcal{A},\epsilon>0} F_\epsilon^\alpha = \text{“}\sim\text{”}$. Suppose that $(y_1,y_2)\in\bigcap_{\alpha\in\mathcal{A},\epsilon>0} V_\epsilon^\alpha$. Then $\{Q(y_1,y_2)\cap F_\epsilon^\alpha \mid \alpha\in\mathcal{A}, \epsilon>0\}$ is a centered system of nonempty closed subsets of $X\times X$. Thus its intersection is nonempty, and $Q(y_1,y_2)\cap\bigcap_{\alpha\in\mathcal{A},\epsilon>0}F_\epsilon^\alpha \neq\emptyset \implies Q(y_1,y_2)\cap\text{“}\sim\text{”}\neq\emptyset \implies y_1=y_2$. Therefore we have $\bigcap_{\alpha\in\mathcal{A},\epsilon>0}V_\epsilon^\alpha = \Delta = \{(y,y)\mid y\in Y\}$. Obviously $V_\epsilon^\alpha\supset\Delta$ for any $\alpha\in\mathcal{A}$, $\epsilon>0$. Moreover, $\operatorname{Int} V_\epsilon^\alpha \supset \Delta$ for any $\alpha\in\mathcal{A}$, $\epsilon>0$. This follows from the inclusion

$$V_\epsilon^\alpha \supset U_\epsilon^\alpha = \{(y_1,y_2)\in Y^2 \mid \exists y_0\in Y\ \forall x_1\in q^{-1}(y_1), \forall x_2\in q^{-1}(y_2), \\ \exists z_1,z_2\in q^{-1}(y_0): \rho_\alpha(x_1,z_1)<\epsilon, \rho_\alpha(x_2,z_2)<\epsilon\}.$$

The upper semicontinuity of Q implies the openness of U_ϵ^α. Obviously $U_\epsilon^\alpha\supset\Delta$. Thus $(V_\epsilon^\alpha)_{\alpha\in\mathcal{A},\epsilon>0}$ is a base of a unique uniformity that generates the topology on Y.

Suppose that $(y_1,y_2),(y_1',y_2')\in V_\epsilon^\alpha$, $\lambda\in I$. Then there exist $x_1,x_2,z_1,z_2,x_1',x_2',z_1',z_2'\in X$ such that $q(x_1)=y_1$, $q(x_2)=y_2$, $z_1\sim z_2$, $\rho_\alpha(x_1,z_1)\leqslant\epsilon$, $\rho_\alpha(x_2,z_2)\leqslant\epsilon$, $q(x_1')=y_1'$, $q(x_2')=y_2'$, $z_1'\sim z_2'$, $\rho_\alpha(x_1',z_1')\leqslant\epsilon$, $\rho_\alpha(x_2',z_2')\leqslant\epsilon$. Then $\rho_\alpha(\lambda(x_1,x_1'),\lambda(z_1,z_1'))\leqslant\epsilon$, $\rho_\alpha(\lambda(x_2,x_2'),\lambda(z_2,z_2'))\leqslant\epsilon$, $\lambda(z_1,z_1')\sim\lambda(z_2,z_2')$. As $q(\lambda(x_1,x_1'))=\lambda(y_1,y_1')$, $q(\lambda(x_2,x_2'))=\lambda(y_2,y_2')$, we obtain $(\lambda(y_1,y_1'),\lambda(y_2,y_2'))\in V_\epsilon^\alpha$.

Thus $(V_\epsilon^\alpha)_{\alpha\in\mathcal{A},\epsilon>0}$ satisfies (4), $Y = X/_\sim$ is a semiconvex compactum and $q : X \to X/_\sim$ preserves semiconvex combination. Therefore if X is strongly semiconvex, then Y is strongly semiconvex as well.

Semiconvex compacta and their continuous mappings which preserve semiconvex combination form a category denoted by $\mathcal{S}Conv$. Strongly semiconvex compacta form a full subcategory $\mathcal{S}sConv \subset \mathcal{S}Conv$.

By $\mathcal{C}omp$ the category of compacta is denoted. Let $U : \mathcal{S}sConv \to \mathcal{C}omp$ and $U' : \mathcal{S}Conv \to \mathcal{C}omp$ be the forgetful functors.

Recall that a functor $L : \mathcal{B} \to \mathcal{C}$ is called *left adjoint* [1] to a functor $R : \mathcal{C} \to \mathcal{B}$ if there are given bijections $\theta(X,Y)$ between arrows from LX to Y in $\mathcal{B}$ and arrows from X to RY in $\mathcal{C}$ for all $X \in Ob\,\mathcal{C}$, $Y \in Ob\,\mathcal{B}$, and these bijections are natural by both arguments, i.e., the diagram

$$\begin{array}{ccc} \mathcal{B}(LX,Y) & \xrightarrow{\theta(X,Y)} & \mathcal{C}(X,RY) \\ \Big\downarrow{\scriptstyle \mathcal{C}(Lf,g)} & & {\scriptstyle \mathcal{C}(f,Rg)}\Big\downarrow \\ \mathcal{B}(LX',Y') & \xrightarrow{\theta(X',Y')} & \mathcal{C}(X',RY') \end{array}$$

commutes for any $X, X' \in Ob\,\mathcal{C}$, $Y, Y' \in Ob\,\mathcal{B}$, $f : X' \to X$, $g : Y \to Y'$.

Theorem 2. *There exist left adjoints to U and U'.*

Proof. An explicit construction of a left adjoint to U was described in [5]. Now an independent proof suitable for both cases will be given. Due to Freyd General Adjoint Functor Theorem [1] for a category $\mathcal{B}$ with all limits and a functor $R : \mathcal{B} \to \mathcal{C}$ the existence of a left adjoint $L : \mathcal{C} \to \mathcal{B}$ is equivalent to the following :

1) R preserves all limits;

2) R satisfies the *solution set condition*, i.e., for any $X \in Ob\,\mathcal{C}$ there exists a set $S \subset \{(Y,f) \mid Y \in Ob\,\mathcal{B},\ f : X \to RY\}$ (solution set) such that for any arrow $g : X \to RZ$, $Z \in Ob\,\mathcal{B}$, there is a pair $(Y,f) \in S$ and an arrow $h : Y \to Z$ in $\mathcal{B}$ such that $g = Rh \circ f$.

It suffices to check the existence of limits in $\mathcal{S}Conv$ and $\mathcal{S}sConv$ and their preservation by U' and U for two partial cases : for products and pairwise equalisers.

If X_α, $\alpha \in \mathcal{A}$ are (strongly) semiconvex compacta, then their product in $\mathcal{S}Conv$ ($\mathcal{S}sConv$) is merely a topological product with semiconvex combination defined by a formula

$$\lambda((x_\alpha),(y_\alpha)) = (\lambda(x_\alpha,y_\alpha)), \quad (x_\alpha),(y_\alpha) \in \prod_{\alpha\in\mathcal{A}} X_\alpha, \lambda \in I.$$

Clearly it is preserved by the forgetful functor.

If $f, g : X \to Y$ is a parallel pair in $\mathcal{S}Conv$ of $\mathcal{S}sConv$, then its equaliser in $\mathcal{C}omp$ is a set $X_0 = \{x \in X \mid f(x) = g(x)\}$ with the embedding $i : X_0 \to X$. This set is closed in X and closed under semiconvex combination. Therefore X_0 with the restriction of semiconvex combination from X and $i : X_0 \to X$ is the equaliser of f, g in $\mathcal{S}Conv$ ($\mathcal{S}sConv$) that is preserved by U' (respectively by U).

Prove that the solution set condition holds. Suppose that Z is a (strongly) semiconvex compactum, $g : X \to Z$ is a continuous map of compacta and $|X| \leqslant \tau$, τ is infinite. Then cardinality of the set

$$\{(\lambda_0,\ldots,\lambda_n)(f(x_0),\ldots,f(x_n)) \mid n\in\mathbb{N},\ x_0,\ldots,x_n\in X,\ (\lambda_0,\ldots,\lambda_n)\in\mathbb{Q}^{n+1}\cap\Delta_n\}$$

is not greater than τ. Its closure Y is the semiconvex hull of $f(X)$ in Z, and $g = h\circ f$, $h : Y \hookrightarrow Z$ is the embedding, $f \equiv g$, $f : X \to Y$. Therefore we can put S to be the set of all continuous maps from X to "representatives" of all (strongly) semiconvex compacta with density not greater than τ.

Any adjunction between $L : \mathcal{C} \to \mathcal{B}$ any $R : \mathcal{B} \to \mathcal{C}$ is uniquely determined by a pair of natural transformations [1] $\eta 1_{\mathcal{C}} \to RL$ (the *unit* of adjunction) and $\epsilon : LR \to 1_{\mathcal{B}}$ (the *counit*) such that $R\epsilon\circ\eta R = 1_R$, $\epsilon L\circ L\eta = 1_L$. Then the functor $T = RL : \mathcal{C}\to\mathcal{C}$ and natural transformations η and $\mu = R\epsilon L : T^2 \to T$ form a *triple* $\mathbb{T} = (T,\eta,\mu)$. This means that diagrams

$$\begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ {\scriptstyle T\eta}\downarrow & \searrow^{1_T} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

commute. Then η is called the *unit* and μ the *multiplication* of $\mathbb{T}$.

For an arbitrary triple $\mathbb{T} = (T,\eta,\mu)$ in $\mathcal{C}$ a pair (X,f), where $f : TX \to X$ is a morphism in $\mathcal{C}$, is called a $\mathbb{T}$-*algebra* iff the following commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta X} & TX \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle f} \\ & & X \end{array} \qquad \begin{array}{ccc} T^2X & \xrightarrow{\mu X} & TX \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f} \\ TX & \xrightarrow{f} & X \end{array}$$

An arrow $\phi : X \to Y$ is called *a map of algebras* $(X,f) \to (Y,g)$ if and only if $g\circ T\phi = \phi\circ f$. Algebras of a triple $\mathbb{T}$ in $\mathcal{C}$ and their maps form a category $\mathcal{C}^{\mathbb{T}}$. There exists a pair of adjoint functors $F^{\mathbb{T}} : \mathcal{C}\to\mathcal{C}^{\mathbb{T}}$ and $U^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$, $F^{\mathbb{T}}X = (TX,\mu X)$, $F^{\mathbb{T}}\phi = T\phi$, $U^{\mathbb{T}}(X,f) = X$, $U^{\mathbb{T}}\phi = \phi$. The triple $\mathbb{T}$ arises from this pair in a way discribed above as well as from original pair L, R. There exist the unique functor (*Eilenberg-Moore comparison functor*) $\Phi : \mathcal{B} \to \mathcal{C}^{\mathbb{T}}$ that makes the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{R} & \mathcal{C} \\ {\scriptstyle L}\uparrow & \searrow^{\Phi} & \uparrow{\scriptstyle U^{\mathbb{T}}} \\ \mathcal{C} & \xrightarrow[F^{\mathbb{T}}]{} & \mathcal{C}^{\mathbb{T}} \end{array}$$

commutative. If Φ is an equivalence of the categories then $\mathcal{B}$ is said to be *tripleable* [1] over $\mathcal{C}$ (with implicit adjoint functors L and R). T. Świrszcz [4] proved that convex compacta are tripleable over compacta (the left adjoint is a probability measure functor). Here is a counterpart for (strongly) semiconvex compacta.

Theorem 3. *Forgetful functors $U' : SConv \to Comp$ and $U : SsConv \to Comp$ are tripleable.*

Proof. Prove the statement for $U' : SConv \to Comp$ (the case of $U : SsConv \to Comp$ is quite analogous). Due to Beck's precise tripleability theorem [1] it is sufficient to prove that

1) U' has a left adjoint;
2) U' reflects isomorphisms;
3) $SsConv$ has and U' preserves coequalizers of U'-contractible coequaliser pairs [1].

(1) is proved above, (2) follows from the fact that isomorphisms in $SConv$ are homeomorphisms that preserve semiconvex combination. Let us prove (3).

Let $f_0, f_1 : X \to Y$ be an U'-contractible coequaliser pair in $SsConv$, i.e. there exists an arrow $t : Y \to X$ in $Comp$ such that $f_0 \circ t = 1_Y$, $f_1 \circ t \circ f_0 = f_1 \circ t \circ f_1$, and the pair f_0, f_1 has a coequaliser $h : Y \to Z$ in $Comp$. Then h is the quotient map of the closed equivalence relation : $y_1 \sim y_2$ for $y_1, y_2 \in Y$ if and only if there exist $x_1, x_2 \in X$ such that $f_0(x_1) = f_0(x_2)$, $y_1 = f_1(x_1)$, $y_2 = f_1(x_2)$. Moreover "$\sim$" is closed under semiconvex combinations. Suppose $y_1 \sim y_1'$, $y_2 \sim y_2'$, $\lambda \in I$. Then there exist $x_1, x_1', x_2, x_2' \in X$ such that $f_0(x_1) = f_0(x_1')$, $y_1 = f_1(x_1)$, $y_1' = f_1(x_1')$, $f_0(x_2) = f_0(x_2')$, $y_2 = f_1(x_2)$, $y_2' = f_1(x_2')$. Put $x = \lambda(x_1, x_2)$, $x' = \lambda(x_1', x_2')$, $y = \lambda(y_1, y_2)$, $y' = \lambda(y_1', y_2')$. Thus $f_0(x) = f_0(x')$, $y = f_1(x)$, $y' = f_1(x')$ implies $\lambda(y_1, y_2) \sim \lambda(y_1', y_2')$. By the first theorem $X/_\sim$ is semiconvex, and h is a coequaliser of f_0, f_1 in $SConv$.

Remark. In fact we have proved that U' and U form [1] coequalizers of U'-contractible (resp. U-contractible) coequaliser pairs. Thus the comparison functors are *isomorphisms* of categories.

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МОНАДИЗОВНІСТЬ КАТЕГОРІЇ (СТРОГО) НАПІВОПУКЛИХ КОМПАКТІВ НАД КАТЕГОРІЄЮ ОПУКЛИХ КОМПАКТІВ

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Поняття (строго) напівопуклого компакта і напівопуклої комбінації узагальнюють поняття опуклого компакта та опуклої комбінації (з різницею, що "відрізок", який з'єднує точку з собою, може бути нетривіальною петлею). Доведено, що фактор-простір (строго) напівопуклого компакта є (строго) напівопуклим за умови, що відповідне відношення еквівалентності замкнене стосовно формування напівопуклих комбінацій. Також доведено монадизовність категорії (строго) напівопуклих компактів над категорією компактів.

Ключові слова: компакт, (сильно) напівопуклий компакт, лівий спряжений функтор, монадичність.

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STANDARD FORM OF PAIRS OF MATRICES WITH RESPECT TO GENERALIZED EQUIVALENCE

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Pairs (A_1, A_2) and (B_1, B_2) of matrices over an adequate ring R are called *generalized equivalent pairs* if $A_1 = UB_1V_1$, $A_2 = UB_2V_2$ for some invertible matrices U, V_1, V_2 over R. A standard form to which a pair of matrices can be reduced by means of generalized equivalent transformations is established. Conditions under which pairs of matrices will be generalized equivalent are founded. Classes of pairs of matrices which have the unique standard form are given.

Key words: pairs of matrices, generalized equivalence, canonical diagonal form, standard form.

Let R be an adequate ring, i.e. R be domain of integrity in which every finitely generated ideal is principal and for every $a, b \in R$ with $a \neq 0$, a can be represented as $a = cd$ where $(c, b) = 1$ and $(d_i, b) \neq 1$ for any non-unit factor d_i of d [1]. Further let $M(n, k, R)$ and $M(n, R)$ be the sets of $n \times k$ and $n \times n$ matrices over R respectively; d_m^A be the greatest common divisor of minors of the order m of the matrix $A \in M(n, k, R)$; D^A be the canonical diagonal form (the Smith normal form) of the matrix A, i.e.

$$D^A = UAV = \operatorname{diag}(\varphi_1, \varphi_2, \dots, \varphi_r, 0, \dots, 0), \quad \varphi_r \neq 0, \quad \varphi_1 \mid \varphi_2 \mid \cdots \mid \varphi_r$$

for certain invertible matrices $U \in GL(n, R)$ and $V \in GL(k, R)$. Pairs of matrices (A_1, B_1) and (A_2, B_2), where $A_1, A_2 \in M(n, k_1, R)$ and $B_1, B_2 \in M(n, k_2, R)$ are called *generalized equivalent pairs* if $A_2 = UA_1V_1$ and $B_2 = UB_1V_2$ for certain matrices $U \in GL(n, R)$ and $U_i \in GL(k, R)$, $i = 1, 2$.

The reducibility of finite sets and pairs of matrices over polynomial and other rings by the same transformations to the triangular forms and their applications is considered in [2-7]. V. Dlab and C. M. Ringel [8] have established the canonical form of the pairs of complex matrices (A_1, A_2) with respect to the transformation $(A_1, A_2)(Q, P_1, P_2) = (QA_1P_1^{-1}, QA_2P_2^{-1})$, where Q is a complex invertible matrix, P_1 and P_2 are real invertible matrices.

The problem of the classification up to generalized equivalence of pairs of matrices over the rings as and the problem of the classification up to equivalence of matrices and of pairs of matrices, is wild [9]. Therefore such classification of pairs of matrices is possible only in some cases.

In this paper some form to which a pair of matrices can be reduced by means of generalized equivalent transformations is established. Conditions for generalized equivalence of pairs of matrices are given.

Lemma 1. *Let $B \in M(n,k,R)$ and $D^B = \operatorname{diag}(\psi_1,\dots,\psi_r,0,\dots,0)$. Then there exist an upper unitriangular matrix $U \in GL(n,R)$ and an invertible matrix $V \in GL(k,R)$ such that*

$$UBV = T^B = \begin{Vmatrix} \psi_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ b_{21}\psi_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{r-1,1}\psi_1 & \dots & \psi_{r-1} & 0 & 0 & \dots & 0 \\ b_{r1}\psi_1 & \dots & b_{r,r-1}\psi_{r-1} & b_{rr}\psi_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n1}\psi_1 & \dots & b_{n,r-1}\psi_{r-1} & b_{nr}\psi_r & 0 & \dots & 0 \end{Vmatrix},$$

where $(b_{rr},\dots,b_{nr}) = 1$.

Proof. By Lemma 1 [7] there exists a row matrix $u = \| 1 \quad u_2 \quad \dots \quad u_n \|$ such that $uB = \| c_1 \quad c_2 \quad \dots \quad c_k \|$, where $(c_1,c_2,\dots,c_k) = d_1^B = \psi_1$. Then for the matrix

$$U_1 = \begin{Vmatrix} 1 & u_2 \ \dots \ u_n \\ 0 & I_{n-1} \end{Vmatrix},$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix, and for some matrix $V_1 \in GL(k,R)$ we get

$$U_1BV_1 = \begin{Vmatrix} \psi_1 & 0 \ \dots \ 0 \\ b_{21}\psi_1 & \\ \dots & B_{n-1,k-1} \\ b_{n1}\psi_1 & \end{Vmatrix} = B_1, \quad B_{n-1,k-1} \in M(n-1,k-1,R).$$

We now carry out similar reasoning on the matrix $B_{n-1,k-1}$ etc. In the end we obtain the matrix B_{r-1}, such that at the lower right corner of B_{r-1} there is a matrix $B_{n-(r-1),k-(r-1)}$ and $\operatorname{rank} B_{n-(r-1),k-(r-1)} = 1$.

Then for some matrix $V_{r-1} \in GL(k-(r-1),R)$ we have

$$B_{n-(r-1),k-(r-1)}V_{r-1} = \begin{Vmatrix} b_{rr}\psi_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ b_{nr}\psi_r & 0 & \dots & 0 \end{Vmatrix}.$$

The lemma is proved.

Corollary 1. *Let $B \in M(n,k,R)$, $\operatorname{rank} B = n$. Then there exist an upper unitriangular matrix $U \in GL(n,R)$ and an invertible matrix $V \in GL(k,R)$ such that $UBV = T^B = TD^B$, where T is a lower unitriangular matrix in $GL(n,R)$.*

Further by $U(R)$ we denote the group of units of the ring R, by R_δ a complete set of residues modulo δ, and by R'_δ the maximal subset of R_δ such that $ua \not\equiv b \pmod{\delta}$ for any $a,b \in R'_\delta$ and every $u \in U(R)$.

Theorem 1. *Let $A \in M(n,k_1,R)$, $B \in M(n,k_2,R)$, $n \leqslant k_1$, $n \leqslant k_2$, and $D^A = \operatorname{diag}(\varphi_1,\dots,\varphi_r,0,\dots,0)$, $D^B = \operatorname{diag}(\psi_1,\dots,\psi_s,0,\dots,0)$, $r \leqslant s \leqslant n$. Then a pair of matrices (A,B) is generalized equivalent to the pair (D^A, T^B), where*

(i) if $s=r=n$, then

$$T^B = \begin{Vmatrix} \psi_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ t_{21}\psi_1 & \psi_2 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_{n1}\psi_1 & t_{n2}\psi_2 & \dots & \psi_n & 0 & \dots & 0 \end{Vmatrix}$$

and $t_{ij} \in R'_{\delta_{ij}}$, where $\delta_{ij} = \left(\dfrac{\varphi_i}{\varphi_j}, \dfrac{\psi_i}{\psi_j}\right)$, $i,j=1,\dots,n$, $i>j$;

(ii) if $r < s \leqslant n$, then

$$T^B = \begin{Vmatrix} \psi_1 & & & & & & & & & \\ t_{21}\psi_1 & \psi_2 & & & & & \mathbf{0} & & & \\ \dots & \dots & \ddots & & & & & & & \\ t_{r1}\psi_1 & t_{r2}\psi_2 & \cdots & \psi_r & & & & & & \\ t_{r+1,1}\psi_1 & t_{r+1,2}\psi_2 & \cdots & t_{r+1,r}\psi_r & \psi_{r+1} & & & & & \\ \dots & \dots & \dots & \dots & \dots & \ddots & & & & \\ t_{s1}\psi_1 & t_{s2}\psi_2 & \cdots & t_{sr}\psi_r & 0 & \cdots & \psi_s & & & \\ t_{s+1,1}\psi_1 & t_{s+1,2}\psi_2 & \cdots & t_{s+1,r}\psi_r & 0 & \cdots & 0 & 0 & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \\ t_{n1}\psi_1 & t_{n2}\psi_2 & \cdots & t_{nr}\psi_r & 0 & \cdots & 0 & 0 & \cdots & 0 \end{Vmatrix}$$

and $t_{ij} \in R'_{\delta_{ij}}$, $i=1,\dots,n$, $j=1,\dots,r$, $i>j$, where

$$\delta_{ij} = \begin{cases} \left(\dfrac{\varphi_i}{\varphi_j}, \dfrac{\psi_i}{\psi_j}\right), & \text{if } i,j=1,\dots,r,\ i>j, \\ \dfrac{\psi_i}{\psi_j}, & \text{if } i=r+1,\dots,s,\ j=1,\dots,r, \\ 0, & \text{if } i=s+1,\dots,s,\ j=1,\dots,r. \end{cases}$$

(iii) if $r=s<n$, then

$$T^B = \begin{Vmatrix} \psi_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ t_{21}\psi_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_{r-1,1}\psi_1 & \dots & \psi_{r-1} & 0 & 0 & \dots & 0 \\ t_{r1}\psi_1 & \dots & t_{r,r-1}\psi_{r-1} & t_{rr}\psi_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_{n1}\psi_1 & \dots & t_{n,r-1}\psi_{r-1} & t_{nr}\psi_r & 0 & \dots & 0 \end{Vmatrix},$$

$(t_{rr},\dots,t_{nr})=1$, and $t_{ij} \in R'_{\delta_{ij}}$, where $\delta_{ij} = \left(\dfrac{\varphi_i}{\varphi_j}, \dfrac{\psi_i}{\psi_j}\right)$, $i,j=1,\dots,r-1$, $i>j$.

Proof. The pair of matrices (A,B) is generalized equivalent to the pair (D^A, B_1), where $D^A = PAQ$, $B_1 = PB$ for some matrices $P \in GL(n,R)$ and $Q \in GL(k_1,R)$.

Further we shall reduce the matrix B_1 to a triangular form by means of admissible equivalent transformations (U,V), $U \in GL(n,R)$ and $V \in GL(k_2,R)$ such that $UD^A = D^A V_1$ for some matrix $V_1 \in GL(k_1,R)$. Then the matrix U has a form $U = \|u_{ij}\|_1^n$, where $u_{ij} = \frac{\varphi_i}{\varphi_j} u'_{ij}$ for all $i > j$, $i,j = 1,\dots,n$.

Let $\operatorname{rank} B_1 = \operatorname{rang} B = s > r$. Then by Lemma 2 there exist an upper unitriangular matrix $U_1 \in GL(n,R)$ and a matrix $V_1 \in GK(k_2,R)$ such that

$$U_1 B V_1 = \begin{Vmatrix} \psi_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ b_{21}\psi_1 & \psi_2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{r1}\psi_1 & b_{r2}\psi_2 & \cdots & \psi_r & 0 & \cdots & 0 \\ b_{r+1,1}\psi_1 & b_{r+1,2}\psi_2 & \cdots & b_{r+1,r}\psi_r & b_{r+1,r+1} & \cdots & b_{r+1,k_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1}\psi_1 & b_{n2}\psi_2 & \cdots & b_{nr}\psi_r & b_{n,r+1} & \cdots & b_{nk_2} \end{Vmatrix} = B_2.$$

Therefore for some matrices $P_{n-r} \in GL(n-r,R)$ and $Q_{k_2-r} \in GK(k_2-r,R)$ we have $P_{n-r}B_{n-r,k_2-r}Q_{k_2-r} = \operatorname{diag}(\psi_{r+1},\dots,\psi_s,0,\dots,0)$, where

$$B_{n-r,k_2-r} = \begin{Vmatrix} b_{r+1,r+1} & \cdots & b_{r+1,k_2} \\ \cdots & \cdots & \cdots \\ b_{n,r+1} & \cdots & b_{nk_2} \end{Vmatrix}.$$

Put $U_2 = I_r \oplus P_{n-r}$, $V_2 = I_r \oplus P_{k_2-r}$. Thus $U_2B_2V_2$ is a lower triangular matrix with the principal diagonal D^B.

Then similarly as in the proof of Theorem 1 [6] we reduce the elements b_{ij} of the matrix T_1^B modules δ_{ij} $i = 2,\dots,n$, $j = 1,\dots,r$, $j < i$. Thus we get a matrix $T_2^B = U_3T_1^BV_3$ whose (i,j)-element is equal to $c_{ij}\psi_j$, where $c_{ij} \in R_{\delta_{ij}}$. If $c_{ij} \notin R'_{\delta_{ij}}$, then there exists $t_{ij} \in R'_{\delta_{ij}}$ such that $u_ic_{ij} \equiv t_{ij}(\operatorname{mod}\delta_{ij})$ for some $u_i \in U(R)$. Then similarly as in the previous case we obtain the matrix $T_3^B = U_4T_2^BV_4$ whose (i,j)-element is equal to $t_{ij}\psi_j$, $i = 2,\dots,n$, $j = 1,\dots,r$, $j < i$, ets. Therefore we get the matrix T^B which is defined in Theorem 1. Since we made admissible equivalent transformations over matrices B_i and T_i^B, $i = 1,2,\dots$ the pair of matrices (A,B) is generalized equivalent to the pair of matrices (D^A,T^B). In cases (i) and (iii) the proofs are similar. Therefore the proof of the theorem is complete.

Definition 1. The pair (D^A,T^B) which defined in Theorem 1 is called *the standard form* of the pair of matrices (A,B).

We remark that the standard form (D^A,T^B) of the pair of matrices (A,B) with respect to generalized equivalence is uniquely determined only in some cases.

Corollary 2. *Let $A \in M(n,k_1,R)$, $B \in M(n,k_2,R)$, $n \leqslant k_1$, $n \leqslant k_2$, and $(d_n^A, d_n^B) = 1$. Then the pair diagonal matrices (D^A,D^B) is the unique standard form of the pair of matrices (A,B).*

Further we shall establish some conditions for generalized equivalence of pairs of matrices. Since for any matrix $A \in M(n,k,R)$, $n < k$ there exists a matrix $V \in GL(k,R)$ such that $AV = \|A_1 \quad 0\|$, where $A_1 \in M(n,R)$ it is sufficient to consider the generalized equivalence of pairs of square $n \times n$-matrices.

Any pair of matrices is generalized equivalent to the pair of matrices of standard form. Hence it is sufficient to consider the generalized equivalence of pairs of matrices of the standard form.

Theorem 2. *Let* (D, T_1) *and* (D, T_2) *be pairs of matrices of standard form, i.e.* $D = \operatorname{diag}(\varphi_1, \dots, \varphi_n)$, $\varphi_n \neq 0$, $\varphi_1 \mid \varphi_2 \mid \cdots \mid \varphi_n$ *and* T_1, T_2 *are lower triangular matrices with the principal diagonal* $\Psi = \operatorname{diag}(\psi_1, \dots, \psi_s, 0, \dots, 0)$, $\psi_1 \mid \psi_1 \mid \dots \psi_s$. *Then the pairs of matrices* (D, T_1) *and* (D, T_2) *are generalized equivalent if and only if the following condition holds:*

(i) the matrices $(\operatorname{adj} D)T_1$ *and* $(\operatorname{adj} D)T_2$ *are equivalent, i.e.*

$$S(\operatorname{adj} D)T_1 = (\operatorname{adj} D)T_2 Q, \quad S, Q \in GL(n, R); \tag{1}$$

(ii) in the set $\mathcal{S} = \{S \mid S(\operatorname{adj} D)T_1 = (\operatorname{adj} D)T_2 Q,\ S, Q \in GL(n, R)\}$ *the exists a matrix*

$$S = \| s_{ij} \|, \text{ such that } s_{ij} = \frac{\varphi_j}{\varphi_i} s'_{ij}, \ i, j = 1, \dots n, \ j > i. \tag{2}$$

Proof. **Necessity.** Let be $UDV_1 = D$ and $UT_1V_2 = T_2$, where $U, V_1, V_2 \in GL(n, R)$. The matrix $V_1 = \| v_{ij} \|_1^n$ has the form (2). Then

$$(\operatorname{adj} D)T_2 = (\operatorname{adj} V_1)(\operatorname{adj} D)(\operatorname{adj} U)UT_1V_2 = (\operatorname{adj} V_1)(\operatorname{adj} D)T_1V_2v, \tag{3}$$

$v \in U(R)$. Since $\operatorname{adj} V_1 = v_1 V_1^{-1}$, $v_1 \in U(R)$, then the equality (3) implies

$$V_1^{-1}(\operatorname{adj} D)T_1 = (\operatorname{adj} D)T_2 Q, \ Q \in GL(n, R).$$

Since the matrix V_1 has a form (2), therefore V_1^{-1} has the same form [10].

Sufficiently. Assume the conditions (1) and (2) hold. It is easily to see, that $S(\operatorname{adj} D) = (\operatorname{adj} D)U$, where

$$U = \| u_{ij} \|_1^n, \ u_{ij} = \frac{\varphi_i}{\varphi_j} u'_{ij}, \ i, j = 1, \dots, n, \ i > j. \tag{4}$$

Then the equality (1) implies $(\operatorname{adj} D)UT_1 = (\operatorname{adj} D)T_2 Q$ or $UT_1 = T_2 Q$. Since the matrix U has the form (4), then $UD = DV$. Therefore the pairs of matrix (D, T_1) and (D, T_2) are generalized equivalent.

Corollary 3. *Let* $A, B \in M(n, R)$ *and* A *be a nonsingular matrix. Then the pair of matrices* (A, B) *is generalized equivalent to the pair of diagonal matrices* (D^A, D^B) *if and only if the matrices* $(\operatorname{adj} A)B$ *and* $(\operatorname{adj} D^A)D^B$ *are equivalent.*

Proof. The pair matrices (A, B) is generalized equivalent to the pair of matrices of standard form (D^A, T^B). The matrix $(\operatorname{adj} D^A)T^B$ is equivalent to the matrix $(\operatorname{adj} A)B$ and hence it is equivalent to $(\operatorname{adj} D^A)D^B$. Then there exist lower unitriangular matrices $S, Q \in GL(n, R$ such that $S(\operatorname{adj} D^A)T^B = (\operatorname{adj} D^A)D^B Q$. Therefore the statement (ii) of Theorem 2 holds for the matrix S. .

Let (D,T_1) and (D,T_2) are pairs of matrices of such form: $D = \operatorname{diag}(1,\dots,1,\varphi_n)$, $\varphi_n \neq 0$,

$$T_i = \begin{Vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \psi_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \psi_{n-1} & 0 \\ t_{i1} & t_{i2}\psi_2 & \dots & t_{i,n-1}\psi_{n-1} & \psi_n \end{Vmatrix}, \quad \psi_2 \mid \psi_3 \mid \cdots \mid \psi_n, \ \psi_n \neq 0, \qquad (5)$$

$i = 1,2$. Without loss of generality we may assume that $\psi_1 = 1$. Further by d_i we denote the greatest common divisor of elements of the last row of the matrix T_i:

$$d_i = (t_{i1}, t_{i2}\psi_2, \dots, t_{i,n-1}\psi_{n-1}, \psi_n), \quad i = 1,2.$$

Lemma 3. *If the pairs of matrices (D,T_1) and (D,T_2) are generalized equivalent, then $(d_1,\varphi_n) = (d_2,\varphi_n)$.*

The proof of lemma follows from Theorem 2.

Lemma 4. *Let (D,T_1) be a pair of matrices of the form (5) and $(d_1,\varphi_n) = 1$. Then the canonical diagonal form of the matrix $(\operatorname{adj} D)T_1$ is equal to the product of canonical diagonal forms of the matrices $\operatorname{adj} D$ and T_1, i.e.*

$$D^{(\operatorname{adj} D)T_1} = D^{\operatorname{adj} D} D^{T_1} = \operatorname{diag}(1, \varphi_n\psi_2, \dots, \varphi_n\psi_n).$$

Proof. By Cauchy-Binet formula the minors of product of matrices we get that $\varphi_n^{p-1}\psi_2\dots\psi_p$ divides every minor of order p of the matrix $(\operatorname{adj} D)T_1$. Since $(d_1,\varphi_n) = 1$ we have $d_p^{(\operatorname{adj} D)T_1} = \varphi_n^{p-1}\psi_2\dots\psi_p$. This implies the statement of the lemma.

Theorem 3. *Let (D,T_1) and (D,T_2) be pairs of matrices of form (5). If $(d_1,\varphi_n) = (d_2,\varphi_n) = 1$, then the pairs of matrices (D,T_1) and (D,T_2) are generalized equivalent.*

Proof. By Lemma 4 the canonical diagonal forms of the matrices $(\operatorname{adj} D)T_1$ and $(\operatorname{adj} D)T_2$ coincide, i.e. the following matrices are equivalent:

$$S(\operatorname{adj} D)T_1 = (\operatorname{adj} D)T_2Q, \quad S,Q \in GL(n,R), \quad S = \| s_{ij} \|_1^n. \qquad (6)$$

Thus we have $\varphi_n \mid s_{in}t_{1j}\psi_j$, for all $j = 1,\dots,n$, $i = 1,\dots,n-1$, where $\psi_1 = t_{1n} = 1$. Since $(d_1,\varphi_n) = 1$ we obtain that $\varphi_n \mid s_{in}$ for all $i = 1,\dots,n-1$, i.e. statement (ii) of Theorem 2 holds for the matrix S. Therefore the proof of the theorem is complete.

Corollary 5. *Let (D,T_1) be a pair of matrices of form (5). If $(d_1,\varphi_n) = 1$, then (D,T_1) is generalized equivalent to the pair $(D,T\Psi)$, where $\Psi = \operatorname{diag}(1,\psi_2,\dots,\psi_n)$,*

$$T = \begin{Vmatrix} & I_{n-1} & & & 0 \\ t & 0 & \dots & 0 & 1 \end{Vmatrix},$$

and

$$t = \begin{cases} 0, & \text{if the matrices } (\operatorname{adj} D)T_1 \text{ and } (\operatorname{adj} D)\Psi \text{ are equivalent;} \\ 1, & \text{otherwise.} \end{cases}$$

Put $R''_\delta = \{a \in R'_\delta \mid (a,\delta) \neq 1 \text{ for all } a \neq 1\}$.
Then Theorems 1 and 3 imply

Corollary 6. *Let $A, B \in M(n, R)$,*

$$D^A = \operatorname{diag}(1, \ldots, 1, \varphi_n), \quad D^B = \operatorname{diag}(1, \psi_2, \ldots, \psi_n),$$

$(\varphi_n, \psi_n) = \delta_n$ and $(\psi_{n-1}, \delta_n) = 1$. Then the pair of matrices (A, B) is generalized equivalent to the pair (D^A, TD^B), where

$$T = \left\| \begin{matrix} & I_{n-1} & & & \mathbf{0} \\ t_1 & t_2 & \ldots & t_{n-1} & 1 \end{matrix} \right\|, \quad t_j \in R''_\delta, \ j = 1, \ldots, n-1.$$

Theorem 4. *Let $A_i, B_i \in M(n, R)$, $D^{A_i} = D^A = \operatorname{diag}(1, \ldots, 1, \varphi_n)$, $D^{B_i} = D^B = \operatorname{diag}(1, \psi_2, \ldots, \psi_n)$, $i = 1, 2$. Let $(\varphi_n, \psi_n) = p$, $(\psi_{n-1}, p) = 1$ and p be a prime element of the ring R. Then*

(i) the pairs of matrices (A_1, B_1) and (A_2, B_2) are generalized equivalent if and only if the matrices $(\operatorname{adj} A_1)B_1$ and $(\operatorname{adj} A_2)B_2$ are equivalent;

(ii) the pair of matrices (A_1, B_1) is generalized equivalent to the pair (D^{A_1}, TD^{B_1}), where

$$T = \left\| \begin{matrix} & I_{n-1} & & \mathbf{0} \\ t & 0 & \ldots \ 0 & 1 \end{matrix} \right\|$$

and

$$t = \begin{cases} 0, & \text{if the matrices } (\operatorname{adj} A_1)B_1 \text{ and } (\operatorname{adj} D^{A_1})D^{B_1} \text{ are equivalent;} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. The pair of matrices (A_i, B_i) $(i = 1, 2)$ is generalized equivalent to the pair $\left(D^{A_i}, T_i^{B_i}\right)$ of form (5) and $t_{ij} \in R_p$, $j = 1, \ldots, n-1$. Then $(d_i, \varphi_n) = 1$ if there exists $t_{ij} \neq 0$, $j = 1, \ldots, n-1$. Further we use Theorem 3, which completes the proof of the statement.

Example. Let $\mathcal{N} = \{(A, B) \mid A, B \in M(2, \mathbb{Z})$ such that $D^A = \operatorname{diag}(1, 25), D^B = \operatorname{diag}(1, 175)\}$. Then $\delta = (25, 175) = 25$, $R_\delta = \{0, 1, \ldots, 24\}$, $R'_\delta = \{0, 1, \ldots, 12\}$ and $R''_\delta = \{0, 1, 5, 10\}$. Then the set $\mathcal{N}$ is partitioned up to the generalized equivalence on four disjoint classes with representations

$$\left(\left\| \begin{matrix} 1 & 0 \\ 0 & 25 \end{matrix} \right\|, \left\| \begin{matrix} 1 & 0 \\ t & 175 \end{matrix} \right\| \right), \quad t = 0, 1, 5, 10.$$

The direct verification shows that the pairs of matrices

$$\left(\left\| \begin{matrix} 1 & 0 \\ 0 & 25 \end{matrix} \right\|, \left\| \begin{matrix} 1 & 0 \\ 5 & 175 \end{matrix} \right\| \right) \text{ and } \left(\left\| \begin{matrix} 1 & 0 \\ 0 & 25 \end{matrix} \right\|, \left\| \begin{matrix} 1 & 0 \\ 10 & 175 \end{matrix} \right\| \right)$$

are not generalized equivalent.

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СТАНДАРТНА ФОРМА ПАР МАТРИЦЬ ВІДНОСНО УЗАГАЛЬНЕНОЇ ЕКВІВАЛЕНТНОСТІ

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Пари матриць (A_1, A_2) і (B_1, B_2) над адекватним кільцем R називаються узагальнено еквівалентними, якщо $A_1 = UB_1V_1, A_2 = UB_2V_2$ для деяких оборотних матриць U, V_1, V_2 над R. Щодо таких перетворень визначено стандартну форму пар матриць та зазначено умови їх загальної еквівалентності. Виділено класи пар матриць, для яких ця форма визначається однозначно.

Ключові слова: пари матриць, узагальнена еквівалентність, канонічно діагональна форма, стандартна форма.

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n-TRIVIAL KNOTS AND THE ALEXANDER POLYNOMIAL

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For each integer $n \geqslant 1$, we construct via the pure braid commutators $(2n-1)$-trivial knots with non-trivial Alexander polynomial. We formulate also a sufficient condition under which an $(n-1)$-trivial knot, $n > 2$, has trivial Alexander polynomial. As a particular case, for each $n > 1$, we describe some class of "geometric" $(n-1)$-trivial knots with trivial Alexander polynomial.

Key words: invariant of finite order, braid commutator, Seifert surface, Alexander polynomial, trivalent diagram, n-trivial knot.

1. Knots K and L are called V_n-equivalent (n-equivalent) if they cannot be distinguished by the Vassiliev invariants (additive Vassiliev invariants, respectively) of order $\leqslant n$, the invariants taking values in any abelian group. Goussarov [6] was the first who has characterized the relation of n-equivalence on knots in combinatorial terms. Later on, it turned out that the relations of V_n-equivalence and n-equivalence coincide on the knots in S^3 (see [7,15]). Habiro [7] has characterized n-equivalence of knots in terms of the so-called C_{n+1}-moves. Stanford [15] has given a description of n-equivalent knots in terms of the pure braid $(n+1)$-commutators. A knot in S^3 is called n-trivial if it is n-equivalent to the trivial one. In [9], Kalfagianni and Lin have introduced for each $n \geqslant 2$ the classes of "geometric" knots, among other the classes of n-hyperbolic, n-elliptic and n-parabolic knots, and showed that all they are n-trivial. Moreover, any n-hyperbolic and n-elliptic knot has the trivial Alexander polynomial [9]. The latter two classes do not exhaust however all n-trivial knots. Kalfagianni and Lin showed (Proposition 6.1 of [9]) that for each integer $n \geqslant 1$ there exists an n-trivial knot with the non-trivial Alexander polynomial. The proof of the proposition is based on Theorem 1 of [3] (which proves the Melvin-Morton-Rozansky conjecture) and is rather of existence character. In the present paper, for each integer $n \geqslant 1$ we indicate in an explicit form the $(2n-1)$-trivial knots having non-trivial Alexander polynomial. Our approach uses in essential way the characterization of n-equivalent knots in terms of the pure braid commutators (see [14] and [15]).

In [12], H. Murakami and T. Ohtsuki have described the filtration on the vector space S over the rationals $\mathbb{Q}$ spanned by Seifert matrices of knots,

$$\mathcal{S} \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \dots,$$

and related this to the Goussarov-Vassiliev filtration of the vector space spanned by knots. They showed that a rational Vassiliev invariant of order n comes from the Alexander polynomial (i.e. can be expressed as a sum of products of the coefficients of the Alexander-Conway polynomial) if and only if it can be factored through the quotient space $\mathcal{S}/\mathcal{S}_{n+1}$. In this paper, using the above mentioned results of H. Murakami and T. Ohtsuki (see [12]) and the results of A. Kricker, B. Spence, and I. Aitchison [10] on the characterization of rational weight systems coming from the Alexander-Conway polynomial, we obtain a sufficient condition for an n-trivial knot, $n \geqslant 2$, to have the trivial Alexander polynomial. As a particular case, for each $n \geqslant 1$ we describe some class of "geometric" n-trivial knots with the trivial Alexander polynomial, where each such "geometric" n-trivial knot is obtained from the trivial one by inserting in it the "double" pure braid $(n+1)$-commutators (see [14] for details).

Now we define some needed notions and review briefly the results on the characterization of n-equivalent knots via pure braid commutators [15] and C_{n+1}-moves [7]. We shall also review the results of H. Murakami and T. Ohtsuki, and A. Kricker, B. Spence, and I. Aitchison on the characterization of the rational Vassiliev invariants and weight systems coming from the Alexander-Conway polynomial (see [12] and [10] for details).

Let $\mathcal{K}$ denote the free abelian group generated by the classes of equivalent oriented knots in S^3 and $\mathcal{K}_n$ the subgroup of $\mathcal{K}$ generated by all n-singular knots, $n \geqslant 1$. Here an n-singular knot we regard as an element of $\mathcal{K}$ so that each double point of this knot is replaced by a difference of the positive and negative crossings (see Fig. 1).

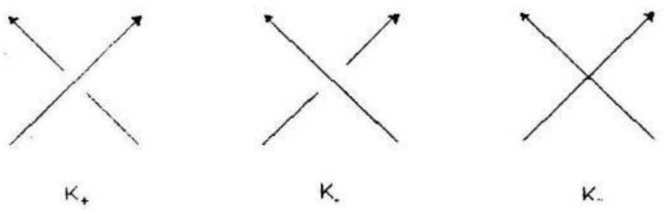

Fig. 1

Let

$$\mathcal{K} \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

be the Vassiliev-Goussarov filtration of $\mathcal{K}$. A Vassiliev invariant of type n, $n \geqslant 0$, taking values in an abelian group H is a map $v: \mathcal{K} \to H$ vanishing on the subgroup $\mathcal{K}_{n+1}$. The smallest number m such that v vanishes on $\mathcal{K}_{m+1}$ is called the order of v. The Vassiliev invariants are called also the invariants of finite type of knots (or links).

A trivalent diagram D of degree n is a connected graph with $2n$ vertices all of which are trivalent. There is a distinguished subgraph which is homeomorphic to a circle, called the external one. Each vertex of D which lies on the external circle is called external, otherwise it is internal. At each internal vertex a of a trivalent diagram one of two possible cyclic ordering of the edges around this vertex (the orientation at a) is chosen. The subgraph of D having the same vertex set as D and the edge set of which consists of all edges of D which do not lie on the external circle is called the internal graph of D. An orientation of the external circle is chosen and the other edges of D are taken to be non-oriented. All trivalent diagrams are considered up to an isomorphism of distinguished graphs that respects the above structures on them. Every trivalent diagram D is pictured in the plane in such a way that each its internal vertex has the counterclockwise orientation. If a trivalent diagram has no internal vertices it is called a chord diagram. Denote by $\mathcal{D}$ and $\mathcal{D}_n$ the collections of all trivalent diagrams and trivalent diagrams of degree n, respectively.

Define $\mathcal{A}_n$ and $\mathcal{A}$ to be the quotient groups, $\mathcal{A}_n = \mathbb{Z}\mathcal{D}_n/\{\text{all STU relations}\}$ and $\mathcal{A} = \mathbb{Z}\mathcal{D}/\{\text{all STU relations}\}$, where STU is the homogeneous relation on $\mathbb{Z}\mathcal{D}$ indicated in Fig. 2.

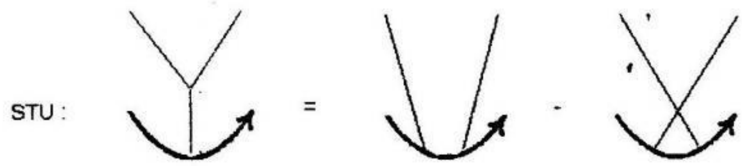

Fig. 2

Note that the graded abelian group $\mathcal{A}$ is naturally isomorphic to the graded abelian group $\mathcal{A}^c$, the quotient of the group freely generated by chords diagrams via 4T-relations (see [2]). A trivalent diagram D is called a tree diagram, if its internal graph is a tree. A trivalent diagram T is called a one-branch tree diagram of degree n, if its internal graph is isomorphic to the standard n-tree. There is a natural one-to-one correspondence between the permutations of the symmetric group S_n and the one-branch tree diagrams of degree n. For a given permutation $\sigma \in S_n$, denote by T_σ the one-branch trivalent diagram of degree n corresponding to σ (see [11]). Note that $\mathcal{A} \otimes \mathbb{Q}$ has an algebra structure with respect to the connected sum of external circles of trivalent diagrams (the product of trivalent diagrams) [2]. The co-product ∇ on $\mathcal{A} \otimes \mathbb{Q}$ is defined in a natural way (see [2]). With respect to these operations, $\mathcal{A} \otimes \mathbb{Q}$ is a commutative and co-commutative Hopf algebra and is generated by the primitive elements of $\mathcal{A} \otimes \mathbb{Q}$ [2]. The primitive subspace $\mathcal{P}$ of $\mathcal{A} \otimes \mathbb{Q}$ is generated (as a graded vector space) by the primitive trivalent diagrams, i.e. the trivalent diagrams with the connected internal graph. It is known (see [5]) that the primitive space $\mathcal{P}_d, d > 1$, is generated by the trivalent diagrams of the two types. The first type of generators consists of the primitive trivalent diagrams whose internal graph has the negative Euler characteristic (see [8]). The second type (only for even $d = 2n$) consists of the so-called "wheel" ω_{2n} (see Fig. 3). Therefore, for odd $d > 1$ the primitive space $\mathcal{P}_d$ is generated by the primitive trivalent diagrams of the first type. The space $\mathcal{P}_1$ is one-dimensional and is generated by a chord diagram D_1 with a single chord. It follows that as an algebra, $\mathcal{A} \otimes \mathbb{Q}$ is generated by D_1, the primitive trivalent diagrams with negative Euler characteristic and the "wheels" $\omega_{2n}, n \geqslant 1$.

An (unframed) $\mathbb{Q}$-valued weight system of degree n is a map $w: \mathcal{A}_n \to \mathbb{Q}$ which vanishes on each trivalent (chord) diagram of degree n having an isolated chord. A split diagram is a diagram which can be decomposed into a product of diagrams of lower order. By Kontsevich's integral, each $\mathbb{Q}$-valued weight system of degree n can be integrated (in a non-unique way) to a $\mathbb{Q}$-valued Vassiliev invariant of order n [2]. A rational-valued Vasiliev invariant v of order n is called canonical if it is determined uniquely by its weight system $w(v)$, of the same degree n [3]. Under the Alexander-Conway polynomial we shall mean a canonical Vassiliev power series $\tilde{C}$ satisfying the following two axioms $\mathbb{A}1$ and $\mathbb{A}2$:

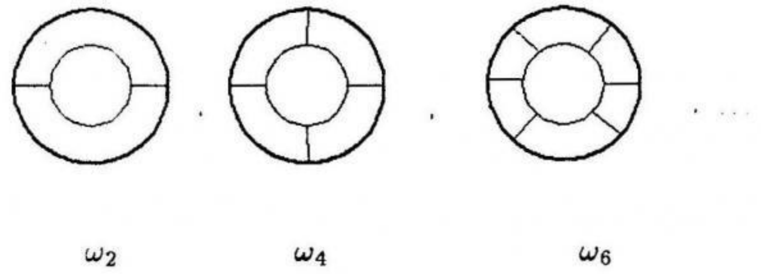

Fig. 3

$\mathbb{A}1$ (the skein-relation). $\tilde{C}(K_+) - \tilde{C}(K_-) = (e^{h/2} - e^{-h/2})\tilde{C}(K_0)$,
for any link diagrams K_+, K_- and K_0 which are the same outside some small disc in the plane where they look as positive crossing, negative crossing and smoothing, respectively;

$\mathbb{A}2$ (the initial data). $\tilde{C}$(c-component unlink)$= 1$ if $c = 1$ and 0, otherwise.

Therefore the Alexander-Conway polynomial is a renormalized and reparametrized version of both the Alexander and Conway polynomials. Bar-Natan and Garoufalidis described the Conway weight systems $w: \mathcal{A}^c \to \mathbb{Z}$ in terms of intersection graphs of chord diagrams (Theorem 3 of [3]) and the universal immanants of such intersection graphs (Theorem 5 of [3]). Chmutov [5] has calculated the Alexander-Conway weight systems on the generators of the space $\mathcal{P}_d, d > 1$. In particular, he showed that every (framed or unframed) Alexander-Conway weight system of degree $n > 1$ vanishes on primitive trivalent diagrams with the internal graph of negative Euler characteristic (see also [10]). Basing on the results of the paper [10], Kricker (Lemma 2.11 of [8]) has described the algebra of (framed) Alexander-Conway weight systems (see also Lemma 2.1 of [12]). We formulate its unframed version as follows.

1.1. Lemma. *If an (unframed) $\mathbb{Q}$-valued weight system w vanishes on every trivalent diagram, the internal graph of which has a component with the negative Euler characteristic, then it can be represented as a sum of products of weight systems coming from the coefficients of the Alexander-Conway polynomial.*

Let B_k be the braid group on k strands and P_k its subgroup of pure braids. For $0 \leqslant i < j \leqslant k-1$ let $p_{i,j} \in P_k$ be the braid that links the ith and jth strands behind the others (see Fig. 4). It is known [4] that the collection of braids $\{p_{i,j}\}_{0 \leqslant i < j \leqslant k-1}$ represents the standard generators of the group P_k.

By a tangle diagram we shall mean a knot diagram K with a single S^1-boundary which intersects each of the strands in the diagram transversely. To put it in another words, K is the closure of an oriented tangle T with $\operatorname{dom} T = \operatorname{codom} T$ which is positioned in $\mathbb{R}^2$ outside of a disc D, the latter being bounded by S^1. As in [16], for a fixed k by a tangle map $T: P_k \to \{\text{knot types}\}$ with $\operatorname{dom} T = \operatorname{codom} T = k$, we shall mean a canonical way of putting a pure braid $p \in P_k$ into a tangle diagram to get an oriented knot $T(p)$ (see also [14] for details).

Let $LCS_n(P_k)$ denote the nth subgroup of the lower central series of the group P_k. For each $\sigma \in S_n$ denote by $p_\sigma \in LCS_n(P_{n+1})$, the pure braid n-commutator of the following form $p_\sigma = [\dots[p_{0,\sigma(n)}, p_{0,\sigma(n-1)}], \dots], p_{0,\sigma(1)}]$, and let $p_n \in LCS_n(P_{n+1})$ be the pure braid n-commutator $p_n = [p_{n-1,n}, [p_{n-2,n-1}, \dots, [p_{1,2}, p_{0,1}] \dots]$. In [14], it is shown that each one-branch simple C_n-move on a knot, defined by Habiro [7], where $n \geqslant 2$, is equivalent to the insertion (in the non-oriented setting) in this knot via some tangle map of the pure braid n-commutator p_n.

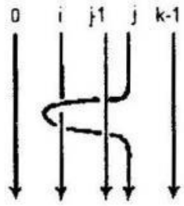

Fig. 4

The operation on oriented knots, inverse to the insertion, is called the deletion (of the pure braids) in knots. As discussed in [14], for each $n \geqslant 2$ the insertions of the coloured pure braid n-commutator p_n in a knot via the tangle maps can be considered as a topological realization of one-branch tree diagrams of degree n. Similarly, the closure $\widehat{p_\sigma}$ of the braid p_σ via the permutation $(1, 2, \dots, n)$, where $\sigma \in S_n$, gives a topological realization of the one-branch tree diagram T_σ.

Let $\mathcal{K}'$ denote the vector space over $\mathbb{Q}$ spanned by all knot types in S^3 and let

$$\mathcal{K}' \supset \mathcal{K}'_1 \supset \mathcal{K}'_2 \supset \mathcal{K}'_3 \supset \dots$$

be the Vassiliev-Goussarov filtration of $\mathcal{K}'$. A rational Vassiliev invariant of type n is a map $\mathcal{K}'/\mathcal{K}'_{n+1} \to \mathbb{Q}$.

Let $\mathcal{M}$ be the set of integer matrices of even size such that $M - M^t$ is unimodular. Denote by $[M]$ the S-equivalence class of matrices in $\mathcal{M}$ which contains M. Let S be the vector space over $\mathbb{Q}$ spanned by the S-equivalence classes of matrices in $\mathcal{M}$. H. Murakami and T. Ohtsuki [12] have defined a filtration of $\mathcal{S}$ in the following way. For a matrix $M \in \mathcal{M}$ of the size $m \times m$ and the integers $i_1, i_2, \dots, i_n$, where $i_j \leqslant n, 1 \leqslant j \leqslant m$, define the alternating sum

$$\sum_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n = 0,1} (-1)^{\varepsilon_1 + \dots + \varepsilon_n} [M + \varepsilon_1 E_{i_1 i_1} + \dots + \varepsilon_n E_{i_n i_n}] \in S,$$

where E_{ii} is the matrix of the size $m \times m$ with (i,i)-entry 1 and the others 0. There is a natural linear map $s: \mathcal{K}' \to \mathcal{S}$ which takes a knot to the S-equivalence class of a Seifert matrix for this knot. The map s respects the filtrations of both the vector spaces $\mathcal{K}'$ and $\mathcal{S}$ and so, induces a map $\mathcal{K}'/\mathcal{K}'_{n+1} \to \mathcal{S}/\mathcal{S}_{n+1}$, denoted also by s. We shall say that a Vassiliev invariant $v: \mathcal{K}'/\mathcal{K}'_{n+1} \to \mathbb{Q}$ comes from Seifert matrices if v can be factored through the map s. H.Murakami and T.Ohtsuki [12] showed that a rational

Vassiliev invariant $v: \mathcal{K}'/\mathcal{K}'_{n+1} \to \mathbb{Q}$ comes from the Alexander-Conway polynomial if and only if it comes from Seifert matrices. As a consequence, any rational Vassiliev invariant of knots coming from Seifert matrices is equal to a linear sum of products of coefficients of Alexander-Conway polynomial.

Now let us recall some needed notions and facts from Habiro's clasper theory [7]. Let K be a knot in S^3. A clasper G for K is a framed uni-trivalent graph embedded in S^3 so that all its univalent vertices (and only they) are positioned on K and all possible intersections between the edges of G and K are transversal. We use the blackboard framing for description of claspers G. The degree $\deg(G)$ of the clasper G is the half of the number of its vertices. Any pair (K, G), where K is a knot and G is a clasper on it, defines a surgery of S^3 and S^3 surgered will be always a three sphere. Denote by K_G a knot obtained from the knot K by surgery of S^3 defined by the pair (K, G). Let $\mathcal{G}_n$ be the vector space over $\mathbb{Q}$ spanned by all the pairs (K, G) with $\deg(G) = n$. Habiro [7] has defined for each $n \geqslant 1$ a natural surjective map $e: \mathcal{G}_n \to \mathcal{K}'_n$. Let $\gamma: \mathcal{G}_n \to \mathcal{A}_n \otimes \mathbb{Q}$ be the map forgetting the embedding, $\varphi: \mathcal{A}_n \otimes \mathbb{Q} \to \mathcal{K}'_n/\mathcal{K}'_{n+1}$ the map which replaces chords by double points and let $p: \mathcal{K}'_n \to \mathcal{K}'_n/\mathcal{K}'_{n+1}$ be the canonical projection. Because of Kontsevich's integral over $\mathbb{Q}$, the map φ is an isomorphism [2]. Habiro actually showed [7] that the equality $p \cdot e = \varphi \cdot \gamma$ holds. As a consequence, the claspers on knots can be considered as topological realization of trivalent diagrams of the same degree.

2. C_n-moves and the Alexander-Conway polynomial.

2.1. Proposition. *Let a pair (K, G), where K is a trivial knot in S^3 and G is a clasper on K of degree $2n$, be a topological realization of the wheel $\omega_{2n}, n \geqslant 1$, and let K_G be the knot obtained by surgery of S^3 defined by the pair (K, G). Then K_G is $(2n-1)$-trivial knot with the non-trivial Alexander polynomial.*

Proof. The proof of the proposition follows from Habiro's clasper theory and the characterization of weight systems coming from the Alexander-Conway polynomial. Indeed, the knot K_G is obtained from the trivial knot K by C_{2n}-move defined by the pair (K, G). By Theorem 6.18 of [7], K_G is $(2n-1)$-trivial knot. Let $w: \mathcal{A}_n \otimes \mathbb{Q} \to \mathbb{Q}$ be the rational weight system of degree $2n$ defined as follows: $w(\omega_{2n}) = 1$ and $w(D) = 0$ for any trivalent diagram of degree $2n$, the internal graph of which has the negative Euler characteristic. By Kontsevich's integral over $\mathbb{Q}$, w can be integrated to a canonical $\mathbb{Q}$-valued Vassiliev invariant of order $2n$ [2]. By the definition of the topological realization of trivalent diagram, we have then $v(K_G) - v(K) = \pm v(\omega_{2n}) = \pm w(\omega_{2n}) = \pm 1$. Therefore, by Lemma 1.1, v is a non-trivial Vassiliev invariant of order $2n$ coming from the coefficients of the Alexander polynomial. It follows that the Alexander polynomial of K_G is non-trivial.

Now, for each $n \geqslant 1$ we indicate explicitly the $(2n-1)$-trivial knots with the non-trivial Alexander polynomial. For this, consider two the following pure braid $2n$-commutators: p_{σ_1} and p_{σ_2}, where $\sigma_1 = (1)(2)(3)\ldots(2n)$ and $\sigma_2 = (1, 2, 3, \ldots, 2n)$. Let $\hat{q}$ denote the closure of the braid $q \in LCS_{2n}(P_{2n+1})$ with the strands $u_0, u_1, u_2, \ldots,$
u_{2n} via the permutation $(0, 1, 2, \ldots, 2n)$ and let K be a trivial knot. Set $p = p_{\sigma_1} \cdot p_{\sigma_2}^{-1}$. Then the knot $K_{2n} = \hat{p}$ is the desired knot. Indeed, the knots K and K_{2n} are $LCS_{2n}(P_{2n+1})$-equivalent. Then, by Theorem 0.2 of [15], they are $(2n-1)$-equivalent.

It follows that K_{2n} is $(2n-1)$-trivial. On the other hand, by Lemma 1.11 of [11], for each Vassiliev invariant v of order $2n$ we have $v(K_{2n}) = v(K) \pm (v(T_{\sigma_1}) - v(T_{\sigma_2})) = \pm(v(T_{\sigma_1}) - v(T_{\sigma_2}))$. Note that in $\mathcal{A}_{2n} \otimes \mathbb{Q}$ we have $T_{\sigma_1} - T_{\sigma_2} = \omega_{2n}$, so $K_{2n} - K$ is a topological realization of the wheel ω_{2n}. Then the same reasoning as in the proof of the Proposition 2.1 shows that K_{2n} has the non-trivial Alexander polynomial.

2.2. Proposition. *Suppose that knots K and L in S^3 are related by a sequence of C_n-moves $M_i, n \geqslant 3$, and possibly isotopies, $K = K_0 \to K_1 \to \cdots \to K_l = L$, with the following properties. Each move $M_i, 1 \geqslant i \geqslant l$, is determined by the pair (K_{i-1}, G_{i-1}), where G_{i-1} is a clasper on the knot K_{i-1} such that the internal graph of the trivalent diagram $\gamma(G_{i-1})$ is connected and has a negative Euler characteristic. Then K and L are $(n-1)$-equivalent and have the same Alexander polynomial.*

Proof. The fact that K and L are $(n-1)$-equivalent follows directly from Theorem 6.18 of [7]. Therefore, we have only to show that the knots K and L share the same Alexander polynomial. The proof of the last assertion is by induction on the number l. Suppose that K and K_i, where $i \leqslant l-1$, have the same Alexander polynomial, $A_K(t) = A_{K_i}(t)$. We now proceed as in the proof of Theorem 1.2 of [12]. Let H_i be the internal graph of the trivalent diagram $D_i = \gamma(K_i, G_i)$. By the assumption, H_i is a connected graph having the negative Euler characteristic. It follows that there exists an internal vertex u of D_i which is not connected to any external vertex of D_i. It follows from the proof of Theorem 1.2 of [12], that the knots K_{i+1} and K_i have S-equivalent Seifert matrices. Since the Alexander-Conway polynomial of a knot is determined by the S-class of its Seifert matrix, the knots K_{i+1} and K_i have the same Alexander polynomial, $A_{K_{i+1}}(t) = A_{K_i}(t)$. Therefore, $A_K(t) = A_{K_{i+1}}(t)$. The induction step completes the proof. Note that the diagram $D = \sum_{i=1}^{l} D_i$, regarded in $\mathcal{A}_n \otimes \mathbb{Q}$, is an integral linear combination of trivalent diagrams, the internal graphs of which have the negative Euler characteristic, i.e. the generators of $\mathcal{A}_n \otimes \mathbb{Q}$ of the first type. Then for each Vassiliev invariant v of order $\leqslant n$ we have $v(L) - v(K) = \sum_{i=1}^{l} v(D_i) = v(\sum_{i=1}^{l} D_i)$.

2.1. Corollary. *Under the assumptions of Proposition 2.2, if the knot K is $(n-1)$-trivial and has the trivial Alexander polynomial, then L is also $(n-1)$-trivial and has the trivial Alexander polynomial.*

2.1 Remark. The restriction $n \geqslant 3$ in Proposition 2.2 is essential. Indeed, there are no trivalent diagram of degree 1 and 2, the internal graph of which has the negative Euler characteristic. On the other hand, it is well known that any two knots K and L in S^3 are related by a sequence of simple one-branch C_2-moves (C_2-move is also called the Δ-unknotting operation, see [13]). On the level of the vector space $\mathcal{A}_n \otimes \mathbb{Q}$, each Δ-operation on knots contributes the value $\pm(1/2)\omega_2$ to the total sum $\sum_{i=1}^{l} D_i$ of trivalent diagrams D_i. It follows that if for a Vassiliev invariant v_2 of order 2 there holds $v_2(K) - v_2(L) = 0$, then l must be even.

2.2. Remark. Recently Traczyk [17] has proved that for any integer $n \geqslant 3$ the Alexander (Conway) polynomial of oriented links is not changed by the rotation operation of Anstee, Przytycki and Rolfsen [1] of order n. Rotants (the pairs of links obtained via rotation operation) are known to share the same Homply polynomial for $n \leqslant 4$ and the same Kauffman polynomial for $n \leqslant 3$ [17]. In this context, it would be interesting to know whether the n-rotants of knots are m-equivalent for appropriate m (depending on n), and if this is the case, whether one can pass from a knot of the pair of rotants to another one of this pair via the particular C_{m+1}-moves just indicated in Proposition 2.2.

3. Band equivalence of knots. In the study of geometric properties of knot invariants of finite order, Kalfagianni and Lin [9] have introduced for each $n \geqslant 2$ several classes of knots, called n-hyperbolic, n-elliptic and n-parabolic. All these knots are characterized by the property that they bound in S^3 the regular Seifert surfaces having certain geometric properties and called n-hyperbolic, n-elliptic and n-parabolic surfaces, respectively. Thus Vassiliev invariants can be thought of the obstructions for knots to bound the regular Seifert surface of the corresponding type. One of the main result of Kalfagianni and Lin in [9] is that all n-hyperbolic, n-elliptic and n-parabolic knots are n-trivial. Kalfagianni and Lin proved also [9] that for each $n \geqslant 2$ all n-hyperbolic and n-elliptic knots have trivial Alexander polynomial, so they do not exhaust entirely the class of n-trivial knots. For example, there exists a 2-parabolic knot with the non-trivial Alexander polynomial. It is unknown likely if n-hyperbolic, n-elliptic and n-parabolic knots exhaust all the class of n-trivial knots. In the present paper, we consider Seifert surfaces for knots (not necessarily regular) in S^3 represented in the disc-band form and some specific moves on them, the band-analogues of insertions in knots of pure braids commutators.

Let K be a knot in $\mathbb{R}^3 \simeq \mathbb{R}^2 \times \mathbb{R}$ and S a Seifert surface for K given in the disc-band form in the projection to the plane $F = \mathbb{R}^2 \times \{0\}$. Suppose that in some disc $D^2 = I \times I \subset F$ the projection of S looks like the geometric trivial braid 1_m with each strand s_i, $i = 1, \dots, m$, being replaced by a thin band b_i (see Fig. 5,a). Each band b_i, $i = 1, \dots, m$, is bounded by two strands u_{2i-1} and u_{2i} (with opposite orientations). All the strands $u_j, 1 \leqslant j \leqslant 2m$, taken together with the appropriate orientations, give a diagram of the trivial braid $1_{2m} \in B_{2m}$ positioned in a disc $D^2 \subset \mathbb{R}^3$. Let p be a geometric pure braid representing an element of the group $LCS_n(P_m)$, where $m \geqslant 3$. We can also thicken the strands s'_i, $i = 1, \dots, m$ of the braid p, replacing s'_i with a thin band b'_i, respecting all under-crossings and over-crossings of the strands s'_i of p. Now we can replace the part of the projection of Seifert surface S contained in the disc D^2 and consisting of m separate bands $b_i, i = 1, \dots, m$ (the "thickened" braid 1_{2m}) with the "thickened" braid p consisting of m band-strands $b'_i, i = 1, \dots, m$. To this operation on Seifert surfaces represented in the disc-band form there corresponds the operation of insertion of the "doubles" of pure braids in a knot diagram K [14]. The orientation of the boundary components u'_{2i-1} and u'_{2i} of b'_i is inherited from the orientation of the surface S (see Fig. 5,b). Denote by K_p and S_p, respectively, the surgered knot and the Seifert surface bounded it in $\mathbb{R}^3$. We shall say that S_p is obtained from S by inserting the thickened pure braid p or the "double" of p in it. The inverse move on the Seifert surfaces, represented in the disc-band form, and on the knots bounded by them, consists in replacement the "double" of the pure braid $p \in LCS_n(P_m), m \geqslant 3$, with the "double" of the trivial one with the same number of strands. Both the moves on Seifert surfaces for knots are called n-elementary moves with m bands, where $m \geqslant 3$. Any two knots K and L are called $LCS_n(P_m)$-band-equivalent, where $m \geqslant 3$, if there is a sequence of knots $K = K_1, K_2, \dots, K_{l-1}, K_l = L$ and pairs of Seifert surfaces

$S_1, S'_1, S_2, S'_2, \ldots, S_{l-1}, S'_{l-1}, S_l$, where S_i and S'_i bound the knot K_i, such that each Seifert surface S_{i+1} is obtained from S'_i by an n-elementary move with m bands or an isotopy. It follows from Proposition 2.3 of [14] that any two $LCS_n(B_m)$-band-equivalent knots are $LCS_n(P_m)$-equivalent. The converse implication does not hold, of cause.

3.1. Proposition. *If the knots K and L are $LCS_n(P_m)$-band-equivalent for some $n \geqslant 2$ and $m \geqslant 3$, then they share the same Alexander polynomial.*

Proof. Suppose that K and L are $LCS_n(P_m)$-band-equivalent for some $n \geqslant 2$ and $m \geqslant 3$. The Alexander polynomial of any knot K' in S^3 is determined by S-equivalence class of Seifert matrices for K'. Let S be any Seifert matrix for K', represented in the disc-band form. It is easy to see that an n-elementary move with m strands on S does not affect its Seifert matrix M. On the other hand, passing from any Seifert surface of K' to an other one, for a knot of the same knot type as K', does not also change the S-equivalence class of M. It follows that K and L have the S-equivalent Seifert matrices, completing the proof.

3.1. Corollary. *If a knot K is $LCS_n(P_m)$-band-equivalent to a trivial one for some $m \geqslant 3$, where $n \geqslant 2$, then K is $(n-1)$-trivial and has the trivial Alexander polynomial.*

Therefore, all the knots which are $LCS_n(P_m)$-band-equivalent to a trivial one for some $m \geqslant 3$, form a class of "geometric" $(n-1)$-trivial knots with trivial Alexander polynomial. Suppose that two knots K and L are related via an n-elementary band move. By Proposition 2.3 of [14], $L-K$, considered modulo $\mathcal{K}'_{n+1}$, can be represented as some integral linear combination of n-singular knots $\sum_i \lambda_i K_i$. Since the map $\varphi: \mathcal{A}_n \otimes \mathbb{Q} \to \mathcal{K}'_n/\mathcal{K}'_{n+1}$ is an isomorphism of vector spaces (see [2]), the diagram $D = \varphi^{-1}(\sum_i \lambda_i K_i)$ in $\mathcal{A}_n \otimes \mathbb{Q}$ is determined uniquely. Then it is not difficult to check directly that in $\mathcal{A}_n \otimes \mathbb{Q}$ the diagram D is a sum of trivalent diagrams, the internal graphs of which have the negative Euler characteristic.

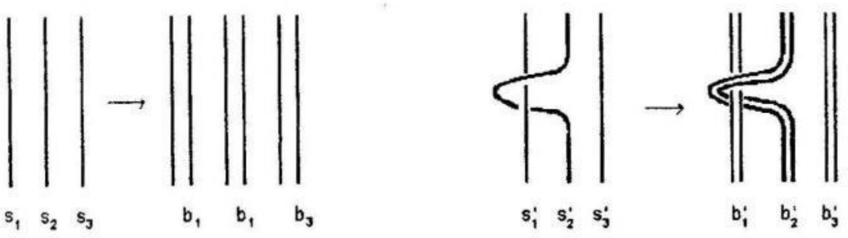

Fig. 5

Question. How does the class of "geometric" n-trivial knots described by Corollary 3.1 relate to the classes of n-hyperbolic and n-elliptic knots?

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n-ТРИВІАЛЬНІ ВУЗЛИ ТА ПОЛІНОМ АЛЕКСАНДЕРА

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Для кожного натурального $n \geqslant 1$, використовуючи комутатори групи чистих кос P_{2n}, побудовано $(2n-1)$-тривіальні вузли з нетривіальним поліномом Александера. Сформульовано достатню умову тривіальності полінома Александера $(n-1)$-тривіального вузла при $n > 2$. Описано клас "геометричних" $(n-1)$-тривіальних вузлів, $n > 1$ з тривіальним поліномом Александера.

Ключові слова: інваріант скінченного типу, комутатор кос, поверхня Зайферта, поліном Александера, тривалентна діаграма, n-тривіальний вузол.

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ON TOTALLY BOUNDED SEMIGROUPS OF CONTINUOUS MAPPINGS

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A semigroup S with the identity e endowed with a topology is called *left (right) totally bounded* if, for every neighborhood U of e, there exists a finite subset F such that $S = FU$ $(S = UF)$. For a topological space X, denote by $C(X)$ the semigroup of all continuous selfmappings of X with the topology of pointwise convergence. We give some sufficient conditions on X under which $C(X)$ is either left or right totally bounded.

Key words: totally bounded semigroup, distal group, homogeneous space.

For a topological space X, let $C(X)$ and $H(X)$ be the semigroup of all continuous selfmappings of X and a group of all homeomorphisms of X with the topology of pointwise convergence (i.e. the topology inherited from the Tychonov product X^X). Every subgroup of $H(X)$ is called a group of homeomorphisms of X. It is well known [1] that $C(X)$ is a semitopological semigroup (i.e. all shifts $x \mapsto sx$, $x \mapsto xs$, $s \in C(X)$ are continuous).

A semigroup S with the identity e endowed with a topology is called *left (right) totally bounded* if, for every neighborhood U of e, there exists a finite subset F such that $S = FU$ $(S = UF)$.

We give some sufficient conditions on X under which $C(X)$ is left or right totally bounded. In particular, we show that $C(X)$ is left and right totally bounded for a Cantor cube X of any weight. Under the Cantor cube of weight α we understand the product $\{0,1\}^\alpha$ of α copies of the discrete space $\{0,1\}$. On the other hand, $H(X)$ is neither left nor right totally bounded for every Cantor cube X of infinite weight.

Theorem 1. *Let X, Y be compact spaces such that X admits a base of the topology consisting of clopen subsets homeomorphic to Y. Then $C(X)$ is left totally bounded.*

Proof. Choose any distinct elements $a_1, a_2, \dots, a_n \in X$ and neighborhoods $A_1, A_2, \dots, A_n$ of $a_1, a_2, \dots, a_n$. Put

$$S = \{s \in C(X) : s(a_i) \in A_i \text{ for } i \leqslant n\}.$$

It suffices to find a finite subset $K \subseteq C(X)$ such that $C(X) = KS$.

For every element $x = (x_1, x_2, \dots, x_n) \in X^n$, choose clopen neighborhoods $V_1, V_2, \dots, V_n$ of $x_1, x_2, \dots, x_n$, homeomorphic to Y. We may suppose that X is infinite, so pick pair-wise disjoint clopen subsets $V_1', V_2', \dots, V_n'$ homeomorphic to Y such that

$$V_i' \subseteq V_i \text{ and } \{a_1, a_2, \dots, a_n\} \cap V_i' = \emptyset$$

for every $i \leqslant n$. Then choose pair-wise disjoint clopen neighborhoods $U_1, U_2, \dots, U_n$ of $a_1, a_2, \dots, a_n$ homeomorphic to Y such that $U_1 \subseteq A_1$, $U_2 \subseteq A_2, \dots, U_n \subseteq A_n$ and

$$(U_1 \cup U_2 \cup \dots \cup U_n) \cap (V_1' \cup V_2' \cup \dots \cup V_n') = \emptyset.$$

Fix any homeomorphisms $h_i : U_i \to V_i$, $t_i : V_i' \to U_i$, $i \leqslant n$, and define the mapping $g_x \in C(X)$ letting

$$g_x(a) = \begin{cases} h_i(a), & \text{if } a \in U_i, \\ t_i(a), & \text{if } a \in V_i', \\ a, & \text{otherwise.} \end{cases}$$

Put $V_x = V_1 \times V_2 \times \dots \times V_n$, consider the clopen cover $\{V_x : x \in X^n\}$ of X^n and choose its finite subcover $\{V_x : x \in F\}$. Put $K = \{g_x : x \in F\}$ and show that $C(X) = KS$.

Let f be an arbitrary element of $C(X)$. Choose $x = (x_1, x_2, \dots, x_n) \in F$ with $(f(a_1), f(a_2), \dots, f(a_n)) \in V_x$. Show that $f = g_x s$ for some element $s \in S$.

For every $i \leqslant n$ choose a clopen neighborhood W_i of a_i such that $f(W_i) \subseteq V_i$ and $W_i \subseteq U_i$. First define the mapping s on $W_1 \cup W_2 \cup \dots \cup W_n$. If $a \in W_i$, put $s(a) = h_i^{-1} f(a)$. Then $f(a) = g_x(s(a))$ and $s(a_i) \in U_i \subseteq A_i$.

To extend the mapping s onto X consider three cases in which $a \notin W_1 \cup W_2 \cup \dots \cup W_n$.

Case 1. $f(a) \notin (U_1 \cup U_2 \cup \dots \cup U_n) \cup (V_1' \cup V_2' \cup \dots \cup V_n')$. Put $s(a) = f(a)$ and note that $f(a) = g_x(s(a))$.

Case 2. $f(a) \in U_1 \cup U_2 \cup \dots \cup U_n$. If $f(a) \in U_i$, put $s(a) = t_i^{-1} f(a)$. Then $f(a) = g_x(s(a))$.

Case 3. $f(a) \in V_1' \cup V_2' \cup \dots \cup V_n'$. If $f(a) \in V_i'$, put $s(a) = h_i^{-1} f(a)$. Then $f(a) = g_x(s(a))$.

By the construction $f = g_x s$ and $s \in S$. □

Question 1. *Is the semigroup $C(X)$ left totally bounded for every zero-dimensional compact homogeneous space?*

A topological space X is called *homogeneous* if, for any points $x_1, x_2 \in X$, there exists a homeomorphism h of X with $h(x_1) = x_2$.

Theorem 2. *Let X be a topological space such that every point of X has a base of clopen neighborhoods homeomorphic to X. Then $C(X)$ is right totally bounded.*

Proof. Take any distinct points $x_1, x_2, \dots, x_n \in X$, choose disjoint open neighborhoods $U_1, U_2, \dots, U_n$ of $x_1, x_2, \dots, x_n$ and put

$$S = \{s \in C(X) : s(x_i) \in U_i \text{ for } i \leqslant n\}.$$

Take a clopen subset $V \subset U_1$ homeomorphic to Y with $x_1 \notin V$. Fix any homeomorphism $f : X \to V$ and show that $C(X) = Sf$. Take any mapping $h \in C(X)$ and define a continuous mapping $s' : V \to X$ such that $h = s'f$. Since X is zero-dimensional, s' can be extended to a mapping $s \in C(X) \cap S$. Hence, $h = sf$. □

Question 2. *Is the semigroup $C(X)$ right totally bounded for every zero-dimensional homogeneous space X?*

Theorem 3. *A semigroup $C(X)$ is left and right totally bounded for the Cantor cube X of any weight.*

Proof. Apply Theorems 1, 2. □

Let X be a topological space and let H be a group of homeomorphisms of X. The pair of distinct points $x_1, x_2 \in X$ is called *H-proximal,* if there exists a point $x \in X$ such that, for every neighborhood V of x there is $h \in H$ with $h(x_1) \in V$, $h(x_2) \in V$. If there are no H-proximal points in X, then H is called *distal.*

Theorem 4. *Let X be a topological space, H be a group of homeomorphisms of X which acts transitively on X. If H is left totally bounded then H is distal.*

Proof. Assume the converse. Since H acts transitively on X, there exist two distinct points $x_1, x_2 \in X$ such that, for every nonempty open subset U of X, there is a homeomorphism $h \in H$ with $h(x_1) \in U$, $h(x_2) \in U$. Choose disjoint open neighborhoods U_1, U_2 of x_1, x_2 and put

$$S = \{s \in H : s(x_1) \in U_1, s(x_2) \in U_2\}.$$

By assumption, there exists a finite subset $F = \{f_1, f_2, \ldots, f_n\}$ of H such that $H = FS$. Put $V_1 = f(U_1)$. If $V_1 \cap f_2(U_1) \neq \emptyset$ put $V_2 = V_1 \cap f_2(U_1)$, otherwise, $V_2 = V_1$. If $V_2 \cap f_3(U_1) \neq \emptyset$ put $V_3 = V_2 \cap f_3(U_1)$, otherwise, $V_3 = V_2$. After n steps put $V = V_n$. By the construction, the subset V has the following property

$$(*) \quad \text{if } V \cap f_k(U_1) \neq \emptyset \text{ then } V \subseteq f_k(U_1),\ k \in \{1, 2, \ldots, n\}.$$

Since H is not distal, there exists $h \in H$ such that $h(x_1) \in V$, $h(x_2) \in V$. Choose $f \in F$ and $s \in S$ with $h = fs$. Taking into account that $h(x_1) = f(s(x_1))$ and $h(x_1) \in V$, $s(x_1) \in U_1$ we conclude that $V \cap f(U_1) \neq \emptyset$. By the condition $(*)$, $V \subseteq f(U_1)$. Since $h(x_2) = f(s(x_2))$ and $h(x_2) \in V$, $s(x_2) \in U_2$ we get $V \cap f(U_2) \neq \emptyset$, a contradiction with $f(U_1) \cap f(U_2) = \emptyset$ and $V \subseteq f(U_1)$. □

Question 3. *Let X be a compact space and let H be a distal group of homeomorphisms of X which acts transitively on X. Is H left totally bounded?*

Theorem 5. *Let X be an infinite topological space and let H be a group of homeomorphisms which acts n-transitively on X for every natural number n. Then H is not right totally bounded.*

Proof. Fix any point $x \in X$ and choose a neighborhood U of x such that the subset $X \setminus U$ is infinite. Put $S = \{s \in H : s(x) \in U\}$ and suppose that there exists a finite subset F of H such that $H = SF$. Let $F^{-1}(x) = \{y_1, y_2, \ldots, y_k\}$. Take any distinct point $z_1, z_2, \ldots, z_k \in X \setminus U$. Choose $h \in H$ such that

$$h(y_1) = z_1, h(y_2) = z_2, \ldots, h(y_k) = z_k.$$

Since $H = SF$, there exists $f \in F$ such that $h = sf$ and thus $hf^{-1} \in S$. Since $f^{-1}(x) \in \{y_1, y_2, \ldots, y_k\}$ we get a contradiction: $hf^{-1} \notin S$. □

Theorem 6. *The group $H(X)$ of all homeomorphisms of a Cantor cube X of infinite weight is neither left nor right totally bounded.*

Proof. Apply Theorems 4, 5. □

Theorem 7. *Let X be an infinite discrete space. Then $C(X)$ is right totally bounded, $H(X)$ is neither left nor right totally bounded.*

Proof. Take any element $x \in X$ and put $S = \{s \in C(X) : s(x) = x\}$. Take any finite subset $F \subset C(X)$. Since $FS(x) = F(x)$ and the subset $F(x)$ is finite, we get $C(X) \neq FS$ so $C(X)$ is not left totally bounded. Put $S' = S \cap H$. The same argument shows that S' is a subgroup of infinite index in $H(X)$ so $H(X)$ is neither left nor right totally bounded.

To show that the semigroup $C(X)$ is right totally bounded, take any distinct elements $x_1, x_2, \dots, x_n \in X$ and put $S = \{s \in C(X) : s(x_i) = x_i \text{ for } i \leqslant n\}$. Fix any bijection $f : X \to X \setminus \{x_1, x_2, \dots, x_n\}$. Clearly, $C(X) = Sf$. □

Theorem 8. *Let X be an infinite discrete space, βX be the Stone-Čech compactification of X. Then $C(\beta X)$ is right but not left totally bounded, $H(\beta X)$ is neither left nor right totally bounded.*

Proof. We prove only that $C(\beta X)$ is right totally bounded. Identify βX with the set of all ultrafilters on X. Given any subset $A \subseteq X$, put $\overline{A} = \{p \in \beta X : A \in p\}$. Then the family $\{\overline{A} : A \in p\}$ is a base of neighborhoods of p. Take any distinct element $p_1, p_2, \dots, p_n \in \beta X$ and pick pairwise disjoint subsets $P_1 \in p_1, P_2 \in p_2, \dots, P_n \in p_n$ such that $|X \setminus (P_1 \cup P_2 \cup \dots \cup P_n)| = |X|$. Put $S = \{s \in C(\beta X) : s(p_i) \in \overline{P_i} \text{ for } i \leqslant n\}$.

Fix any bijection $f : X \to X \setminus (P_1 \cup P_2 \cup \dots \cup P_n)$ and denote by $\overline{f}$ the extension of f to βX. Then, $C(\beta X) = S\overline{f}$. □

Added in Proofs. Recently, T. Banakh and O. Hryniv answered Questions 1 and 2 in negative: they proved that the semigroup $C(X)$ of the paratopological first-countable zero-dimensional homogeneous compactum X constructed by E. van Douwen [2] is neither left nor right totally bounded; moreover, the homeomorhism group $H(X)$ of X is neither left nor left totally bounded in $C(X)$.

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ПРО ЦІЛКОМ ОБМЕЖЕНІ НАПІВГРУПИ НЕПЕРЕРВНИХ ВІДОБРАЖЕНЬ

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Напівгрупа S з одиницею e, наділена топологією, називається *цілком обмеженою (зліва) справа*, якщо для кожного околу U одиниці e існує така скінченна підмножина F, що $S = FU$ $(S = UF)$. Через $C(X)$ позначимо напівгрупу всіх неперервних відображень топологічного простору X в себе, наділену топологією поточкової збіжності. Знайдено достатні умови на топологічний простір X, при яких напівгрупа $C(X)$ цілком обмежена зліва чи справа.

Ключові слова: цілком обмежена напівгрупа, дистальна група, однорідний простір.

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ON H-CLOSED PARATOPOLOGICAL GROUPS

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A Hausdorff paratopological group is H-closed if it is closed in every Hausdorff paratopological group containing it as a paratopological subgroup. We give a criterion of the H-closedness of an abelian topological group for some classes of abelian paratopological groups are obtained simple criteria of the H-closedness.

Key words: paratopological group, minimal topological group, absolutely closed topological group.

All topological spaces considered in this paper are Hausdorff, if the opposite is not stated. We shall use the following notations. Let A be a subset of a group and n be an integer. Put $A^n = \{a_1 a_2 \cdots a_n : a_i \in A\}$ and $nA = \{a^n : a \in A\}$. For a group topology τ the closure of a set A is denoted by $\overline{A}^{\tau}$ and $\mathcal{B}_\tau$ stands for a neighborhood base of the unit.

A topological space X is *C-closed* in a class C of topological spaces provided X is closed in any space $Y \in C$ containing X as a subspace. It is well known that when C is the class of Tychonoff spaces, then the C-closedness coincides with the compactness. For the class of Hausdorff spaces the following conditions for a space X are equivalent [1, 3.12.5]:

1) The space X is H-closed;

2) If $\mathcal{V}$ is a centered family of open subsets of X then $\bigcap\{\overline{V} : V \in \mathcal{V}\} \neq \varnothing$;

3) Every ultrafilter in the family of all open subsets of X is convergent;

4) Every cover $\mathcal{U}$ of the space X contains a finite subfamily $\mathcal{V}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$.

The group G with a topology τ is called a *paratopological group* if the multiplication on the group G is continuous. If the inversion on the group G is continuous, then (G, τ) is a *topological group*. A group (G, τ) is paratopological if and only if the following conditions (known as Pontrjagin conditions) are satisfied for a neighborhood base $\mathcal{B}$ at unit e of G [4,5]:

1. $\bigcap\{UU^{-1} : U \in \mathcal{B}\} = \{e\}$;
2. $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B}) : W \subset U \cap V$;
3. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^2 \subset U$;
4. $(\forall U \in \mathcal{B})(\forall u \in U)(\exists V \in \mathcal{B}) : uV \subset U$;
5. $(\forall U \in \mathcal{B})(\forall g \in G)(\exists V \in \mathcal{B}) : g^{-1}Vg \subset U$.

The paratopological group G is a topological group if and only if

6. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^{-1} \subset U$.

A topological group is *absolutely closed* if it is closed in every Hausdorff topological group containing it as a topological subgroup. A topological group G is closed in the class of topological groups if and only if it is *Rajkov-complete*, that is complete with respect to the upper uniformity which is defined as the least upper bound $\mathcal{L} \vee \mathcal{R}$ of the left and the right uniformities on G. Recall that the sets $\{(x, y) : x^{-1}y \in U\}$, where U runs over a base at unit of G, constitute a base of entourages for the left uniformity $\mathcal{L}$ on G. In the case of the right uniformity $\mathcal{R}$, the condition $x^{-1}y \in U$ is replaced by $yx^{-1} \in U$. The *Rajkov completion* $\hat{G}$ of a topological group G is the completion of G with respect to the upper uniformity $\mathcal{L} \vee \mathcal{R}$. For every topological group G the space $\hat{G}$ has a natural structure of a topological group. The group $\hat{G}$ can be defined as a unique (up to an isomorphism) Rajkov complete group containing G as a dense subgroup.

A paratopological group is *H-closed* if it is closed in every Hausdorff paratopological group containing it as a subgroup. In the present section we shall consider H-closed paratopological groups.

Question. Let G be a regular paratopological group which is closed in every regular paratopological group containing it as a subgroup. Is G H-closed?

1. Lemma. *Let (G, τ) be a paratopological group. If there exists a paratopology σ on the group $G \times \mathbb{Z}$ such that $\sigma|G \subset \tau$ and $e \in \overline{(G,1)}^{\sigma}$ then (G, τ) is not H-closed.*

Proof. We shall build the paratopology ρ on the group $G \times \mathbb{Z}$ such that $\rho|G = \tau$ and $\overline{G}^{\rho} \neq G$. Determine a base of unit $\mathcal{B}_\rho$ as follows. Let $S = \{(x, n) : x \in G, n > 0\}$. For every neighborhoods $U_1 \in \tau$, $U_2 \in \sigma$ such that $U_1 \subset U_2$ put $(U_1, U_2) = U_1 \cup (U_2 \cap S)$. Put $\mathcal{B}_\rho = \{(U_1, U_2) : U_1 \in \mathcal{B}_\tau, U_2 \in \mathcal{B}_\sigma\}$. Verify that $\mathcal{B}_\rho$ satisfies the Pontrjagin conditions.

1. It is satisfied since $(U_1, U_2) \subset U_2$.

2. It is satisfied since $(U_1 \cap V_1, U_2 \cap V_2) \subset (U_1, U_2) \cap (V_1, V_2)$.

3. Select $V_2 \in \mathcal{B}_\sigma$ and $V_1 \in \mathcal{B}_\tau$ such that $V_2^2 \subset U_2$, $V_1^2 \subset U_1$ and $V_1 \subset V_2$. Let $y_1, y_2 \in (V_1, V_2)$. The following cases are possible

A. $y_1, y_2 \in V_1$. Then $y_1 y_2 \in V_1^2 \in (U_1, U_2)$.

B. $y_1 \in V_1, y_2 \in V_2 \cap S$. Then $y_1 y_2 \in V_2^2 \in U_2$. Since $y_1 \in G$ and $y_2 \in S$, we get $y_1 y_2 \in S$ and hence $y_1 y_2 \in U_2 \cap S$.

C. $y_1 \in V_2 \cap S, y_2 \in V_1$ is similar to the case B.

D. $y_1, y_2 \in V_2 \cap S$. Since S is a semigroup, $y_1 y_2 \in U_2 \cap S$.

4. Let $y \in (U_1, U_2)$. There exist $V_2 \in \mathcal{B}_\sigma$ and $V_1 \in \mathcal{B}_\tau$ such that $yV_2 \subset U_2$ and $V_1 \subset V_2$. The following cases are possible:

A. $y \in U_1$. We may suppose that $yV_1 \subset U_1$. Since $y \in G$, $y(V_2 \cap S) \subset U_2 \cap S$.

B. $y \in U_2 \cap S$. Since $V_1 \subset G$ then $yV_1 \in U_2 \cap S$. Since S is a semigroup and $y \in S$ then $y(V_2 \cap S) \subset U_2 \cap S$. Therefore $y(V_1, V_2) \subset (U_1, U_2)$.

5. Let $(g, n) \in G \times \mathbb{Z}$. There exist $V_2 \in \mathcal{B}_\sigma$ and $V_1 \in \mathcal{B}_\tau$ such that $V_1 \subset V_2$, $g^{-1}V_1 g \subset U_1$ and $g^{-1}V_2 g \subset U_2$. Then $(g, n)^{-1}(V_1, V_2)(g, n) = g^{-1}(V_1, V_2)g = g^{-1}(V_1 \cup (V_2 \cap S)g \subset U_1 \cup (U_2 \cap S) = (U_1, U_2)$.

Therefore (H, ρ) is a paratopological group. Taking into account that $(U_1, U_2) \cap G = U_1$ we get $\rho|G = \tau$.

Since $e \in \overline{(G,1)}^{\sigma}$, for every $U_2 \in \mathcal{B}_\sigma$ there exists $g \in G$ such that $(g, 1) \in U_2$. Then $g \in (e, -1)(U_2 \cap S)$ and therefore $(e, -1) \in \overline{G}^{\rho}$. □

A group topology τ_1 on the group G is called *complementable* if there exist a nondiscrete group topology τ_2 on G and neighborhoods $U_i \in \tau_i$ such that $U_1 \cap U_2 = \{e\}$. In this case we say that τ_2 is a *complement* to τ_1. Proposition 1.4 from [1] implies that in this case the topology $\tau_1 \wedge \tau_2$ is Hausdorff.

A *Banach measure* is a real function μ defined on the family of all subsets of a group G and satisfies the following conditions:

a) $\mu(G) = 1$;

b) if $A, B \subset G$ and $A \cap B = \varnothing$ then $\mu(A \cup B) = \mu(A) + \mu(B)$;

c) $\mu(gA) = \mu(A)$ for every element $g \in G$ and for every subset $A \subset G$.

2. Lemma. [3, p.37]. *Let G be an abelian group and let μ be a Banach measure on G. Let τ be a group topology on G. Suppose that the set nG is U-unbounded for some natural number n and for some neighborhood U of zero in (G, τ). Then $\mu(\{x \in G : nx \in gW\}) = 0$ for every element $g \in G$ and for every neighborhood W of zero satisfying $WW^{-1} \subset U$.*

Let U be a neighborhood of zero in a topological group (G, τ). We say that a subset $A \subset G$ is U-unbounded if $A \not\subset KU$ for every finite subset $K \subset G$.

Given any elements $a_0, a_1, \ldots, a_n$ of an abelian group G put

$$Y(a_0, a_1, \ldots, a_n) = \{a_0^{x_0} a_1^{x_1} \cdots a_n^{x_n} : 0 \leqslant x_i \leqslant i+1, i \leqslant n, \sum x_i^2 > 0\},$$

$$X(a_0, a_1, \ldots, a_n) = \{a_0^{x_0} a_1^{x_1} \cdots a_n^{x_n} : -(i+1) \leqslant x_i \leqslant i+1, i \leqslant n\}.$$

Then $X(a_0, a_1, \ldots, a_n) = Y(a_0, a_1, \ldots, a_n) Y(a_0, a_1, \ldots, a_n)^{-1}$.

3. Lemma. *Let (G, τ) be an abelian paratopological group of infinite exponent. If there exists a neighborhood $U \in \mathcal{B}_\tau$ such that the group nG is UU^{-1}-unbounded for every natural number n, then the paratopological group (G, τ) is not H-closed.*

Proof. Define a seminorm $|\cdot|$ on the group G such that $|xy| \leqslant |x| + |y|$ for all $x, y \in G$. Suppose that there exists a non periodic element $x_0 \in G$. Determine a map $\phi_0 : \langle x_0 \rangle \to \mathbb{Z}$ putting $\phi_0(x_0^n) = n$. Since $\mathbb{Q}$ is a divisible group, the map ϕ_0 can be extended to a homomorphism $\phi : G \to \mathbb{Q}$. Put $|x| = |\phi(x)|$ for every element $x \in G$. If G is periodic, then put $|e| = 0$ and $|x| = [\ln ord(x)] + 1$, where $ord(x)$ denotes the order of the element x.

Fix a neighborhood $V \in \mathcal{B}_\tau$ such that $V^2 \subset U$ and put $W = VV^{-1}$. We shall construct a sequence $\{a_n\}$ such that

a) $|a_n| > n$;

b) $W \cap X(a_0, a_1, \ldots, a_n) = \{e\}$;

c) $Y(a_0, a_1, \ldots, a_n) \not\ni e$;

d) if $-n \leqslant k \leqslant n, k \neq 0$ then $a_n^k \notin 2X(a_0, a_1, \ldots, a_{n-1})$.

Take any element $a_0 \notin W$. Suppose that the elements $a_0, \ldots, a_n$ have been chosen to satisfy conditions (a) and (b). Put

$$B_n = \{x \in G : (\forall g \in X(a_0, a_1, \ldots, a_{n-1}))(\forall k \in \mathbb{Z}\backslash\{0\} : -e^{n+1} \leqslant k \leqslant e^{n+1}) : kx \notin gW\}$$

If the group G is periodic, then $|x| > n$ for every element $x \in B_n$. Lemma 2 implies that $\mu(B_n) = 1$. If the group G is not periodic, then the construction of the seminorm $|\cdot|$ implies that $\mu(\{x \in G : |x| \leqslant n\}) = \mu(\phi^{-1}[-n;n]) = 0$. In both cases there exists an element $a_n \in B_n$ such that $|a_n| > n$. Then $W \cap X(a_0, a_1, \ldots, a_n) = \varnothing$. Considering a subsequence and applying condition (a) we can satisfy conditions (c) and (d) also.

Define a base $\mathcal{B}_{\tau\{a_n\}}$ at the unit of group topology $\tau\{a_n\}$ on the group $G \times \mathbb{Z}$ as follows. Put $A_n^+ = \{(e,0)\} \cap \{(a_k, 1) : k \geqslant n\}$. For every increasing sequence $\{n_k\}$ put $A[n_k] = \bigcup_{l \in \mathbb{N}} A_{n_1}^+ \cdots A_{n_l}^+$. Put $\mathcal{B}_{\tau\{a_n\}} = \{A[n_k]\}$. We claim that $(G \times Z, \tau\{a_n\})$ is a zero dimensional paratopological group.

Put $F = \bigcup_{n \in \omega} X(a_0, a_1, \ldots, a_n)$. Let $A[n_k] \in \mathcal{B}_{\tau\{a_n\}}, (x, n_x) \notin A[n_k]$. If $x \notin F$, then $(x, n_x)A[n] \cap A[n_k] = \varnothing$. Let $x \in X(a_0, a_1, \ldots, a_m)$. Put $m_k = m + k$. Suppose that $(x, n_x)A[m_k] \cap A[n_k] \neq \varnothing$. Select the minimal k such that $(x, n_x)(A_{m_1}^+ \cdots A_{m_k}^+) \cap A[n_k] \neq \varnothing$. Let

$$(*) \qquad (x, n_x)(a_{l_1}, 1) \cdots (a_{l_k}, 1) = (a_{l'_1}, 1) \cdots (a_{l'_{k'}}, 1)$$

and for all i, i' holds $m_i \leqslant l_i \leqslant l_{i+1}$, $n'_i \leqslant l'_{i'} \leqslant l'_{i'+1}$. Remark that a member a_q occurs in each part of the equality $(**)$ no more than q times. If $l_k > l'_{k'}$ then if we move all members which are not equal to $(a_{l_k}, 1)$ from the left side of the equality (*) to the right one, we obtain contradiction to condition (d). The case $l_k < l'_{k'}$ is considered similarly. Therefore $l_k = l'_{k'}$, a contradiction with the choice of k as a minimal number satisfying the equality (*). It is showed similarly that if $x \neq e$ and $m_k = m + k + 1$, then $(x, n_x) \notin A[m_k]$. If $x = e$ and $n_x \neq 0$, then the condition (c) implies that $A[n] \not\ni (x, n_x)$. Hence Pontrjagin condition 1 for $\mathcal{B}_{\tau\{a_n\}}$ is satisfied. Since $A[n_{2k}]^2 \subset A[n_k]$, Pontrjagin condition 3 is satisfied. All the other Pontrjagin conditions are obvious.

Condition (b) implies that $A[n]A[n]^{-1} \cap VV^{-1} = \{(e,0)\}$. Therefore the topology $\tau\{a_n\}_g$ is a complement to the topology $(\tau \times \{0\})_g$, where $\tau \times \{0\}$ is the product topology on the group $(G, \tau) \times \mathbb{Z}$. Therefore the topology $\sigma = \tau\{a_n\}(\tau \times \{0\})$ is Hausdorff. Since $(e, 0) \in \overline{(G,1)}^{\tau\{a_n\}} \subset \overline{(G,1)}^{\sigma}$ we can apply Lemma 1 to show that (G, τ) is not H-closed. □

We shall need the following lemma.

4. Lemma. *Let G be a paratopological group and H be a normal subgroup of G. If H and G/H are topological groups then so is the group G.*

Proof. Let U be an arbitrary neighborhood of the unit. There exist neighborhoods V, W of the unit such that $V \subset U$, $(V^{-1})^2 \cap H \subset U$ and $W \subset V$, $W^{-1} \subset VH$. If $x \in W^{-1}$, then there exist elements $v \in V, h \in H$ such that $x = vh$. Then $h = v^{-1}x \in V^{-1}W^{-1} \cap H \subset U$. Therefore $x \in VU \subset U^2$. Hence G is a topological group. □

The following criterion was suggested by T. Banakh.

5. Theorem. *An abelian topological group (G, τ) is H-closed if and only if (G, τ) is Rajkov complete and for every group topology $\sigma \subset \tau$ on G the quotient group $\hat{G}/G$ is periodic, where $\hat{G}$ is the Rajkov completion of the group (G, σ).*

Proof. Suppose that there exists a group topology $\sigma \subset \tau$ on G such that the quotient group $\hat{G}/G$ is not periodic, where $\hat{G}$ is the Rajkov completion of the group

(G,σ). Select a non periodic element $x \in \hat{G}$ such that $\langle x\rangle \cap G = \{e\}$. Then $G \times \langle x\rangle$ is naturally isomorphic to the group $G \times \mathbb{Z}$ and Lemma 1 implies that the group (G,τ) is not H-closed.

Let a paratopological group (H,τ') contains (G,τ) as non closed subgroup. Since G is abelian, $\overline{G}$ is an abelian semigroup. Choose an arbitrary element $x \in \overline{G}\backslash G$. Then the group hull $F = \langle G, x\rangle$ with the topology $\tau'|F$ is an abelian paratopological group. Then the group G is dense in $(F,{\tau'}_g)$. Since the Rajkov completion $\hat{F}$ of the topological group $(F,\tau'|F_g)$ is periodic, there exists a natural number n such that $x^n \in G$. Therefore $F^n \subset G$. Lemma 4 implies that F is a topological group and therefore G is closed in $(F,{\tau'}_g)$, a contradiction. □

6. Corollary. *A Rajkov completion of a isomorphic condensation of H-closed abelian topological group is H-closed.*

7. Proposition. *Let G be a Rajkov complete topological group, H be H-closed paratopological subgroup of the group G. If a group G/H has finite exponent then G is an H-closed paratopological group.*

Proof. Select a number n such that $g^n \in H$ for every element $g \in G$. Let $F \supset G$ be a paratopological group. Since H is closed in F then for every element $g \in \overline{G}$ we get $g^n \in H$. Denote the continuous maps $\phi : \overline{G} \to \overline{G}$ as $\phi(g) = g^{n-1}$ and $\psi : \overline{G} \to H$ as $\psi(g) = (g^n)^{-1}$. Then for every element $g \in \overline{G}$ we get $g^{-1} = \phi(g)\psi(g)$ and hence the inversion on the group $\overline{G}$ is continuous. Since $\overline{G}$ is a topological group and G is Rajkov complete, $\overline{G} = G$. □

8. Proposition. *Let G be a paratopological group and K be a compact normal subgroup of the group G. If the group G/K is H-closed then the group G is H-closed.*

Proof. Suppose that there exists a paratopological group F containing the group G such that $\overline{G} \neq G$. Since K is compact then F/K is a Hausdorff paratopological group by Proposition 1.13 from [4]. Let $\pi : F \to F/K$ be the quotient homomorphism map. Then $\overline{G/K} \supset \pi(\overline{\pi^{-1}(G/K)}) \supset \pi(\overline{G}) \neq \pi(G) = G/K$. This implies that the group G/K is not H-closed, a contradiction. □

Let G be a topological group, N be a closed normal subgroup of the group G. If N and G/N are Rajkov complete, then so is the group G [5]. This suggests the following

9. Question. Let G be a paratopological group, N be a closed normal subgroup of the group G and the groups N and G/N are H-closed. Is the group G H-closed?

Let (G,τ) be a paratopological group. Then there exists the finest group topology τ_g coarser than τ (see [2]), which is called *the group reflection* of the topology τ.

10. Proposition. *Let (G,τ) be an abelian paratopological group. If (G,τ_g) is H-closed then (G,τ) is H-closed. If (G,τ) is H-closed and (G,τ_g) is Rajkov complete then (G,τ_g) is H-closed.*

Proof. Suppose that the group (G,τ_g) is H-closed and (G,τ) is not. Suppose a paratopological group $(H,\hat{\tau})$ contains (G,τ) as a non closed subgroup. Without loss of generality we may suppose that there exists an element $x \in H\backslash G$ such that $H = \langle G, x\rangle$ and the group H is abelian. Let $\hat{\tau}_g$ be the group reflection of the topology $\hat{\tau}$. Since $\hat{\tau}_g|G \subset \tau_g$, Theorem 5 implies that the group H/G is periodic. Without loss of generality we may suppose that $x^p \in G$ for some prime p.

Denote by $\mathcal{B}_{\hat{\tau}}$ the family of neighborhoods at unit in the topology τ. Let $U \in \mathcal{B}_{\hat{\tau}}$. If $U \cap xG = \varnothing$ then there exists a neighborhood V of unit such that $V^p \subset U$ and thus $V \subset G$ and G is open in $(H, \hat{\tau})$. Therefore a set $\mathcal{F} = \{x^{-1}(xG \cap U) : U \in \mathcal{B}_{\tau}\}$ is a filter. Let $U \in \mathcal{B}_{\hat{\tau}}$. There exists $V \in \mathcal{B}_{\hat{\tau}}$ such that $V^p \subset U$. Then $(xG \cap V)^p \subset U$. Let $xg \in (xG \cap V)$. Then $x^{-1}(xG \cap V) \subset x^{-1}((xg)^{1-p}(xG \cap V)^p) \cap G \subset x^{-p}g^{1-p}(U \cap G)$ and hence $\mathcal{F}$ is a Cauchy filter in the group (G, τ_g). Let $h \in G$ be a limit of the filter $\mathcal{F}$ on the group (G, τ_g). But then for every neighborhood of the unit U in the topology $\hat{\tau}_g$ we get $U \cap xhU \supset U \cap xh(U \cap G) \neq \varnothing$ and therefore $(H, \hat{\tau}_g)$ is not Hausdorff, a contradiction.

Let (G, τ_g) is Rajkov complete and (G, τ_g) is not H-closed. Then Theorem 5 implies that there exists a group topology $\sigma \subset \tau$ on G such that the quotient group $\hat{G}/G$ of the Rajkov completion $\hat{G}$ of the group (G, σ) is not periodic. Then Lemma 1 implies that a group (G, τ) is not H-closed. □

11. Lemma. *Let topological group (H, σ_H) be a closed subgroup of an abelian topological group (G, τ) and $\sigma_H \subset \tau|H$. Then there exists a group topology $\sigma \subset \tau$ on the group G such that $\sigma|H = \sigma_H$.*

Proof. Let $\mathcal{B}_\tau$ and $\mathcal{B}_{\sigma H}$ be bases of unit of (G, τ) and (H, σ_H) respectively.

Put $\mathcal{B}_\sigma = \{U_1U_2 : U_1 \in \mathcal{B}_\tau, U_2 \in \mathcal{B}_{\sigma H}\}$. Verify that the family $\mathcal{B}_\sigma$ satisfies the Pontrjagin conditions.

2. It is satisfied since $(U_1 \cap V_1)(U_2 \cap V_2) \subset U_1U_2 \cap V_1V_2$.

3. Select $V_2 \in \mathcal{B}_{\sigma H}$ and $V_1 \in \mathcal{B}_\tau$ such that $V_2^2 \subset U_2$, $V_1^2 \subset U_1$. Then $(V_1V_2)^2 \subset U_1U_2$.

4. Let $y \in U_1U_2$. Then there exist points $y_1 \in U_1$ and $y_2 \in U_2$ such that $y = y_1y_2$. Therefore there exist a neighborhoods $V_1 \in \mathcal{B}_\tau$ and $V_2 \in \mathcal{B}_{\sigma H}$ such that $y_iV_i \subset U_i$. Then $yV_1V_2 \subset U_1U_2$.

5. It is satisfied since G is abelian.

6. $(U_1^{-1}U_2^{-1})^{-1} \subset U_1U_2$.

1. Since all others Pontrjagin conditions are satisfied, it suffices to show that $\bigcap \mathcal{B}_\sigma = \{e\}$. Let $x \in G$ and $x \neq e$. If $x \in H$ then there exists $U_2 \in \mathcal{B}_{\sigma H}$ such that $U_2^2 \not\ni x$ and $U_1 \in \mathcal{B}_\sigma$ such that $U_1 \cap H \subset U_2$. Then $U_1U_2 \cap \{x\} = U_1U_2 \cap \{x\} \cap H \subset U_2^2 \cap \{x\} = \varnothing$. If $x \notin H$ then $(G \backslash xH)H \not\ni x$.

Therefore (G, σ) is a topological group. Since $U_1U_2 \cap H = (U_1 \cap H)U_2$, we conclude $\sigma|H = \sigma_H$. □

12. Proposition. *A closed subgroup of an H-closed abelian topological group is H-closed.*

Proof. Let H be a closed subgroup of an H-closed abelian group (G, τ). Then G and H are Rajkov complete. Let $\sigma_H \subset \tau|H$ be a group topology on the group H. Lemma 11 implies that there exists a group topology σ on the group G such that $\sigma|H = \sigma_H$. Let $(\hat{G}, \hat{\sigma})$ be the Rajkov completion of the group (G, σ). Then a closure $\overline{H}^{\hat{\sigma}}$ of the group H in the group $(\hat{G}, \hat{\sigma})$ is a Rajkov completion of the group (H, σ_H). Let $x \in \overline{H}^{\hat{\sigma}}$. Theorem 5 implies that there exists $n > 0$ such that $x^n \in G$. Since $\overline{H}^{\hat{\sigma}} \cap G = H$ then $x^n \in H$. Therefore Theorem 5 implies that H is H-closed. □

13. Proposition. *Let G be a H-closed abelian topological group. Then $K = \bigcap_{n\in\mathbb{N}} \overline{nG}$ is compact and for each neighborhood U of zero in G there exists a natural n with $\overline{nG} \subset KU$.*

Proof. Let Φ be the filter on G generated by base $\{\overline{nG} : n \in \mathbb{N}\}$, and Ψ be an arbitrary ultrafilter on G with $\Psi \supset \Phi$. Let U be a closed neighborhood of the unit in G. Lemma 2 implies that there exists a number n such that the set $\overline{nG}$ is U-bounded. Since $\overline{nG} \in \Phi$ and Ψ is an ultrafilter, there exists $g \in G$ with $gU \in \Psi$. Hence Ψ is a Cauchy filter on G. By the completeness of G, Ψ is convergent. Therefore each ultrafilter Ψ on G with $\Psi \supset \Phi$ converges. In particular each ultrafilter on K is convergent, and since K is closed, K is compact.

To show that there exists a number n with $\overline{nG} \subset KU$, it suffices to prove that $KU \in \Phi$. Assume that $KU \notin \Phi$. Then there exists an ultrafilter $\Psi \supset \Phi$ with $G \setminus KU \in \Psi$. As we have proved, Ψ is convergent. Clearly $\lim \Psi \in K$. Therefore $KU \in \Psi$ which is a contradiction. Hence $KU \in \Phi$, and this completes the proof. □

14. Corollary. *A divisible abelian H-closed topological group is compact.* □

15. Proposition. *Every H-closed abelian topological group is a union of compact groups.*

Proof. Let G be such a group. It suffice to show that every element $x \in G$ is contained in a compact subgroup. Let X be the smallest closed subgroup of G containing the element x. Then $X = \bigcup_{k=0}^{n}(kx + \overline{nX})$ for every natural n. Let U be an arbitrary neighborhood of the zero. By Lemma 15 there exists a natural number n such that nG is U-bounded. Then X is also U-bounded. Hence X is a precompact group. Since X is Rajkov complete then X is compact. □

16. Conjecture. An abelian topological group G is H-closed if and only if G is Rajkov complete and nG is precompact for some natural n.

17. Proposition. *The Conjecture 16 is true provided the group (G, τ) satisfies the following two conditions:*

1) There exists a σ-compact subgroup L of G such that G/L is periodic.

2) There exists a group topology $\tau' \subset \tau$ such that the Rajkov completion $\hat{G}$ of the group (G, τ') is Baire.

Proof. Let G be such a group and $L = \bigcup_{k\in\mathbb{N}} L_k$ be a union of compact subsets L_k. Put $G(n,k) = \{x \in \hat{G} : nx \in L_k\}$ for every natural n and k. Then every set $G(n,k)$ is closed. By Theorem 5 $\hat{G} = \bigcup_{n,k\in\mathbb{N}} G(n,k)$. Since $\hat{G}$ is Baire, there exist natural numbers n and k such that $\operatorname{int} G(n,k) \neq \varnothing$. Then $F = G(n,k) - G(n,k)$ is a neighborhood of the zero. By Corollary 6 the group $\hat{G}$ is H-closed. Put $K = \bigcap_{n\in\mathbb{N}} \overline{n\hat{G}}$. By Proposition 13 there exists a natural m such that $m\hat{G} \subset F + K$. Then $mnG \subset mn\hat{G} \subset L_k - L_k + K$ and hence the group mnG is precompact. □

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ПРО H-ЗАМКНЕНІ ПАРАТОПОЛОГІЧНІ ГРУПИ

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Гаусдорфова паратопологічна група називається H-замкненою, якщо вона замкнена у довільній гаусдорфовій паратопологічній групі, що її містить. Отримано критерій H-замкненості абелевої топологічної групи і для деяких класів абелевих паратопологічних груп одержано прості критерії H-замкненості.

Ключові слова: паратопологічна група, мінімальна топологічна група, абсолютно замкнена топологічна група.

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ELEMENTARY ROW TRANSFORMATIONS OVER RINGS OF STABLE RANK $\leqslant 2$

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It is proved that for a ring R of stable rank $\leqslant 2$ any row $(a, b, c) \in R^3$ with $aR + bR + cR = R$ can be reduced to $(1, 0, 0)$ by elementary transformations. Also it is shown that for a right Bezout ring R of stable rank $\leqslant 2$ any row $(a, b, c) \in R^3$ can be reduced to $(\alpha, \beta, 0)$, $\alpha, \beta \in R$, by means of elementary transformations.

Key words: stable rank, Bezout ring, elementary transformations.

In [1] B.V. Zabavsky has posed a problem of a complete description of the rings over which any matrix can be reduced to the diagonal form by elementary transformations. In this note we consider elementary transformations of rows over rings of stable rank 1 and 2.

Throughout this paper R will denote an associative ring with $1 \neq 0$.

Let us introduce the necessary definitions.

An *elementary matrix* with entries from a ring R is a square matrix of one of the following three types [2]: a diagonal matrix with invertible elements on the diagonal; a matrix differing from the identity matrix by a unique nonzero element outside of the main diagonal; permutation matrix, i.e., the identity matrix with its rows or columns permuted arbitrarily.

Denote by $GE_n(R)$ the group generated by elementary $(n \times n)$ matrices.

A ring R is called a *right Bezout ring* [3] if any finitely generated right ideal in R is principal.

A row $(a_1, a_2, \ldots, a_n)$ of elements of a ring R is a *right unimodular row* if there are elements $x_i \in R$, $1 \leqslant i \leqslant n$, with $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1$. A positive integer d is called a *the stable rank* [4] of R if for any unimodular row $(a_1, a_2, \ldots, a_n)$ with $n > d$, there are elements b_i, $1 \leqslant i \leqslant n-1$ such that the row $(a_1 + a_nb_1, \ldots, a_{n-1} + a_nb_{n-1})$ is again right unimodular.

Theorem 1. *Let R be a ring of stable rank 1. Then for any elements $a, b \in R$ with $aR + bR = R$ there is a matrix $M \in GE_2(R)$ such that*

$$(a, b)M = (1, 0).$$

Proof. Since R is a ring of stable rank 1, there are elements $t, w \in R$ such that

$$(a + bt)w = 1.$$

Then

$$(a,b)\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}\begin{pmatrix}1 & w(1-b)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -(a+bt) & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=(1,0).$$

The proof in complete.

Theorem 2. *Let R be a ring of stable rank 2. Then for any elements $a,b,c\in R$ with $aR+bR+cR=R$ there is a matrix $M\in GE_3(R)$ such that*

$$(a,b,c)M=(1,0,0).$$

Proof. Since R is a ring of a stable rank $\leqslant 2$, there exist $x,y,p,q\in R$ such that

$$(a+cx)p+(b+cy)q=1.$$

Then

$$(a,b,c)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ x & y & 1\end{pmatrix}=(a+cx,b+cy,c)$$

and

$$(a+cx,b+cy,c)\begin{pmatrix}1 & 0 & p(1-c)\\ 0 & 1 & q(1-c)\\ 0 & 0 & 1\end{pmatrix}=(a+cx,b+cy,1).$$

It is clear that the row $(a+cx,b+cy,1)$ can be transformed into $(1,0,0)$ by elementary transformations.

The proof in complete.

Corollary 1. *Let R be a ring of stable rank 1. Then for any elements $a,b,c\in R$ with $aR+bR+cR=R$ there is a matrix $M\in GE_3(R)$ such that*

$$(a,b,c)M=(1,0,0).$$

Theorem 3. *Let R be a right Bezout ring of stable rank 1. Then for any elements $a,b\in R$ there is a matrix $M\in GE_2(R)$ such that*

$$(a,b)M=(\alpha,0),\quad \alpha\in R.$$

Proof. Since R is a right Bezout ring, the ideal $aR+bR$ is principal and thus equal to dR for some $d\in R$. Then $a=da_0$, $b=db_0$, $au+bv=d$, $a_0,b_0,u,v\in R$. Let $e_0=1-a_0u-b_0v$. Then $de_0=0$ and $a_0R+b_0R=R$.

Since R is a ring of stable rank 1, there are elements $t,w\in R$ such that $(a_0+b_0t)w=1$. Then

$$d(a_0+b_0t)w=(a+bt)w=d.$$

Thus

$$(a,b)\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}\begin{pmatrix}1 & w(1-b_0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -w\\ 0 & 1\end{pmatrix}=(a+bt,0).$$

which finishes the proof.

Theorem 4. *Let R be a right Bezout ring of stable rank 2. Then for any elements $a, b, c \in R$ there is a matrix $M \in GE_3(R)$ such that*

$$(a, b, c)M = (\alpha, \beta, 0), \quad \alpha, \beta \in R.$$

Proof. Let $aR + bR + cR = dR$, $d \in R$. Then $a = da_0$, $b = db_0$, $c = dc_0$, $au + bv + cw = d$, $a_0, b_0, c_0, u, v, w \in R$. Let $e_0 = 1 - a_0u - b_0v - c_0w$. Then $de_0 = 0$ and $a_0R + b_0R + c_0R = R$.

Since R is a ring of stable rank 2, there exist $x, y, p, q \in R$ such that

$$(a_0 + c_0x)p + (b_0 + c_0y)q = 1.$$

Then

$$d(a_0 + c_0x)p + d(b_0 + c_0y)q =$$

$$= (a + cx)p + (b + cy)q = ap + bq + c(xp + yq) = d.$$

Thus

$$(a, b, c)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} = (a + cx, b + cy, c),$$

$$(a + cx, b + cy, c)\begin{pmatrix} 1 & 0 & p(1 - c_0) \\ 0 & 1 & q(1 - c_0) \\ 0 & 0 & 1 \end{pmatrix} = (a + cx, b + cy, d)$$

and

$$(a + cx, b + cy, d)\begin{pmatrix} 1 & 0 & -p \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{pmatrix} = (a + cx, b + cy, 0),$$

which finishes the proof.

Corollary 2. *Let R be a right Bezout ring of stable rank 1. Then for any elements $a, b, c \in R$ there is a matrix $M \in GE_3(R)$ such that*

$$(a, b, c)M = (\alpha, \beta, 0), \quad \alpha, \beta \in R.$$

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ЕЛЕМЕНТАРНІ ПЕРЕТВОРЕННЯ РЯДКІВ НАД КІЛЬЦЯМИ СТАБІЛЬНОГО РАНГУ $\leqslant 2$

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Доведено, що над кільцем R стабільного рангу $\leqslant 2$ довільний рядок $(a, b, c) \in R^3$ такий, що $aR + bR + cR = R$, елементарними перетвореннями зводиться до вигляду $(1, 0, 0)$. Показано, що над правим кільцем Безу R стабільного рангу $\leqslant 2$ довільний рядок $(a, b, c) \in R^3$ за допомогою елементарних перетворень зводиться до вигляду $(\alpha, \beta, 0)$, $\alpha, \beta \in R$.

Ключові слова: стабільний ранг, кільце Безу, елементарна редукція.

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ON DECOMPOSITION OF COMPLETE LINEAR GROUP INTO PRODUCT OF SOME ITS SUBGROUPS

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The group G_Φ of invertible matrices quasicommuting with the diagonal matrix Φ is considered. It is shown that the complete linear group over some Bezout domain decomposes into the product of G_Φ, lower, and upper unitriangulars groups. Nessesary and suffisient conditions for the equolity $GL(n,R) = G_\Psi^T G_\Phi$, where T denotes the transposition, are obtained. Some applications of these results are considered.

Key words: complete linear group, decomposition, subgroup, divisor of matrices.

Let R be a commutative Bezout domain in which for all $a, b, c \in R$ with $(a,b,c) = 1$, $c \neq 0$, there exists element $r \in R$, such that $(a + rb, c) = 1$. As an example of such rings one can consider the Euclidean rings, principal ideal rings, adequate rings. Let $\Phi = \operatorname{diag}(\varphi_1, \dots, \varphi_n)$ be a nonsingular d-matrix, i.e. a matrix in which $\varphi_i \mid \varphi_{i+1}$, $i = 1, \dots, n-1$. We will consider a group of matrices

$$G_\Phi = \{H \in GL_n(R) \mid H\Phi = \Phi S, \quad S \in GL_(n,R)\},$$

which consist of all invertible matrices of the form $\|h_{ij}\|_1^n$, where $h_{ij} = \frac{\varphi_i}{\varphi_j} k_{ij}$, $i = 2, \dots, n$, $j = 1, \dots, n-1$, $i > j$. In the papers [1, 2, 3] it was shown that the group G_Φ play the main role in the description of the nonassociative divisors of matrices. This paper is devoted to an investigation of this group. Let $U_{up}(n,R)$ and $U_{lw}(n,R)$ be groups of upper and lower $n \times n$ unitriangular matrices over R, respectively.

Theorem 1. $GL(n,R) = G_\Phi U_{lw}(n,R) U_{up}(n,R)$.

In order to prove this Theorem we establish a series of facts.

Lemma 1. *Let $A \in GL(n-1,R)$, $a = \| a_1 \ \dots \ a_{n-1} \|^T$ then there exists a column $x = \| x_1 \ \dots \ x_{n-1} \|^T$ such that*

$$\left\| \begin{matrix} 1 & 0 \\ a & E_{n-1} \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ 0 & A \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ 0 & A \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ x & E_{n-1} \end{matrix} \right\|.$$

Proof. It is easy to see that $x = A^{-1}a$. □

Lemma 2. *Let $\varphi \neq 0$ be any fixed element of R, $(a_1, \dots, a_n) = 1$, $(a_1, \varphi) = 1$. Then the row $\| a_1 \ \ \dots \ \ a_n \|$ can be complemented to an invertible matrix of the form*

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & 1 & 0 & \dots & 0 & u_n \\ 0 & 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 0 & u_4 \\ 0 & 0 & 0 & & 1 & u_3 \\ \varphi u_1 & 0 & 0 & \dots & 0 & u_2 \end{matrix} \right\| . \tag{1}$$

Proof. Observe that $(a_1, \varphi a_2, \dots, \varphi a_n) = 1$ and use results of paper [4], which without loss of generality can be extended to our ring, complement the row $\| a_1 \ \ \dots \ \ a_n \|$ to an invertible matrix of form (1). □

We will consider a group of matrices

$$G_\Phi^T = \{H \in GL_n(R) \mid \Phi H = S\Phi, \quad S \in GL_n(R)\},$$

which consists of all invertible matrices of the form $\|h_{ij}\|_1^n$, where $h_{ij} = \frac{\varphi_j}{\varphi_i} k_{ij}$, $i = 1, \dots, n-1$, $j = 2, \dots, n$, $i < j$.

Lemma 3. *Let $(a_1, \dots, a_n) = 1$, $n \geqslant 2$ and $\left(a_1, \frac{\varphi_2}{\varphi_1} a_2, \dots, \frac{\varphi_n}{\varphi_1} a_n\right) = \delta$. Then in the groups G_Φ, G_Φ^T there exist matrices H, L such that*

$$\| a_1 \ \ \dots \ \ a_n \| H = \| \delta \ \ * \ \ \dots \ \ * \|,$$

$$L \| a_1 \ \ \dots \ \ a_n \|^T = \| \delta \ \ * \ \ \dots \ \ * \|^T .$$

Proof. There are elements $u_1, \dots, u_n$ sush that

$$a_1 u_1 + \frac{\varphi_2}{\varphi_1} a_2 u_2 + \dots + \frac{\varphi_n}{\varphi_1} a_n u_n = \delta.$$

By property 4 from [4] the element u_1 can be chosen so that $\left(u_1, \frac{\varphi_n}{\varphi_1}\right) = 1$. Hence,

$$\left(u_1, \frac{\varphi_n}{\varphi_1} (u_2, \dots, a_n)\right) = 1.$$

Since $\frac{\varphi_i}{\varphi_1} \mid \frac{\varphi_n}{\varphi_1}$, $i = 2, \dots, n$, then $\left(u_1, \frac{\varphi_2}{\varphi_1} u_2, \dots, \frac{\varphi_n}{\varphi_1} u_n\right) \mid \left(u_1, \frac{\varphi_n}{\varphi_1} u_2, \dots, \frac{\varphi_n}{\varphi_1} u_n\right) = 1$. Consequently

$$\left(u_1, \frac{\varphi_2}{\varphi_1} u_2, \dots, \frac{\varphi_n}{\varphi_1} u_n\right) = 1.$$

By a Theorem from [5] in the group G_Φ there exists a matrix H with the first row $\left\| u_1 \ \ \frac{\varphi_2}{\varphi_1} u_2 \ \ \dots \ \ \frac{\varphi_n}{\varphi_1} u_n \right\|^T$. The second part of our assertion can be proved by analogy. □

Lemma 4. *Let A be a $k \times l$ matrix and α the greatest common divisor of all elements of this matrix. If A is a submatrix of the $n \times n$ matrix B and $k+l \geqslant n+1$ than $\alpha \mid \det B$.*

Proof. Without loss of generality we can suppose that the matrix A is in left lower corner of the matrix $B = \|b_{ij}\|_1^n$. Hence

$$\left\| \begin{matrix} b_{s1} & \dots & b_{sl} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nl} \end{matrix} \right\|,$$

where $s = n-k+1$. Since $k+l \geqslant n+1$, we obtain $l \geqslant n-k+1 = s$. It means that the diagonal element b_{ss} is an element of the first row of the matrix A. By Lemma from [3], $\alpha \mid \det B$. □

Proof of Theorem 1. Let be $A \in GL(2,R)$. Then $\det A = e \in U(R)$. In the group G_Φ there exists a matrix $H_1 = \mathrm{diag}(1, e^{-1})$. Denote $H_1 A = \|a_{ij}\|_1^2$. Since $(a_{11}, a_{21}) = 1$, there exist elements u_1, u_2 such that

$$u_1 a_{11} + u_2 a_{21} = 1.$$

For each element $r \in R$ we have

$$(u_1 + a_{21} r) a_{11} + (u_2 - a_{11} r) a_{21} = 1.$$

Since $(u_1, a_{21}) = 1$ we see that $\left(u_1, a_{21}, \frac{\varphi_2}{\varphi_1}\right) = 1$. Thus there exists r_0 such that

$$\left(u_1 + a_{21} r_0, \frac{\varphi_2}{\varphi_1}\right) = 1.$$

We denote by $\overline{u}_1 = u_1 + a_{21} r_0, \overline{u}_2 = u_2 + a_{11} r_0$. Then

$$\left(\overline{u}_1, \frac{\varphi_2}{\varphi_1}\overline{u}_2\right) = 1,$$

so there exist x, y such that

$$\overline{u}_1 y - \frac{\varphi_2}{\varphi_1}\overline{u}_2 x = 1.$$

It means that in the group G_Φ there exists a matrix

$$H_2 = \left\| \begin{matrix} \overline{u}_1 & \overline{u}_2 \\ \frac{\varphi_2}{\varphi_1} x & y \end{matrix} \right\|.$$

Then

$$H_2 H_1 A = \left\| \begin{matrix} 1 & a \\ b & c \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ b & 1 \end{matrix} \right\| \left\| \begin{matrix} 1 & a \\ 0 & 1 \end{matrix} \right\|.$$

Therefore

$$A = H \left\| \begin{matrix} 1 & 0 \\ b & 1 \end{matrix} \right\| \left\| \begin{matrix} 1 & a \\ 0 & 1 \end{matrix} \right\|,$$

where $H = (H_2 H_1)^{-1} \in G_\Phi$. Hence the result holds for $n = 2$. Let $n \geqslant 3$, and suppose that the result is established for $k < n$. Since $(a_{11}, \dots, a_{n1}) = 1$, it follows that there exist elements $u_1, \dots, u_n$ such that

$$u_1 a_{11} + \dots + u_n a_{n1} = 1,$$

where the element u_1 satisfies the condition

$$\left(u_1, \frac{\varphi_n}{\varphi_1}\right) = 1.$$

By Lemma 2 the row $\| u_1 \ \ \dots \ \ u_n \|$ is complementable to an invertible matrix of form (1), where $\varphi = \frac{\varphi_n}{\varphi_1}$. It is obvious that $H \in G_\Phi$. Then

$$HA = \left\| \begin{matrix} 1 & b \\ a & A_{n-1} \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ a & E_{n-1} \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ 0 & B_{n-1} \end{matrix} \right\| \left\| \begin{matrix} 1 & b \\ 0 & E_{n-1} \end{matrix} \right\|,$$

$a = \| a_1 \ \ \dots \ \ a_{n-1} \|^T$, $b = \| b_1 \ \ \dots \ \ b_{n-1} \|$, $B_{n-1} \in GL(n-1, R)$. By the induction hypothesis $B_{n-1} = H_{n-1}UV$, where $H_{n-1} \in G_{\Phi_1}$, $\Phi_1 = \operatorname{diag}(\varphi_2, \dots, \varphi_n)$, $U \in U_{lw}(n-1, R)$, $V \in U_{up}(n-1, R)$. By Lemma 1 there exists column $x = \| x_1 \ \ \dots \ \ x_{n-1} \|^T$ such that

$$\left\| \begin{matrix} 1 & 0 \\ a & E_{n-1} \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ 0 & H_{n-1} \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ 0 & H_{n-1} \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ x & E_{n-1} \end{matrix} \right\|.$$

Then

$$HA = \left\| \begin{matrix} 1 & 0 \\ 0 & H_{n-1} \end{matrix} \right\| \left(\left\| \begin{matrix} 1 & 0 \\ x & E_{n-1} \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ 0 & U \end{matrix} \right\| \right) \left(\left\| \begin{matrix} 1 & 0 \\ 0 & V \end{matrix} \right\| \left\| \begin{matrix} 1 & b \\ 0 & E_{n-1} \end{matrix} \right\| \right) =$$

$$= \left\| \begin{matrix} 1 & 0 \\ 0 & H_{n-1} \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ x & U \end{matrix} \right\| \left\| \begin{matrix} 1 & b \\ 0 & V \end{matrix} \right\|.$$

Hence,

$$A = \left(H^{-1} \left\| \begin{matrix} 1 & 0 \\ 0 & H_{n-1} \end{matrix} \right\| \right) \left\| \begin{matrix} 1 & 0 \\ x & U \end{matrix} \right\| \left\| \begin{matrix} 1 & b \\ 0 & V \end{matrix} \right\|.$$

Taking into account that $\left\| \begin{matrix} 1 & 0 \\ 0 & H_{n-1} \end{matrix} \right\| \in G_\Phi$, we see that our statement is true. □

Let A be an $n \times n$ matrix over R. Since R is a commutative elementary divisor domain [6], there exist invertible matrices P and Q such that

$$PAQ = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_n) = \Psi,$$

which is a d-matrix. The matrix Ψ is named canonical diagonal form of the matrix A. Denote by $K(f)$ the set of representatives of the conjugate class of the factor-ring R/Rf, where $f \in R$. Let

$$V(\Psi, \Phi) = \{V = \|v_{ij}\|_1^n \in U_{lw}(n, R) \mid v_{ij} = \frac{\varphi_i}{(\varphi_i, \varepsilon_j)} k_{ij}, k_{ij} \in K\left(\frac{(\varphi_i, \varepsilon_j)}{\varphi_j}\right)\},$$

$i = 2, \dots, n$, $j = 1, \dots, n-1$, $i > j$.

Corollary 1. *$(V(\Psi, \Phi)U_{up}(n, R)P)^{-1}\Phi$ is the set of left divisors of the matrix A which contain all left nonassociative by right divisors of this matrix with canonical diagonal form Φ.*

Proof. We define

$$L(\Psi, \Phi) = \{L \in GL(n, R) \mid L\Psi = \Phi S, \quad S \in M(n, R)\}.$$

By Corollary 3 from [3] the set $L(\Psi,\Phi)$ consists of all invertible matrices of the form $\|l_{ij}\|_1^n$, where $l_{ij} = \frac{\varphi_i}{(\varphi_i,\varepsilon_j)}k_{ij}$, $i = 2,\ldots,n$, $j = 1,\ldots,n-1$, $i > j$. From Proposition from [3] it follows that the set $(L(\Psi,\Phi)P)^{-1}\Phi$ is the set of left divisors of the matrix A which contain all left nonassociative by right divisors of this matrix with canonical diagonal form Φ. Let $T \in V(\Psi,\Phi)$, $N \in U_{up}(n,R)$. Since $U_{up}(n,R) \subset G_\Psi$, we have

$$TN\Psi = T\Psi S_1 = \Phi S_2 S_1, S_1 \in GL(n,R), S_2 \in M(n,R).$$

Therefore $V(\Psi,\Phi)U_{up}(n,R) \subset L(\Psi,\Phi)$. Consequently, $(V(\Psi,\Phi)U_{up}(n,R)P)^{-1}\Phi$ is the set of left divisors of the matrix A with canonical diagonal form Φ. We will show that this set is contain all left nonassociative by right divisors of the matrix A with canonical diagonal form Φ.

Let $L \in L(\Psi,\Phi)$, it means that the matrix $B = (LP)^{-1}\Phi$ is the left divisor of the matrix A. By Theorem 1, $L = HUV$, where $H \in G_\Phi$, $U \in U_{lw}(n,R)$, $V \in U_{up}(n,R)$. Hence, $U = H^{-1}LV^{-1}$. Since $V^{-1} \in G_\Psi$, it follows that

$$U\Psi = H^{-1}LV^{-1}\Psi = H^{-1}L\Psi S_1 = H^{-1}\Phi S_2 S_1 = \Phi(S_3 S_2 S_1).$$

Thus $U \in L(\Psi,\Phi)$. By Lemma 3 from [7] in the group G_Φ there exists a matrix H_1 such that $H_1 U = T_1 \in L(\Psi,\Phi)$. Consequently,

$$B = (LP)^{-1}\Phi = (HUVP)^{-1}\Phi = \left(HH_1^{-1}(H_1U)VP\right)^{-1}\Phi =$$

$$= (T_1VP)^{-1}\left(H_1H^{-1}\right)^{-1}\Phi = (T_1VP)^{-1}\Phi S = B_1 S,$$

$S \in GL(n,R)$ where $B_1 = (T_1VP)^{-1}\Phi S \in (V(\Psi,\Phi)U_{up}(n,R)P)^{-1}\Phi$. It means that every left divisor of the matrix A with canonical diagonal form Φ in the set $(V(\Psi,\Phi)U_{up}(n,R)P)^{-1}\Phi$ have associative by right matrix. The proof of the Corollary is complete. □

Theorem 2. *Let* $\Phi = \operatorname{diag}(\varphi_1,\ldots,\varphi_n)$ *and* $\Psi = \operatorname{diag}(\varepsilon_1,\ldots,\varepsilon_n)$ *be nonsingular d-matrices. In order that* $GL(n,R) = G_\Psi^T G_\Phi$, *it is necessary and sufficient that* $\left(\det\frac{1}{\varphi_1}\Phi, \det\frac{1}{\varepsilon_1}\Psi\right) = 1$.

Proof. **Necessity.** Let $\left(\det\frac{1}{\varphi_1}\Phi, \det\frac{1}{\varepsilon_1}\Psi\right) = \delta$ and φ_s, ε_r are the first diagonal elements of the matrices Φ, Ψ with the property $\delta|\frac{\varphi_s}{\varphi_1}$, $\delta|\frac{\varepsilon_r}{\varepsilon_1}$, $2 \leqslant r,s \leqslant n$. Then $\delta|\frac{\varphi_i}{\varphi_1}$, $i = s, s+1,\ldots,n$ and $\delta\frac{\varepsilon_j}{\varepsilon_1}$, $j = r, r+1,\ldots,n$. Consequently, in the lower left corner of every matrix of the group G_Φ there is an $(n-s+1)\times(s-1)$ submatrix all elements of which are divisible by δ. And in the upper right corner every matrix of the group G_Ψ^T is $(r-1)\times(n-r+1)$ submatrix all elements of which are divisible by δ. Since $GL(n,R) = G_\Psi^T G_\Phi$ then

$$LH = \begin{Vmatrix} 0 & & 1 \\ & \vdots & \\ 1 & & 0 \end{Vmatrix} = T,$$

where $L \in G_\Psi^T$, $H \in G_\Phi$. Thus, $L = TH^{-1}$. Therefore, in the left upper corner of the matrix L there is an $(n-s+1)\times(s-1)$ submatrix all elements of which are divisible by δ. Taking into account structure of elements of the group G_Ψ^T we come to the conclusion that $s < r$, otherwise all elements of first row of the matrix L would be divisible by δ. Possible cases:

1. $n-s+1 \leqslant r-1$. Then the matrix L has $(n-s+1) \times ((s-1)+(n-r+1))$ submatrix all elements of which are divisible by δ. Since,

$$(n-s+1)+(s-1)+(n-r+1)=(n+1)+(n-r) \geqslant n+1$$

by Lemma 4, $\delta | \det L \in U(R)$. Therefore $\delta = 1$.

2. $n-s+1 > r-1$. Then the matrix L contains an $(r-1) \times ((s-1)+(n-r+1))$ submatrix all elements of which are divisible by δ. Since,

$$(r-1)+(s-1)+(n-r+1)=n+s-1=(n+1)+(s-2) \geqslant n+1,$$

as above $\delta = 1$.

Sufficiency. Let $A = \|a_{ij}\|_1^2 \in GL(2,R)$ and $\left(a_{11}, \frac{\varphi_2}{\varphi_1} a_{12}\right) = \delta$. By Lemma 3 there exists $H \in G_\Phi$ such that

$$AH = \begin{Vmatrix} \delta & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

Since $\left(\frac{\varphi_2}{\varphi_1}, \frac{\varepsilon_2}{\varepsilon_1}\right) = 1$ and $\delta | \frac{\varphi_2}{\varphi_1}$, it follows that $\left(\delta, \frac{\varepsilon_2}{\varepsilon_1}\right) = 1$. Therefore, there exists $L \in G_\Psi^T$ such that $\det LAH = 1$ and

$$LAH = \begin{Vmatrix} 1 & a \\ b & c \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ b & 1 \end{Vmatrix} \begin{Vmatrix} 1 & a \\ 0 & 1 \end{Vmatrix}.$$

Consequently,

$$A = \left(L^{-1} \begin{Vmatrix} 1 & 0 \\ b & 1 \end{Vmatrix}\right) \left(\begin{Vmatrix} 1 & a \\ 0 & 1 \end{Vmatrix} H^{-1}\right).$$

Hence the result holds for $n=2$.

Let $n \geqslant 3$, and suppose that the result is established for $k < n$. Let $A = \|a_{ij}\|_1^n \in GL(n,R)$. By analogy we can find matrices $L \in G_\Psi^T$, $H \in G_\Phi$ such that

$$LAH = \begin{Vmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A_{n-1} \end{Vmatrix}.$$

By the induction hypothesis $A_{n-1} = L_{n-1} H_{n-1}$, where $L_{n-1} \in G_{\Psi_1}^T$, $H_{n-1} \in G_{\Phi_1}$, $\Psi_1 = \operatorname{diag}(\varepsilon_2, \ldots, \varepsilon_n)$, $\Phi_1 = \operatorname{diag}(\varphi_2, \ldots, \varphi_n)$. Hence,

$$A = \left(L^{-1} \begin{Vmatrix} 1 & \mathbf{0} \\ \mathbf{0} & L_{n-1} \end{Vmatrix}\right) \left(\begin{Vmatrix} 1 & \mathbf{0} \\ \mathbf{0} & H_{n-1} \end{Vmatrix} H^{-1}\right).$$

Since $\begin{Vmatrix} 1 & \mathbf{0} \\ \mathbf{0} & L_{n-1} \end{Vmatrix} \in G_\Psi^T$ and $\begin{Vmatrix} 1 & \mathbf{0} \\ \mathbf{0} & H_{n-1} \end{Vmatrix} \in G_\Phi$, the proof of our statement is complete. □

Corollary 2. *Let A, B be matrices with the canonical diagonal form $\Psi = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, $\Phi = \operatorname{diag}(\varphi_1, \ldots, \varphi_n)$, respectively. If $\left(\det \frac{1}{\varphi_1}\Phi, \det \frac{1}{\varepsilon_1}\Psi\right) = 1$ then the matrix AB has the canonical diagonal form $\Psi\Phi$.*

Proof. Since, $A = P_A^{-1} \Psi Q_A^{-1}$, $B = P_B^{-1} \Phi Q_B^{-1}$, where $P_A, Q_A, P_B, Q_B \in GL(n,R)$, we have $AB = P_A^{-1} \Psi \left(Q_A^{-1} P_B^{-1}\right) \Phi Q_B^{-1}$. By Theorem 2, $Q_A^{-1} P_B^{-1} = UV$, where $U \in G_\Psi^T$,

$V \in G_\Phi$. Consequently,

$$AB = P_A^{-1}(\Psi U)(V\Phi)Q_B^{-1} = \left(P_A^{-1}S_1\right)\Psi\Phi\left(S_2Q_B^{-1}\right) \sim \Psi\Phi.$$

□

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ПРО РОЗКЛАД ПОВНОЇ ЛІНІЙНОЇ ГРУПИ В ДОБУТОК ДЕЯКИХ ЇЇ ПІДГРУП

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Розглянуто групу G_Φ-оборотних матриць, які квазікомутують з діагональною матрицею Φ. Показано, що над деякою областю Безу повна лінійна група розкладається в добуток G_Φ груп нижніх і верхніх унітрикутних матриць. Зазначено необхідні та достатні умови для того, щоб $GL(n,R) = G_\Psi^T G_\Phi$, де T – знак транспонування. Одержані результати використано для опису дільників матриць.

Ключові слова: повна лінійна група, розклад, підгрупа, дільники матриць.

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ON 2-TORSION OF BRAUER GROUPS OF HYPERELLIPTIC CURVES OVER PSEUDOLOCAL FIELDS

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It is proved an analogue of the Yanchevskii and Margolin's theorem about the 2-torsion of the Brauer group of hyperelliptic curve over a complete discretely valued field with pseudofinite residue field.

Key words: hyperelliptic curve, Brauer group, local field, pseudofinite field.

By a pseudolocal field K we mean a complete with respect to a discrete valution field with pseudofinite [1] residue fields k. Recall that an infinite field is called *pseudo-finite* if it is perfect, pseudoalgebraically closed and possesses exactly one extension of each degree. If K_s is a separable closure of a field K, let $G_K = Gal(K_s/K)$ be its Galois group. Then $H^i(K,M)$ denotes the Galois cohomology group of G_K-module M. C_n and C/nC stand for the kernel and cokernel of multiplication by n in an abelian group C. For an abelian variety A defined over K by $A(K)$ (respectively $K(A)$) we denote the group of K-rational points of A (respectively the function field on A).

Consider the homomorphism $\mu : K(A)^* \to Div(A)$, which sends a function from $K(A)^*$ to its divisor. The map μ induces the corresponding homomorphism in Galois cohomology

$$\mu_* : H^2(G, K(A)^*) \to H^2(G, Div(A)).$$

The kernel of μ_* called the Brauer group of A, is denoted by BrA. It is known [1], that the group BrA consists of the classes of similar central simple K -algebras, which are unramified at all the valutions v of K.

Recall that two central simple K-algebras C_1, C_2 are *similar* if there exist two natural numbers m, n such that $C_1 \otimes M_n(K)$ and $C_2 \otimes M_m(K)$ are isomorphic. Recall also that a generalized quaternion algebra $(\frac{a,b}{K})$ over a field K is a K-algebra, generated by 1, x, y, $z \in K^*$, where $x^2 = a$, $y^2 = b$, $z^2 = -ab$, and $xy = -yx$, $xz = -zx$, $yz = -zx$. $[\frac{a,b}{K}]$ is the element of BrK with the representative $(\frac{a,b}{K})$.

The 2-torsion part of the Brauer group of an elliptic or hyperelliptic curve over a local field was described by V.I.Yanchevskii and G.L.Margolin [2]. In their description every element of $(BrA)_2$ is represented by a quaternion algebra over $K(A)$. It turns out that the analogous results remain true for hyperelliptic curves over a pseudolocal field.

Let K be a pseudolocal field. Denote by O_K the ring of integer of K. Let π be an uniformizing element of K, α be a unit of K which is not a square, n be a prime number, $n \neq chark$ and $|A|$ denote the order of finite set A.

Let A be a hyperelliptic curve over a preudolocal field K, that is a curve defined by the equation $y^2 = f(x)$, $f(x) = \beta f_0(x)$, where $f_0(x) \in O_K$ is a monic polynomial, $m = deg f(x)$, $\beta \in \{1, \alpha, \pi\}$.

Theorem 1. *Let A be either an elliptic or a hyperelliptic curve with good reduction defined over a pseudolocal field K with a pseudofinite residue field $\bar{K}$, $char K \neq 2$, $m \not\equiv 0 (mod 4)$. Then $(BrA)_2$ consists of the folowing pairwise distinct elements:*

$$\left[\left(\frac{\pi, 1}{K(A)}\right)\right], \left[\left(\frac{\pi, \alpha}{K(A)}\right)\right], \left[\left(\frac{\pi, g(x)}{K(A)}\right)\right] \left[\left(\frac{\pi, \alpha g(x)}{K(A)}\right)\right],$$

where

(i) if m is odd then $g(x)$ runs over all monic divisors of $f(x)$ of degree less than $m/2$.

(ii) if m is even then $g(x)$ runs over all monic divisors of $f(x)$ of even degree less than $m/2$.

To prove this result we need some preliminary statements which are of interest by its own right.

First, we will need the following three lemmas from [2].

Lemma 1. *Let $g(x)$ be a divisor of $f(x)$ and let either m be odd or $deg\, g(x)$ even. Then the quaternion algebra $\left(\frac{B, g(x)}{K(A)}\right)$ is unramified over $K(A)$ for any $B \in K^*$.*

Lemma 2. *Let β be either 1 or α, $g = Bg_0$, B either 1 or α and $g_0 \in O_K[x]$ be a monic divisor of $f(x)$, $f_0 = g_0\hat{g_0}$. If $\bar{g}_0 \notin \bar{K}[x]^2$, $\bar{\hat{g_0}} \notin \bar{K}[x]^2$, where $\bar{K}$ is residue field of K, $\bar{g}_0$ and $\bar{\hat{g_0}}$ are polinomial g_0 and $\hat{g_0}$, regarded over the residue field $\bar{K}$. Then $\left(\frac{\pi, g(x)}{K(A)}\right) \not\simeq 1$.*

Lemma 3. *Let K be a general local field and A has good reduction. If $\left(\frac{\pi, \alpha}{K}\right)$ is the quaternion division algebra over K, then $\left(\frac{\pi, \alpha}{K(A)}\right)$ is a division algebra.*

Note, that the statements and the proofs of these three results remain true for any complete discretely valued field.

Lemma 4. *Let K be a pseudolocal field, A be an abelian variety defined over K. Suppose that A has good reduction. Then $|A(K)/nA(K)| = |A(K)_n|$ for any n, $(n, char \bar{K}) = 1$.*

For local fields this was proved by V. I. Yanchevskii and G. L. Margolin [2]. The case of elliptic curves over pseudolocal fields was considered in [5]. The case of abelian variety with good reduction was investigated by V. I. Andrijchuk [3].

For completeness sake we sketch briefly the corresponding arguments. Any principal homogeneous space for A over K has a K-rational point, so $H^1(k, A) = 0$. Thus the Kummer exact sequence corresponding to multiplication by n yields the isomorphism of the groups $A(k)/nA(k)$ and $H^1(K, A_n)$. Besides, since the absolute Galois group of k is isomorphic to $\hat{Z}$, we obtain $|H^1(k, A_n)| = |H^0(k, A_n)|$ (see [6] for more details). Thus we have $|A(k)/nA(k)| = |A(k)_n|$. To finish the proof it is sufficient to use the reduction exact sequence and the snake lemma together with the fact that the kernel of the reduction map is uniquelly divisible by n.

Lemma 5. *Let A be an abelian variety with good reduction, defined over a pseudolocal field K, $\hat{A}$ be its dual variety. Then for any n, $(n, char k) = 1$ the Tate-Shafarevich pairing induces a nondegenerate pairing $A(K)/nA(K) \otimes H^1(K, A)_n \longrightarrow \mathbb{Z}/n\mathbb{Z}$. If A is an elliptic curve with bad reduction, then the last pairing is nondegenerate in the case of general local field.*

For an elliptic curve this was proved in [4]. According to [3] this fact remains true for an abelian variety of any dimention with good reduction.

To prove that the pairing $A(K)/nA(K) \otimes H^1(K, \hat{A}_n) \longrightarrow \mathbb{Z}/n\mathbb{Z}$ is non-degenerate, we consider the commutative diagram

$$\begin{array}{ccccc} H^1(K, A_n) & \times H^1(K, \hat{A}_n) & \xrightarrow{W} & \mathbb{Q}/\mathbb{Z} \\ \uparrow i_n & \downarrow j_n & & \| \\ A(K)/nA(K) & \times H^1(K, \hat{A})_n & \xrightarrow{W} & \mathbb{Q}/\mathbb{Z}, \end{array}$$

where i_n and j_n are the homomorphisms from the Kummer exact sequences for A, W is induced by the Weil pairing, and T is induced by the Tate-Shafarevich pairing. The homomorphis i_n is injective and it is known that the pairing W is non-degenerate. Since i_n is a monomorphism and j_n is an epimorphism, it follows that T is nondegenerate on the left. To prove that it is a duality, it suffices to prove that $|A(K)/nA(K)| = |H^1(K, \hat{A}_n)|$. But this follows from the equalities $|A(K)_n| = |\hat{A}(K)_n|$, $|H^1(K, A_n)| = |H^0(K, A_n)| \times |H^2(K, A_n)|$ and $|H^1(K, A_n)| = |H^1(K, \hat{A}_n)|$, which hold for any complete discretely valued field with quasifinite residue fields (the first of them holds for any field).

Proof of Theorem 1. We denote by $PicA$ (respectively Pic^0A) the Picard group (respectively its subgroup of divisor classes of degree zero). As in [2], we begin by considering the following exact sequence

$$0 \to PicA \to H^0(K, Pic\bar{A}) \to BrK \to BrA \to H^1(K, Pic\bar{A}) \to H^3(K, K_s^*). \quad (1)$$

In this sequence $H^3(K, K_s^*) = 0$, since the cohomological dimension of K is 2, as it is follows from [6, Prop.12, p.105]. Since A has a K-rational point, the index of A is equal 1. Thus the homomorphism $Pic^0 A \to H^0(K, Pic^0 A)$ is surjective, and we obtain the exact sequence

$$0 \longrightarrow BrK \longrightarrow BrA \longrightarrow H^1(K, PicA) \longrightarrow 0. \quad (2)$$

Using $BrK \cong \mathbb{Q}/\mathbb{Z}$ for any general local field K and passing to n-torsion in the exact sequence (1) we obtain the following equality:

$$|(BrA)_n| = n|H^1(K, Pic\bar{A})|. \quad (3)$$

Since the period of A divides the index of A, it is 1, so passing to cohomology in the exact sequence

$$0 \longrightarrow Pic^0\bar{A} \longrightarrow Pic\bar{A} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

we obtain the isomorphism $H^1(K, Pic^0\bar{A}) \to H^1(K, Pic\bar{A})$ which induces the isomorphism $H^1(K, Pic^0\bar{A})_n \to H^1(K, Pic\bar{A})_n$

Now, Lemma 5 implies, that $|H^1(K, Pic^0\bar{A})| = |A(K)/nA(K)|$, and by the equality (3), $|(BrA)_n| = n|H^1(K, Pik^0\bar{A})| = n|Pic^0A/nPic^0A| = n|(Pic^0A)_n|$. Here the last equality follows from Lemma 4.

The order of $(BrA)_2$ is $2|Pic^0(A)_2|$. But, by [2, Corol. 4, p.19], there is bijective correspondence between elements of $(Pic^0\bar{A})_2$ and monic divisors of $f(x)$ defined over K of degree less than $m/2$ in the case of odd m and monic divisors of $f(x)$ defined over K of even degree less than $m/2$ in the case of even m. Thus, to finish the proof, it remains to show that all algebras from the statement of Theorem 1 are nontrivial, unramified and pairwise not isomorphic. But this follows from Lemmas 1, 2, 3 and from the fact that the tensor products of two such algebras is similar to an algebra of the same form.

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2-КРУЧЕННЯ ГРУП БРАУЕРА ГІПЕРЕЛІПТИЧНИХ КРИВИХ НАД ПСЕВДОЛОКАЛЬНИМИ ПОЛЯМИ

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Доведено аналог теореми В. І. Янчевського та Г. Л. Марголіна про 2-кручення групи Брауера гіпереліптичної кривої над повним дискретно нормованим полем з псевдоскінченним полем лишків.

Ключові слова: гіпереліптична крива, група Брауера, локальне поле, псевдоскінченне поле.

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ZERO-KNOWLEDGE PROOFS OF THE CONJUGACY FOR PERMUTATION GROUPS

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We design a perfect zero-knowledge proof system for recognition if two permutation groups are conjugate. It follows, answering a question posed by O. G. Ganyushkin, that this recognition problem is not NP-complete unless the polynomial-time hierarchy collapses.

Key words: interactive proof system, zero-knowledge proof, NP-completeness, permutation group, conjugacy.

1. Let S_m be a symmetric group of order m. We suppose that an element of S_m, a permutation of the set $\{1, 2, \dots, m\}$, is encoded by a binary string of length $l = \lceil \log_2 m! \rceil$, $m(\log_2 m - O(1)) \leqslant l \leqslant m \log_2 m$. Given $v \in S_m$, $y \in S_m$, and $Y \subseteq S_m$, we denote $y^v = v^{-1}yv$ and $Y^v = \{ y^v : y \in Y\}$. Two subgroups G and H of S_m are *similar* if their actions on $\{1, 2, \dots, m\}$ are isomorphic or, equivalently, if $G = H^v$ for some $v \in S_m$. If $X \subseteq S_m$, let $\langle X \rangle$ denote the group generated by elements of X.

We address the following algorithmic problem.

SIMILITUDE OF PERMUTATION GROUPS
Given: $A_0, A_1 \subseteq S_m$.
Recognize if: A_0 and A_1 are similar.

Note that the EQUALITY OF PERMUTATION GROUPS problem, that is, recognition if $\langle A_0 \rangle = \langle A_1 \rangle$ reduces to recognition, given $X \subseteq S_m$ and $y \in S_m$, if $y \in \langle X \rangle$. Since the latter problem is known to be solvable in time bounded by a polynomial of the input length [20, 10], so is EQUALITY OF PERMUTATION GROUPS. As a consequence, SIMILITUDE OF PERMUTATION GROUPS belongs to NP, the class of decision problems whose yes-instances have polynomial-time verifiable certificates. The similitude of $\langle A_0 \rangle$ and $\langle A_1 \rangle$ is certified by a permutation v such that $\langle A_1 \rangle = \langle A_0^v \rangle$.

Another problem, ISOMORPHISM OF PERMUTATION GROUPS, is to recognize if $\langle A_0 \rangle$ and $\langle A_1 \rangle$ are isomorphic. This problem also belongs to NP (E. Luks, see [5, Corollary 4.11]). Furthermore, it is announced [7] that ISOMORPHISM OF PERMUTATION GROUPS belongs to the complexity class coAM (see Section 2 for the definition). By [8] this implies that ISOMORPHISM OF PERMUTATION GROUPS is not NP-complete unless the polynomial-time hierarchy collapses to its second level (for the background on computational complexity theory the reader is referred to [12]).

O. G. Ganyushkin [11] posed a question if a similar non-completeness result can be obtained for SIMILITUDE OF PERMUTATION GROUPS. In this paper we answer this question in affirmative. We actually prove a stronger result of independent interest, namely, that SIMILITUDE OF PERMUTATION GROUPS has a perfect zero-knowledge interactive proof system. It follows from [1] that SIMILITUDE OF PERMUTATION GROUPS belongs to coAM and is therefore not NP-complete unless the polynomial-time hierarchy collapses.

Informally speaking, a zero-knowledge proof system for a recognition problem of a language L is a protocol for two parties, the prover and the verifier, that allows the prover to convince the verifier that a given input belongs to L, with high confidence but without communicating the verifier any information (the rigorous definitions are in Section 2). Our zero-knowledge proof system for SIMILITUDE OF PERMUTATION GROUPS uses the underlying ideas of the zero-knowledge proof systems designed in [16] for the QUADRATIC RESIDUOSITY and in [14] for the GRAPH ISOMORPHISM problem. In particular, instead of direct proving something about the input groups $\langle A_0\rangle$ and $\langle A_1\rangle$, the prover prefers to deal with their conjugates $\langle A_0\rangle^w$ and $\langle A_1\rangle^w$ via a random permutation w. The crucial point is that these random groups are indistinguishable by the verifier because they are identically distributed, provided $\langle A_0\rangle$ and $\langle A_1\rangle$ are similar. However, we here encounter a complication: the verifier may actually be able to distinguish between $\langle A_0\rangle^w$ and $\langle A_1\rangle^w$ based on particular representations of these groups by their generators. Overcoming this complication, which does not arise in [16, 14], is a novel ingredient of our proof system.

Our result holds true even for a more general problem of recognizing if $\langle A_0\rangle$ and $\langle A_1\rangle$ are conjugated via an element of the group generated by a given set $U \subseteq S_m$. We furthermore observe that a similar perfect zero-knowledge proof system works also for the ELEMENT CONJUGACY problem of recognizing, given $a_0, a_1 \in S_m$ and $U \subseteq S_m$, if $a_1 = a_0^v$ for some $v \in \langle U\rangle$. A version of this problem where $a_0, a_1 \in \langle U\rangle$ was proved to be in coAM in [5, Corollary 12.3 (i)]. Note that the proof system developed in [5] uses different techniques and is not zero-knowledge.

2. Preliminaries. Every decision problem under consideration can be represented through a suitable encoding as a recognition problem for a language L over the binary alphabet. We denote the *length* of a binary word w by $|w|$.

An *interactive proof system* $\{V, P\}$, further on abbreviated as IPS, consists of two probabilistic Turing machines, a polynomial-time *verifier* V and a computationally unlimited *prover* P. The input tape is common for the verifier and the prover. The verifier and the prover also share a communication tape which allows message exchange between them. The system works as follows. First both the machines V and P are given an input w and each of them is given an individual random string, r_V for V and r_P for P. Then P and V alternatingly write messages to one another in the communication tape. V computes its i-th message a_i to P based on the input w, the random string r_V, and all previous messages from P to V. P computes its i-th message b_i to V based on the input w, the random string r_P, and all previous messages from V to P. After a number of message exchanges V terminates interaction and computes an output based on w, r_V, and all b_i. The output is denoted by $\{V, P\}(w)$. Note that, for a fixed w, $\{V, P\}(w)$ is a random variable depending on both random strings r_V and r_P.

Let $\epsilon(n)$ be a function of a natural argument taking on positive real values. We say that $\{V, P\}$ is an *IPS for a language L with error $\epsilon(n)$* if the following two conditions are fulfilled.

Completeness. If $w \in L$, then $\{V, P\}(w) = 1$ with probability at least $1 - \epsilon(|w|)$.
Soundness. If $w \notin L$, then, for an arbitrary interacting probabilistic Turing machine P^*, $\{V, P^*\}(w) = 1$ with probability at most $\epsilon(|w|)$.

We will call any prover P^* interacting with P on input $w \notin L$ *cheating*. If in the completeness condition we have $\{V, P\}(w) = 1$ with probability 1, we say that $\{V, P\}$ has *one-sided error* $\epsilon(n)$.

An IPS is *public-coin* if the concatenation $a_1 \dots a_k$ of the verifier's messages is a prefix of his random string r_V. A *round* is sending one message from the verifier to the prover or from the prover to the verifier. The class AM consists of those languages having IPSs with error 1/3 and with number of rounds bounded by a constant for all inputs. A language L belongs to the class coAM iff its complement $\{0,1\}^* \setminus L$ belongs to AM.

2.1. Proposition (Goldwasser-Sipser [17]). *Every IPS for a language L can be converted into a public-coin IPS for L with the same error at cost of increasing the number of rounds in 2.*

Given an IPS $\{V, P\}$ and an input w, let $\text{view}_{V,P}(w) = (r'_V, a_1, b_1, \dots, a_k, b_k)$ where r'_V is a part of r_V scanned by V during work on w and $a_1, b_1, \dots, a_k, b_k$ are all messages from P to V and from V to P (a_1 may be empty if the first message is sent by P). Note that the verifier's messages $a_1, \dots, a_k$ could be excluded because they are efficiently computable from the other components. For a fixed w, $\text{view}_{V,P}(w)$ is a random variable depending on r_V and r_P.

An IPS $\{V, P\}$ is *perfect zero-knowledge on L* if for every interacting polynomial-time probabilistic Turing machine V^* there is a probabilistic Turing machine M_{V^*}, called a *simulator*, that on every input $w \in L$ runs in expected polynomial time and produces output $M_{V^*}(w)$ which, if considered as a random variable depending on a random string of M_{V^*}, is distributed identically with $\text{view}_{V^*,P}(w)$. This notion formalizes the claim that the verifier gets no information during interaction with the prover: everything that the verifier gets he can get without the prover by running the simulator. According to the definition, the verifier learns nothing even if he deviates from the original program and follows an arbitrary probabilistic polynomial-time program V^*. We will call the verifier V *honest* and all other verifiers V^* *cheating*. If, for all V^*, M_{V^*} is implemented by the same simulator M running V^* as a subroutine, we say that $\{V, P\}$ is *black-box simulation* zero-knowledge.

We call $\epsilon(n)$ *negligible* if $\epsilon(n) < n^{-c}$ for every c and all n starting from some $n_0(c)$. The class of languages L having IPSs that are perfect zero-knowledge on L and have negligible error is denoted by PZK.

2.2. Proposition (Aiello-Håstad [1]). *PZK $\subseteq$ coAM.*

The *$k(n)$-fold sequential composition* of an IPS $\{V, P\}$ is the IPS $\{V', P'\}$ in which V' and P' on input w execute the programs of V and P sequentially $k(|w|)$ times, each time with independent choice of random strings r_V and r_P. At the end of interaction V' outputs 1 iff $\{V, P\}(w) = 1$ in all $k(|w|)$ executions. The initial system $\{V, P\}$ is called *atomic*.

2.3. Proposition.
1)If $\{V', P'\}$ is the $k(n)$-fold sequential composition of $\{V, P\}$, then

$$\max_{P^*} \mathbf{P}\left[\{V', P^*\}(w) = 1\right] = \left(\max_{P^*} \mathbf{P}\left[\{V, P^*\}(w) = 1\right]\right)^{k(|w|)}.$$

Consequently, if $\{V, P\}$ is an IPS for a language L with one-sided constant error ϵ, then $\{V', P'\}$ is an IPS for L with one-sided error $\epsilon^{k(n)}$.
2) (Goldreich-Oren [15], see also [13, Lemma 6.19]) If in addition $\{V, P\}$ is black-box simulation perfect zero-knowledge on L, then $\{V', P'\}$ is perfect zero-knowledge on L.

In the *$k(n)$-fold parallel composition* $\{V'', P''\}$ of $\{V, P\}$, the program of $\{V, P\}$ is executed $k(|w|)$ times in parallel, that is, in each round all $k(|w|)$ versions of a message are sent from one machine to another at once as a long single message. In every parallel execution V'' and P'' use independent copies of r_V and r_P. At the end of interaction V'' outputs 1 iff $\{V, P\}(w) = 1$ in all $k(|w|)$ executions.

2.4. Proposition. *If $\{V'', P''\}$ is the $k(n)$-fold parallel composition of $\{V, P\}$, then*

$$\max_{P^*} \mathbf{P}\left[\{V'', P^*\}(w) = 1\right] = \left(\max_{P^*} \mathbf{P}\left[\{V, P^*\}(w) = 1\right]\right)^{k(|w|)}.$$

3. Group Conjugacy. We consider the following extension of Similitude of Permutation Groups.

Group Conjugacy
Given: $A_0, A_1, U \subseteq S_m$.
Recognize if: $\langle A_1\rangle = \langle A_0\rangle^v$ for some $v \in \langle U\rangle$.

3.1. Theorem. Group Conjugacy *is in PZK.*

Designing a perfect zero-knowledge interactive proof system for Group Conjugacy, we will make use of the following facts due to Sims [20,10].
1) There is a polynomial-time algorithm that, given $X \subseteq S_m$ and $y \in S_m$, recognizes if $y \in \langle X\rangle$. As a consequence, there is a polynomial-time algorithm that, given $X \subseteq S_m$ and $Y \subseteq S_m$, recognizes if $\langle X\rangle = \langle Y\rangle$.
2) There is a probabilistic polynomial-time algorithm that, given $X \subseteq S_m$, outputs a random element of $\langle X\rangle$. Here and further on, by a *random element* of a finite set Z we mean a random variable uniformly distributed over Z.
Given $A \subseteq S_m$ and a number k, define

$$G(A, k) = \{\,(x_1, \ldots, x_k) : x_i \in S_m, \langle x_1, \ldots, x_k\rangle = \langle A\rangle\}\,.$$

In the sequel, the length of the binary encoding of an input $A_0, A_1, U \subseteq S_m$ will be denoted by n. We set $k = 4m$. On input (A_0, A_1, U), the IPS we design is the n-fold sequential repetition of the following 3-round system. We will say that the verifier V *accepts* if $\{V, P\}(A_0, A_1, U) = 1$ and *rejects* otherwise.

If (A_0, A_1, U) is yes-instance of Group Conjugacy, P finds an element $v \in \langle U\rangle$ such that $\langle A_1\rangle = \langle A_0\rangle^v$.

1st round.

P generates a random element $u \in \langle U \rangle$, computes $A = A_1^u$, chooses a random element $(a_1, \dots, a_k)$ in $G(A, k)$, and sends $(a_1, \dots, a_k)$ to V. V checks if all $a_i \in S_m$ and, if not (this is possible in the case of a cheating prover), halts and rejects.

2nd round.
V chooses a random bit $\beta \in \{0, 1\}$ and sends it to P.

3rd round.
Case $\beta = 1$. P sends V the permutation $w = u$. V checks if $w \in \langle U \rangle$ and if $\langle a_1, \dots, a_k \rangle = \langle A_1^w \rangle$.
Case $\beta \neq 1$ (this includes the possibility of a message $\beta \notin \{0, 1\}$ produced by a cheating verifier). P computes $w = vu$ and sends w to V. V checks if $w \in \langle U \rangle$ and if $\langle a_1, \dots, a_k \rangle = \langle A_0^w \rangle$.
V halts and accepts if the conditions are checked successfully and rejects otherwise.

We now need to prove that this system is indeed an IPS for GROUP CONJUGACY and, moreover, that it is perfect zero-knowledge.

Completeness. To show that the prover is able to follow the above protocol, we have to check that $G(A, k) \neq \emptyset$ for $k = 4m$. The latter is true by the fact that every subgroup of S_m can be generated by at most $m - 1$ elements [18]. If $\langle A_0 \rangle$ and $\langle A_1 \rangle$ are conjugate via an element of $\langle U \rangle$ and the prover and the verifier follow the protocol, then $\langle a_1, \dots, a_k \rangle = \langle A \rangle = \langle A_1^u \rangle = \langle A_0^{vu} \rangle$. Therefore, the verifier accepts with probability 1 both in the atomic and the composed systems.

Soundness. Assume that $\langle A_0 \rangle$ and $\langle A_1 \rangle$ are not conjugate via an element of $\langle U \rangle$ and consider an arbitrary cheating prover P^*. Observe that if both $\langle a_1, \dots, a_k \rangle = \langle A_1^u \rangle$ and $\langle a_1, \dots, a_k \rangle = \langle A_0^w \rangle$ with $u, w \in \langle U \rangle$, then $\langle A_1 \rangle = \langle A_0 \rangle^{wu^{-1}}$. It follows that V rejects for at least one value of β and, therefore, in the atomic system V accepts with probability at most $1/2$. By Proposition 2.3 (1), in the composed system V accepts with probability at most 2^{-n}.

Zero-knowledge. We will need the following fact.

3.2. Lemma. *Let G be a subgroup of S_m and $a_1, \dots, a_k$ be random independent elements of G.*
1) If $k = 4m$, then $\langle a_1, \dots, a_k \rangle = G$ with probability more than $1/2$.
2) If $k = 8m$, then $\langle a_1, \dots, a_k \rangle = G$ with probability more than $1 - 2^{-m}$.

Proof. We will estimate from above the probability that $\langle a_1, \dots, a_k \rangle \neq G$. This inequality is equivalent with the condition that all $\langle a_1 \rangle, \langle a_1, a_2 \rangle, \dots, \langle a_1, \dots, a_k \rangle$ are proper subgroups of G. Assume that this condition is true. Since every subgroup chain in S_m has length less than $2m$ (see [3, 9]), less than $2m - 1$ inclusions among $\langle a_1 \rangle \subseteq \langle a_1, a_2 \rangle \subseteq \dots \subseteq \langle a_1, \dots, a_k \rangle$ are proper. In other words, less than $2m - 1$ of the events $a_2 \notin \langle a_1 \rangle$, $a_3 \notin \langle a_1, a_2 \rangle$, $\dots$, $a_k \notin \langle a_1, \dots, a_{k-1} \rangle$ occur. Equivalently, there occur more than $k - 2m$ of the events $a_2 \in \langle a_1 \rangle$, $a_3 \in \langle a_1, a_2 \rangle$, $\dots$, $a_k \in \langle a_1, \dots, a_{k-1} \rangle$.

Let $p = |H|/|G|$ be the maximum density of a proper subgroup H of G. Given $a_1, \dots, a_i \in G$, define $E(a_1, \dots, a_i)$ to be an arbitrary subset of G fixed so that

(i) $E(a_1, \dots, a_i)$ has density p in G, and
(ii) $E(a_1, \dots, a_i)$ contains $\langle a_1, \dots, a_i \rangle$ if the latter is a proper subgroup of G.

If $\langle a_1, \dots, a_k \rangle \neq G$, there must occur more than $k - 2m$ of the events

$$a_2 \in E(a_1),\ a_3 \in E(a_1, a_2), \dots, a_k \in E(a_1, \dots, a_{k-1}). \quad (1)$$

It suffices to show that the probability of so many occurrences in (1) is small enough. Set $X_i(a_1, \dots, a_k)$ to be equal to 1 if $a_{i+1} \in E(a_1, \dots, a_i)$ and to 0 otherwise. In these terms, we have to estimate the probability that

$$\sum_{i=1}^{k-1} X_i > k - 2m. \quad (2)$$

It is easy to calculate that an arbitrary set of l events in (1) occurs with probability p^l. Hence events (1) as well as the random variables $X_1, \dots, X_{k-1}$ are mutually independent, and $X_1, \dots, X_{k-1}$ are successive Bernoulli trails with success probability p.

If $k = 4m$, inequality (2) implies that strictly more than a half of all the trails are successful. Since $p \leqslant 1/2$, this happens with probability less than $1/2$ and item 1 of the lemma follows.

If $k = 8m$, inequality (2) implies

$$\frac{1}{k-1} \sum_{i=1}^{k-1} X_i > p + \epsilon$$

with deviation $\epsilon = 1/4$ from the mean value $p = \mathbf{E}\left[\frac{1}{k-1} \sum_{i=1}^{k-1} X_i\right]$. By the Chernoff bound [2, Theorem A.4], this happens with probability less than $\exp\left(-2\epsilon^2(k-1)\right) = \exp(-m + \frac{1}{8}) < 2^{-m}$. This proves item 2 of the lemma. □

By Proposition 2.3 (2) it suffices to show that the atomic system is black-box simulation perfect zero-knowledge. We describe a probabilistic simulator M that uses the program of V^* as a subroutine and, for each V^*, runs in expected polynomial time. Assume that the running time of V^* is bounded by a polynomial q in the input size. On input (A_0, A_1, U) of length n, M will run the program of V^* on the same input with random string r, where r is the prefix of M's random string of length $q(n)$. In all other cases of randomization, M will use the remaining part of its random string.

Having received an input (A_0, A_1, U), the simulator M chooses a random element $w \in \langle U \rangle$ and a random bit $\alpha \in \{0, 1\}$. Then M randomly and independently chooses elements $a_1, \dots, a_k$ in $\langle A_\alpha^w \rangle$ and checks if

$$\langle a_1, \dots, a_k \rangle = \langle A_\alpha^w \rangle. \quad (3)$$

If (3) is not true, M repeats the choice of $a_1, \dots, a_k$ again and again until (3) is fulfilled. By Lemma 3.2 (1), M succeeds in at most 2 attempts on average. The resulting sequence $(a_1, \dots, a_k)$ is uniformly distributed on $G(A_\alpha^w, k)$. Then M computes $\beta = V^*(A_0, A_1, U, r, a_1, \dots, a_k)$, the message that V^* sends P in the 2-nd round after receiving P's message $a_1, \dots, a_k$. If β and α are simultaneously equal to or different from 1, M halts and outputs $(r', a_1, \dots, a_k, \beta, w)$, where r' is the prefix of r that V^* actually uses after reading the input (A_0, A_1, U) and the prover's message $a_1, \dots, a_k$. If exactly one of β and α is equal to 1, then M restarts the same program from the very beginning with another independent choice of w, α, and $a_1, \dots, a_k$. Notice that it might happen that in unsuccessful attempts V^* used a prefix of r longer than r'.

We first check that, for each V^*, the simulator M terminates in expected polynomial time whenever A_0 and A_1 are conjugated via an element of $\langle U \rangle$. Since V^* is polynomial-time, one attempt to pass the body of M's program takes time bounded by a polynomial of n. Observe that α and $(r, a_1, \dots, a_k)$ are independent. Really, independently of whether $\alpha = 0$ or $\alpha = 1$, r is a random string of length $q(n)$ and $(a_1, \dots, a_k)$ is a random element of $G(A, k)$, where A itself is a random element of the orbit $\{ A_0^w : w \in \langle U \rangle \} = \{ A_1^w : w \in \langle U \rangle \}$ under the conjugating action of $\langle U \rangle$ on subsets of S_m. It follows that α and β are independent and therefore an execution of the body of M's program is successful with probability 1/2. We conclude that on average M's program is executed twice and this takes expected polynomial time.

We finally need to check that, whenever A_0 and A_1 are conjugated via an element of $\langle U \rangle$, for each V^* the output $M(A_0, A_1, U)$ is distributed identically with $\mathrm{view}_{V^*, P}(A_0, A_1, U)$. Notice that both the random variables depend on V^*'s random string r. It therefore suffices to show that the distributions are identical when conditioned on an arbitrary fixed r. Denote these conditional distributions by $D_M(A_0, A_1, U, r)$ and $D_{V^*, P}(A_0, A_1, U, r)$. We will show that they are both uniform on the set

$$S = \Big\{ (a_1, \dots, a_k, \beta, w) \ : \ w \in \langle U \rangle, \ \beta = V^*(A_0, A_1, U, r, a_1, \dots, a_k), \\ (a_1, \dots, a_k) \in G(A^w_{\delta(\beta)}, k) \Big\},$$

where $\delta(\beta)$ is equal to 1 if $\beta = 1$ and to 0 otherwise.

Let $v \in \langle U \rangle$, such that $\langle A_1 \rangle = \langle A_0 \rangle^v$, be chosen by the prover P on input (A_0, A_1, U). Given $x_1, \dots, x_k \in G(A_1, k)$ and $u \in \langle U \rangle$, define $\phi(x_1, \dots, x_k, u) = (a_1, \dots, a_k, \beta, w)$ by $a_i = x_i^u$ for all $i \leqslant k$, $\beta = V^*(A_0, A_1, U, r, a_1, \dots, a_k)$, and $w = v^{1-\delta(\beta)} u$. As easily seen, $\phi(x_1, \dots, x_k, u) \in S$.

Claim. The map $\phi : G(A_1, k) \times \langle U \rangle \to S$ is one-to-one.

Proof. Define $\psi(a_1, \dots, a_k, \beta, w) = (x_1, \dots, x_k, u)$ by $u = v^{\delta(\beta)-1} w$ and $x_i = a_i^{u^{-1}}$ for all $i \leqslant k$. It is not hard to check that the map ψ is the inverse of ϕ. $\square$

Observe now that if $(x_1, \dots, x_k, u)$ is chosen at random uniformly in $G(A_1, k) \times \langle U \rangle$, then $\phi(x_1, \dots, x_k, u)$ has distribution $D_{V^*, P}(A_0, A_1, U, r)$. By Claim we conclude that $D_{V^*, P}(A_0, A_1, U, r)$ is uniform on S.

As a yet another consequence of Claim, observe that if a random tuple $(a_1, \dots, a_k, \beta, w)$ is uniformly distributed on S, then its prefix $(a_1, \dots, a_k)$ is a random element of $G(A, k)$, where A is a random element of the orbit $\{ A_0^w : w \in \langle U \rangle \} = \{ A_1^w : w \in \langle U \rangle \}$ under the conjugating action of $\langle U \rangle$ on subsets of S_m. This suggests the following way of generating a random element of S. Choose uniformly at random $\alpha \in \{0, 1\}$, $w \in \langle U \rangle$, $(a_1, \dots, a_k) \in G(A^w_\alpha, k)$ and, if

$$\delta\left(V^*(A_0, A_1, U, r, a_1, \dots, a_k)\right) = \alpha, \tag{4}$$

output $(a_1, \dots, a_k, V^*(A_0, A_1, U, r, a_1, \dots, a_k), w)$; otherwise repeat the same procedure once again independently. Under the condition that (4) is fulfilled for the first time in the i-th repetition, the output is uniformly distributed on S. Notice now that this sampling procedure coincides with the description of $D_M(A_0, A_1, U, r)$. It follows that $D_M(A_0, A_1, U, r)$ is uniform on S. The proof of the perfect zero-knowledge property of our proof system for GROUP CONJUGACY is complete.

The following corollary immediately follows from Theorem 3.1 by Proposition 2.2 and the result of [8].

3.3. Corollary. GROUP CONJUGACY *is in coAM and is therefore not NP-complete unless the polynomial-time hierarchy collapses.*

We also give an alternative proof of this corollary that consists in direct designing a two-round IPS $\{V, P\}$ with error 1/4 for the complement of GROUP CONJUGACY and applying Proposition 2.1. More precisely, we deal with the GROUP NON-CONJUGACY problem of recognizing, given $A_0, A_1, U \subseteq S_m$, if there is no $v \in \langle U \rangle$ such that $\langle A_1 \rangle = \langle A_0 \rangle^v$.

Set $k = 8m$. The below IPS is composed twice in parallel.

1st round.
V chooses a random bit $\alpha \in \{0, 1\}$, a random element $u \in \langle U \rangle$, and a sequence of random independent elements $a_1, \ldots, a_k \in \langle A_\alpha^u \rangle$. Then V sends $(a_1, \ldots, a_k)$ to P.

2nd round.
P determines β such that $\langle a_1, \ldots, a_k \rangle$ and $\langle A_\beta \rangle$ are conjugate via an element of $\langle U \rangle$ and sends β to V.

V accepts if $\beta = \alpha$ and rejects otherwise.

Completeness. By Lemma 3.2 (2), $\langle a_1, \ldots, a_k \rangle = \langle A_\alpha^u \rangle$ with probability at least $1 - 2^{-m}$. If this happens and if $\langle A_0 \rangle$ and $\langle A_1 \rangle$ are not conjugated via $\langle U \rangle$, the group $\langle a_1, \ldots, a_k \rangle$ is conjugated via $\langle U \rangle$ with precisely one of $\langle A_0 \rangle$ and $\langle A_1 \rangle$. In this case P is able to determine α correctly. Therefore V accepts with probability at least $1 - 2^{-m}$ in the atomic system and with probability at least $1 - 2^{-m+1}$ in the composed system.

Soundness. If $\langle A_0 \rangle$ and $\langle A_1 \rangle$ are conjugated via $\langle U \rangle$, then for both values $\alpha = 0$ and $\alpha = 1$, the vector $(a_1, \ldots, a_k)$ has the same distribution, namely, it is a random element of A^k, where A is a random element of the orbit $\{A_0^w : w \in \langle U \rangle\} = \{A_1^w : w \in \langle U \rangle\}$ under the conjugating action of $\langle U \rangle$ on subsets of S_m. It follows that, irrespective of his program, P guesses the true value of α with probability 1/2. With the same probability V accepts in the atomic system. By Proposition 2.4, in the composed system V accepts with probability 1/4.

Note that $\{V, P\}$ is perfect zero-knowledge only for the honest verifier but may reveal a non-trivial information for a cheating verifier.

4. Element Conjugacy. This section is devoted to the following problem.

ELEMENT CONJUGACY
Given: $a_0, a_1 \in S_m$, $U \subseteq S_m$.
Recognize if: $a_1 = a_0^v$ for some $v \in \langle U \rangle$.

L. Babai [5] considers a version of this problem with $a_0, a_1 \in \langle U \rangle$ and proves that it belongs to coAM. His result holds true not only for permutation groups but also for arbitrary finite groups with efficiently performable group operations, in particular, for matrix groups over finite fields. It is easy to see that Theorem 3.1 carries over to ELEMENT CONJUGACY.

4.1. Theorem. ELEMENT CONJUGACY *is in PZK*.

The proof system designed in the preceding section for GROUP CONJUGACY applies to ELEMENT CONJUGACY as well. Moreover, the proof system for ELEMENT CONJUGACY is considerably simpler. In place of groups $\langle A_0^u \rangle$ and $\langle A_1^u \rangle$ we now deal with single elements a_0^u and a_1^u and there is no complication with representation of $\langle A_0^u \rangle$ and $\langle A_1^u \rangle$ by generating sets.

We now notice relations of ELEMENT CONJUGACY with the following problem considered by E. Luks [19] (see also [6, Section 6.5]). Given $x \in S_m$, let $C(x)$ denote the centralizer of x in S_m.

CENTRALIZER AND COSET INTERSECTION
Given: $x, y \in S_m$, $U \subseteq S_m$.
Recognize if: $C(x) \cap \langle U \rangle y \neq \emptyset$.

Since, given a permutation x, one can efficiently find a list of generators for $C(x)$, this is a particular case of the COSET INTERSECTION problem of recognizing, given $A, B \subseteq S_m$ and $s, t \in S_m$, if the cosets $\langle A \rangle s$ and $\langle B \rangle t$ intersect.

4.2. Proposition. ELEMENT CONJUGACY *and* CENTRALIZER AND COSET INTERSECTION *are equivalent with respect to the polynomial-time many-one reducibility.*

Proof. We first reduce ELEMENT CONJUGACY to CENTRALIZER AND COSET INTERSECTION. Given permutations a_0 and a_1, it is easy to recognize if they are conjugate in S_m and, if so, to find an s such that $a_1 = a_0^s$. The set of all $z \in S_m$ such that $a_1 = a_0^z$ is the coset $C(a_0)s$. It follows that $\langle U \rangle$ contains v such that $a_1 = a_0^v$ iff $C(a_0)$ and $\langle U \rangle s^{-1}$ intersect.

A reduction from CENTRALIZER AND COSET INTERSECTION to ELEMENT CONJUGACY is based on the fact that $C(x)$ and $\langle U \rangle y$ intersect iff x and yxy^{-1} are conjugated via an element of $\langle U \rangle$. □

Note that, while the reduction we described from ELEMENT CONJUGACY to CENTRALIZER AND COSET INTERSECTION works only for permutation groups, the reduction in the other direction works equally well for arbitrary finite groups with efficiently performable group operations, in particular, for matrix groups over finite fields.

We now have three different ways to prove that ELEMENT CONJUGACY is in coAM and is therefore not NP-complete unless the polynomial-time hierarchy collapses. First, this fact follows from Theorem 4.1 by Proposition 2.2. Second, one can use Proposition 4.2 and the result of [5, Corollary 12.2 (d)] that COSET INTERSECTION is in coAM. Finally, one can design a constant-round IPS for the complement of ELEMENT CONJUGACY as it is done in the preceding section for the complement of GROUP CONJUGACY.

We conclude with two questions.

4.3. Question. Is there any reduction between GROUP CONJUGACY and COSET INTERSECTION? We are not able to prove an analog of Proposition 4.2 for groups because, given $A_0, A_1 \subseteq S_m$ such that $\langle A_1 \rangle = \langle A_0 \rangle^v$ for some $v \in S_m$, we cannot efficiently find any v with this property (otherwise we could efficiently recognize the SIMILITUDE OF PERMUTATION GROUPS).

4.4. Question. Does ELEMENT CONJUGACY reduce to GROUP CONJUGACY? Whereas Corollary 3.3 gives us an evidence that GROUP CONJUGACY is not NP-complete, we have no formal evidence supporting our feeling that GROUP CONJUGACY is not solvable efficiently. A reduction from Element Conjugacy could be considered such an evidence as Element Conjugacy is not expected to be solvable in polynomial time [4, page 1483].

Note that the conjugacy of permutations a_0 and a_1 via an element of a group $\langle U\rangle$ does not reduce to the conjugacy of the cyclic groups $\langle a_0\rangle$ and $\langle a_1\rangle$ via $\langle U\rangle$ because $\langle a_0\rangle$ and $\langle a_1\rangle$ can be conjugated by conjugation of another pair of their generators, while such a new conjugation may be not necessary via $\langle U\rangle$. For example, despite the groups $\langle(123)\rangle$ and $\langle(456)\rangle$ are conjugated via $\langle(14)(26)(35)\rangle$, the permutations (123) and (456) are not.

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ДОВЕДЕННЯ БЕЗ РОЗГОЛОШЕННЯ ДЛЯ СПРЯЖЕНОСТІ ГРУП ПІДСТАНОВОК

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Описано досконалу систему доведення без розголошення для задачі розпізнавання спряженості двох груп підстановок. Звідси випливає, відповідаючи на запитання О. Г. Ганюшкіна, що ця задача розпізнавання не є NP-повною за умови невиродженості поліноміальної ієрархії.

Ключові слова: подібність груп перестановок, алгоритмічні задачі розпізнавання, NP-повнота, системи доведень без розголошення.

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DIAGONALIZATION OF MATRICES OVER RING WITH FINITE STABLE RANK

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In the present work we construct a theory of diagonalizability for matrices over rings with finite stable rank. We prove that if R is a regular ring, then every $m \times k$ and $k \times m$ matrices, where $m \geqslant bsr(R) + 2$, admits a diagonal reduction. If R is a directly finite regular ring, then R_n is directly finite for all $n \geqslant bsr(R) + 2$. We obtain an affirmative answer in greater generality to the question of Henriksen: if R is a right Bezout ring and $R/J(R)$ is a right Hermite ring, then R is right Hermite. An affirmative answer to this question implies that a commutative Bezout ring is an elementary divisor ring if and only if $R/J(R)$ is an elementary divisor ring.

Key words: stable rank, Bezout ring, elementary transformations, Hermite ring.

1. The aim of this paper is to study the question of diagonalizability for matrices over ring. In [1] Henriksen proved that if R is a unit regular ring, then every matrix over R admits diagonal reduction. The diagonalizability question for matrices was answered by Menal and Moncasi [2, Theorem 7], they showed that all matrices over regular ring R admit diagonal reductions if only if R is Hermite. Further, the stable rank (in the sense of K-theory) of a regular ring satisfying the above condition is at most 2 [2, Proposition 8].

We construct a theory of diagonalizability for matrices over rings with finite stable rank. We provide that if R is a regular ring with finite stable rank $bsr(R)$, then every $k \times m$ and $m \times k$ matrices over R, where $m \geqslant bsr(R) + 2$, admit diagonal reduction. We provide an answer to a question in [4]: if R is a directly finite regular ring, is R_n directly finite? We prove that if R is directly finite regular ring with finite stable rank $bsr(R)$, then R_n is directly finite for all $n \geqslant bsr(R) + 2$. We also obtain an affirmative answer to a question of Henriksen [6, Question 2]: if R is an right Bezout ring and $R/J(R)$ is a right Hermite ring, then R is right Hermite. An affirmative answer to this question implies that a commutative Bezout ring is an elementary divisor ring if and only if $R/J(R)$ is an elementary divisor ring.

All rings we consider are supposed to be associative with $1 \neq 0$. By a right Bezout ring we will mean a ring in which all finitely generated right ideals are principal, and by a Bezout ring a ring which is both right and left Bezout. We recall that a module is uniserial if its lattice of submodules forms a chain. A ring is right serial if as a right module over itself, it is a direct sum of uniserial modules. A ring is serial if it both right ad left serial [5].

We shall call two matrices A and B over a ring R equivalent, if there exist invertible matrices P,Q such that $B = PAQ$. An matrix A admits diagonal reduction if A is equivalent to a diagonal matrix. If every $1 \times n$ ($n \times 1$) matrix over R admits diagonal reduction, then R is n-right (left) Hermite. A right (left) Hermite ring is a ring which is n-right (left) Hermite, for any $n \geqslant 1$. A ring which is both right and left Hermite is an Hermite ring. Obviously a right Hermite ring is right Bezout. A ring R is said to be regular if for every $a \in R$ there exists $x \in R$ such that $axa = a$. It is easy to see that a regular ring is Bezout [4]. A row $(a_1, \dots, a_n)$ over a ring R is called right unimodular, if $a_1R + \dots + a_nR = R$. If $(a_1, \dots, a_n)$ is a right unimodular n-row over a ring R, then we say that $(a_1, \dots, a_n)$ if reducible if there exists an $(n-1)$-row $(b_1, \dots, b_{n-1})$ such that the $(n-1)$-row $(a_1 + a_nb_1, \dots, a_{n-1} + a_nb_{n-1})$ is a right unimodular $(n-1)$-row. A ring R is said to have stable rank $n \geqslant 1$, if n is the least positive integer such that every right unimodular $(n+1)$-row is reducible. This number is denoted by $bsr(R)$. A ring R is directly finite if $xy = 1$ implies $yx = 1$ for all $x, y \in R$.

We denote by R_n the ring of all $n \times n$ matrices over R, and by $GL_n(R)$ its group of unities. We write $GE_n(R)$ for the subgroup of $GL_n(R)$ generated by elementary matrices. The Jacobson radical of a ring R will be denoted by $J(R)$. Denote by $U(R)$ the group of unities of R.

2. Diagonalization of matrices over ring with finite stable rank.

Proposition 1. *Let R be a right Bezout ring with finite stable rank $bsr(R)$. Then any right unimodular row of length m over R, where $m \geqslant bsr(R)+1$, can be completed to an invertible matrix in $GE_m(R)$.*

Proof. If $a_1R + \dots + a_{n+1}R = R$, then there exists an m-row $(c_1, \dots, c_m)$ with

$$(a_1 + a_{m+1}c_1)R + \dots + (a_m + a_{m+1}c_m)R = R.$$

There exist $u_1, \dots, u_m \in R$ such that

$$(a_1 + a_{m+1}c_1)u_1 + \dots + (a_m + a_{m+1}c_m)u_m = 1.$$

Set

$$P_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \dots & c_m & 1 \end{pmatrix} \in GE_{m+1}(R),$$

$$P_2 = \begin{pmatrix} 1 & 0 & \dots & 0 & u_1(1-a_{m+1}) \\ 0 & 1 & \dots & 0 & u_2(1-a_{m+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & u_m(1-a_{m+1}) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in GE_{m+1}(R).$$

We see that for a row $(a_1, \dots, a_{m+1})P_1P_2$ there exists a matrix $P_3 \in GE_{m+1}(R)$ such that $(a_1, \dots, a_{m+1})P_1P_2P_3 = (1, 0, \dots, 0)$. Thus we obtain a matrix $P \in GE_{m+1}(R)$ such that $(a_1, \dots, a_{m+1})P = (1, 0, \dots, 0)$. Then $(a_1, \dots, a_{m+1})$ is the first row of the matrix P^{-1}. For any right unimodular row of length $> m+1$ the result follows by induction.

Proposition 2. *Let R be a right Bezout ring with finite stable rank $bsr(R)$, then R is an m-right Hermite ring, for any $m \geqslant bsr(R)+1$.*

Proof. Since R is a right Bezout ring, then for any $a_1, \dots, a_m \in R$ there exists $d \in R$ such that $a_1R + \dots + a_mR = dR$. Say $a_1u_1 + \dots + a_mu_m = d$, $a_1 = db_1$, $\dots$, $a_m = db_m$. From these relations we get $d(b_1u_1 + \dots + b_mu_m - 1) = 0$ so that $b_1R + \dots + b_mR + cR = R$ for some $c \in R$ such that $dc = 0$. Since $m \geqslant bsr(R)+1$, we have $(b_1 + cx_1)R + \dots + (b_m + cx_m)R = R$, where $x_1, \dots, x_n \in R$. By Proposition 1, we can find an invertible matrix $P \in GE_m(R)$ of the form

$$P = \begin{pmatrix} b_1 + cx_1, & \dots, & b_m + cx_m \\ & * & \end{pmatrix}.$$

Clearly $(a_1, \dots, a_m)P^{-1} = (d, 0, \dots, 0)$, some R is m-right Hermite.

Now we are ready to prove a result which characterizes the regular rings which have finite stable rank.

Theorem 1. *Let R be a regular ring with finite stable rank $bsr(R)$. Then for every $k \times m$ ($m \times k$) matrices A over R, where $m \geqslant bsr(R)+2$, there exist invertible matrices $P \in GE_k(R)$ ($P \in GE_m(R)$), $Q \in GE_m(R)$ ($Q \in GE_k(R)$) such that PAQ is a diagonal matrix.*

Proof. In order to prove that A admits diagonal reduction, we proceed by induction on k. If $k = 1$, the result follows by Proposition 2. If $k > 1$ it follows similarly as the proof of Theorem 9 [2].

Thus we provide an answer to Henriksen's question [1], whether a regular ring can be an elementary divisor ring without being unit regular.

Theorem 2. *Let R be a directly finite ring. If every $n \times n$ matrix over R is equivalent to a diagonal matrix, then R_n is a directly finite ring.*

Proof. Let $A, B \in R_n$ and $AB = E$, the identity n-matrix. If

$$PAQ = \begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \varepsilon_n \end{pmatrix} = \varepsilon,$$

where $P, Q \in GL_n(R)$, then $PAQQ^{-1}BP^{-1} = \varepsilon Q^{-1}BP^{-1} = E$. Since R is directly finite, we see that $\Phi = Q^{-1}BP^{-1}$ is a diagonal matrix. Since R is directly finite, we obtain $\Phi\varepsilon = \varepsilon\Phi = E$ and $\varepsilon \in GL_n(R)$. Thus $A = P^{-1}\varepsilon Q^{-1} \in GL_n(R)$ and $BA = E$ and hence R_n is directly finite.

Theorem 3. *Let R be a directly finite regular ring with finite stable rank $bsr(R)$. Then R_m is directly finite for every $m \geqslant bsr(R)+2$.*

This theorem follows from Theorem 1 and Theorem 2.

Theorem 2.5 in [3] provides a large class of regular rings over which all square matrices are diagonalizable, these rings are separative regular rings. Then we have

Theorem 4. *Let R be directly finite separative regular ring. Then R_n is directly finite for all n.*

Levy in [5] proved that all square matrices over serial rings are diagonalizable. Then we have

Theorem 5. *Let R be a directly finite serial ring. Then R_n is directly finite for all n.*

We obtain an affirmative answer to a question of Henriksen [6, Question 2].

Theorem 6. *Let R be a right Bezout ring, and $R/J(R)$ is a right Hermite ring. Then R is right Hermite.*

Proof. We show first that any right unimodular row over R can be completed to an invertible matrix. Set $\overline{R} = R/J(R)$. Let $aR + bR = R$, then $\overline{a}\overline{R} + \overline{b}\overline{R} = \overline{R}$. Since $\overline{R}$ is a right Hermite ring, the right unimodular row $(\overline{a}, \overline{b})$ over $\overline{R}$ can be completed to an invertible matrix

$$\overline{A} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{u} & \overline{v} \end{pmatrix}.$$

Thus $\overline{AC} = \overline{CA} = \overline{E}$. Let

$$\overline{C} = \begin{pmatrix} \overline{c} & \overline{x} \\ \overline{d} & \overline{y} \end{pmatrix}.$$

Then $ac + bd = 1 + j_1$, $ax + by = j_2$, $uc + vd = j_3$, $ux + vy = 1 + j_4$, for any $j_1, j_2, j_3, j_4 \in J(R)$. Set

$$A = \begin{pmatrix} a & b \\ u & v \end{pmatrix}, \quad C = \begin{pmatrix} c & x \\ d & y \end{pmatrix},$$

then

$$AC = \begin{pmatrix} 1 + j_1 & j_2 \\ j_3 & 1 + j_4 \end{pmatrix} = J.$$

Since $1 + j_1 \in U(R)$, then $J \in GL_2(R)$ and $A \in GL_2(R)$.

Now we prove that R is right Hermite ring. Suppose that we are given $a, b \in R$, then $aR + bR = dR$, say $a = da_0$, $b = db_0$, $d = au + bv$. From these relations we get $d(a_0u + b_0v - 1) = 0$, so $a_0R + b_0R + c_0R = R$ for some $c_0 \in R$ such that $dc_0 = 0$. Since $\overline{R}$ is a right Hermite ring, then $bsr(\overline{R}) \leqslant 2$ [2, Proposition 8]. Since for the ring R the following assertion hold: $u \in U(R)$ if and only if $u + J(R) \in U(\overline{R})$, then $bsr(R) \leqslant 2$. Thus $(a_0 + c_0x)R + (b_0 + c_0y)R = R$, where $x, y \in R$. By the above argument, we can find an invertible matrix of the form

$$P = \begin{pmatrix} a_0 + c_0x & b_0 + c_0y \\ * & * \end{pmatrix}.$$

Clearly $(a, b)P^{-1} = (d, 0)$, so R is right Hermite.

Theorem 7. *A commutative Bezout ring is an elementary divisor ring if and only if $R/J(R)$ is an elementary divisor ring.*

Proof. Obviously, every homomorphic image of an elementary divisor ring is an elementary divisor ring, so we have only to prove the sufficiency. Let $R/J(R)$ be an elementary divisor ring, then by Theorem 6, R is Hermite. By [6, Theorem 3] R is an elementary divisor ring.

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ДІАГОНАЛІЗАЦІЯ МАТРИЦЬ НАД КІЛЬЦЯМИ СКІНЧЕННОГО СТАБІЛЬНОГО РАНГУ

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Побудовано теорію діагоналізації матриць над кільцями скінченного стабільного рангу. Доведено таке: якщо R – регулярне кільце, то довільні $m \times k$ і $k \times m$ матриці над R, де $m \geqslant ст.р.(R) + 2$, володіють діагональною редукцією. Якщо R прямо скінченне регулярне кільце, то кільце матриць R_n є прямо скінченне для довільного $n \geqslant ст.р.(R) + 2$. Показано таке: якщо R праве кільце Безу таке, що $R/J(R)$ є правим кільцем Ерміта, тоді R праве кільце Ерміта. Одержали, що комутативне кільце Безу є кільцем елементарних дільників тоді і тільки тоді, коли $R/J(R)$ кільце елементарних дільників.

Ключові слова: стабільний ранг, кільце Безу, елементарна редукція, кільце Ерміта.

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ASYMPTOTIC CATEGORY AND SPACES OF PROBABILITY MEASURES

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An example is provided of a proper metric space whose space of probability measures is not an absolute extensor for the asymptotic category in the sense of Dranishnikov.
Key words: asymptotic category, probability measures.

The functor of probability measures P in the asymptotic topology was first considered by A. Dranishnikov [D]. It is remarked in [D] that the space $P(X)$ is an absolute extensor for the class of asymptotically Lipschitz maps defined on the proper metric spaces of finite asymptotic dimension and, more generally, of slow dimension growth. Here a metric space X is said to be of *slow dimension growth* if $\lim_{L\to\infty} m(L)/L = 0$; by $m(L)$ the minimal multiplicity of a uniformly bounded cover of X with the Lebesgue number $> L$ is denoted. The problem whether the space $P(X)$ is an absolute extensor for the category of all proper metric spaces and asymptotically Lipschitz maps was formulated in [D] (Problem 12). As remarked in [D], an affirmative solution of this problem would allow to prove a homotopy extension theorem in the asymptotic category in full generality. In this paper we provide a negative solution of this problem (see Section 3).

In Section 4 we consider another problem mentioned in [D], namely that of relationship between the cone (in the sense of Dranishnikov) of a proper metric space X and the join $X * \mathbb{R}_+$. It turns out that these objects are not always isomorphic as objects of the asymptotic category (see the definition below).

This paper was finished when the author visited the University of Florida.

1. Preliminaries A typical metric will be denoted by d. A map $f\colon X \to Y$ between metric spaces is called (λ,ε)-*Lipschitz* for $\lambda > 0$, $\varepsilon \geqslant 0$ if $d(f(x), f(x')) \leqslant \lambda d(x,x') + \varepsilon$ for every $x, x' \in X$. A map is called *asymptotically Lipschitz* if it is (λ,ε)-Lipschitz for some $\lambda, \varepsilon > 0$.

The $(1,0)$-Lipschitz maps are also called *Lipschitz* or *short*. By $\mathrm{Lip}(X)$ we denote the set of all Lipschitz functions on X.

A metric space X is called *proper* if every closed ball in X is compact.

The *asymptotic category* $\mathcal{A}$ is introduced by A. Dranishnikov [D]. The objects of $\mathcal{A}$ are proper metric spaces and the morphisms are proper asymptotically Lipschitz maps.

We also need a notion of asymptotic Lipschitz equivalence, which is a weaker notion than that of isomorphism in $\mathcal{A}$. Two proper metric spaces, X and Y, are *asymptotically Lipschitz equivalent* if there exist proper asymptotically Lipshchitz maps $f: X \to Y$ and $g: Y \to X$ such that the compositions gf and fg are of finite distance (in the sup-metric) from the corresponding identity maps.

A metric space X is said to be an *absolute extensor (AE)* for $\mathcal{A}$ if for every proper asymptotically Lipshitz map $f: A \to X$ defined on a closed subset A of a proper metric space Y there is a proper asymptotically Lipshitz extension $\bar{f}: Y \to X$.

A metric space is called C*-connected*, for $C > 0$ if for every $x, y \in X$ there exists a sequence $x = z_0, z_1, \dots, z_{k-1}, z_k = y$ such that $d(z_{i-1}, z_i) \leqslant C$ for every $i = 1, \dots, k$.

Lemma. *Suppose that $f: X \to Y$ is an asymptotically Lipschitz map of proper metric spaces and X is a C-connected space, for some $C > 0$. Then there exists $C' > 0$ such that the space Y is C'-connected.*

Proof. It is easy to see that Y is C'-connected with $C' = C\lambda + s$.

A metric space X is said to be a *geodesic metric space* if for every two points $x, y \in X$ there is an isometric embedding $j: [0, d(x, y)] \to X$ (the segment $[0, d(x, y)]$ is endowed with the euclidean metric) such that $j(0) = x$ and $j(d(x, y) = y$.

The following proposition is a version of Proposition 1.4 from [D] (see also [R]).

Proposition. *Let $f: X \to Y$ be a map of metric spaces. If X is a geodesic metric space and there exists $C > 0$ such that $d(f(x), f(y)) \leqslant C$ for any $x, y \in X$ with $d(x, y) \leqslant 1$, then f is asymptotically Lipschitz.*

Proof. The proof of Proposition 1.4 from [D] also works in our situation.

1.1. Spaces of probability measures. For a metric space X let $P(X)$ denote the space of probability measures on X with compact supports. We identify the measures with the corresponding functionals on the set $C(X)$ of continuous real-valued functions on X. For $x \in X$ by δ_x we denote the Dirac measure concentrated at X. There are different metrizations of the space of probability measures (see, e. g., [H, S, Z]). Following [H] we endow the space $P(X)$ with the following metric:

$$d(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| : \varphi \in \mathrm{Lip}(X)\}.$$

In general, the metric space $P(X)$ is not locally compact for a proper metric space X. We complete it with respect to the defined metric and preserve the denotation $P(X)$ for the completed space. However, even this complete space is not, in general, proper, as the following example shows. Let $X = \{0\} \cup \mathbb{N}$, with the standard metric. For every $n \in \mathbb{N}$ denote by μ_n the probability measure $(1 - 2^{-n})\delta_0 + 2^{-n}\delta_{2^n}$. For every $m \in \mathbb{N}$ denote by φ_m the function defined by the formula $\varphi_m(x) = \max\{0, x - 2^m\}$. Then

$$d(\mu_n, \delta_0) = \sup\{2^{-n}|\varphi(0) - \varphi(2^n)| \mid \varphi \in \mathrm{Lip}(X)\} = 1.$$

On the other hand, if $m, n \in \mathbb{N}$, $m < n$, then

$$\begin{aligned}(\mu_m, \mu_n) \geqslant& |(2^{-m} - 2^{-n})\varphi_m(0) + 2^{-m}\varphi_m(2^m) - 2^{-n}\varphi_m(2^n)| \\ =& 2^{-n}(2^n = 2^m) \geqslant 1/2,\end{aligned}$$

and therefore the set $\{\mu_n \mid n \in \mathbb{N}\}$ is a $1/2$-discrete infinite subset of the 1-ball in $P(X)$ centered at δ_0.

This example also demonstrates that the spaces $P_n(X)$ of probability measures with supports of cardinality $\leqslant n$ are not objects of the category $\mathcal{A}$.

Note that this lack of local compactness for the spaces of probability measures causes some difficulties in defining the notion of convexity in the asymptotic category.

Suppose $\mu \in P(\mathbb{R}^n)$, $\mu = \sum_{i=1}^k \alpha_i \delta_{x_i}$. Put $b(\mu) = \sum_{i=1}^k \alpha_i x_i$.

1.1. Lemma. *Let $f: X \to Y$ be a short map. Then the map $P(f)$ defined by the formula $Pf(\sum_{i=1}^k \alpha_i x_i) = \sum_{i=1}^k \alpha_i f(x_i)$ is a short map from the set of probability measures with finite supports on X to $P(Y)$.*

Proof. Obvious.

The lemma allows us to extend the map $P(f)$ to a short map of $P(X)$ into $P(Y)$. We preserve the notation $P(f)$ for this extended map.

1.2. Lemma. *The map b is a short map from the set of all probability measures on $\mathbb{R}^n$ with finite supports into $\mathbb{R}^n$.*

Proof. Suppose $\mu = \sum_{i=1}^k \alpha_i \delta_{x_i}, \nu = \sum_{j=1}^l \beta_j \delta_{y_j} \in P(\mathbb{R}^n)$ and $b(\mu) \neq b(\nu)$. Denote by $p: \mathbb{R}^n \to \mathbb{R}$ the orthogonal projection onto the direction of the vector $b(\mu) - b(\nu)$. Then

$$\begin{aligned} \|b(\mu) - b(\nu)\| =& |p(b(\mu)) - p(b(\nu))| = \sum_{i=1}^k \alpha_i p(x_i) - \sum_{j=1}^l \beta_j p(y_j) \\ =& |\mu(p) - \nu(p)| \leqslant d(\mu, \nu), \end{aligned}$$

because $p \in \mathrm{Lip}(X)$.

Lemma 1.2 allows us to extend the map b to a short map from $P(\mathbb{R}^n)$ to $\mathbb{R}^n$. This extended map will be also denoted by b. The map b is called the *barycenter map*.

3. Space of probability measures which is not an absolute extensor. For every n, the euclidean space $\mathbb{R}^n$ is naturally identified with the subspace $\{(x_i) \mid x_i = 0 \text{ for all } j > n\}$ of the space ℓ^2 of square-summable sequences.

We endow the subspace $X = \bigcup_{n \in \mathbb{N}} \{n\} \times \mathbb{R}^n \subset \mathbb{R} \times \ell^2$ with the metric

$$d(n, (x_i)), (m, (y_i)) = (|m - n|^2 + \|(x_i) - (y_i)\|^2)^{1/2}.$$

Obviously, X is a proper metric space. For every n we denote by $p_n: X \to \mathbb{R}^n$ a map defined by the formula $p_n(m, (x_i)) = (x_1, \dots, x_n)$. Clearly, p_n is a short map.

We are going to show that the space $P(X)$ is not an absolute extensor in the category $\mathcal{A}$.

It is shown in [L] (see Theorem 1.5 therein) that for any $n \geqslant 2$ there exists a metric space extension X_n of the euclidean space $\mathbb{R}^n$ such that there is no (λ, ε)-Lipschitz retraction from X_n onto $\mathbb{R}^n$ with $\lambda < n^{1/4}$. For the sake of completeness we provide the details of the construction. Following [L], for every natural k and natural $n \geqslant 2$ we define graphs $G_{n,k}$ as follows: the set of vertices $V(G_{n,k})$ is the union of $I(G_{n,k})$ and $T(G_{n,k})$, where

$$\begin{aligned} I(G_{n,k}) =& \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| = k \text{ for all } i\}, \\ T(G_{n,k}) =& \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| = 2k \text{ for all } i\}; \end{aligned}$$

the set of edges $E(G_{n,k})$ is defined by the condition: $\{x, y\} \in E(G_{n,k})$ if and only if $x, y \in V(G_{n,k})$ and either $\|x - y\| = 2k$ or $y = 2x$ (we suppose that the spaces $\mathbb{R}^n$ are endowed with the Euclidean metric).

The set $V(G_{n,k})$ is equipped with the metric $d = d_{n,k}$,

$$d(x,y) = \inf\{\sum_{i=1}^{l} \|x_{i-1}, x_i\| : (x = x_0, x_1, \ldots, x_l = y) \text{ is a path in } \dot{G}_{n,k}\}.$$

Denote by Y the metric space defined as follows. Put

$$Y = \left(X \sqcup \left(\bigcup_{k=1}^{\infty}\bigcup_{n=2}^{\infty} V(G_{n,k})\right)\right)/\sim,$$

where the equivalence relation $\sim$ is defined by identification of every $x \in T(G_{n,k})$ with $x \in \mathbb{R}^n$. The metric on Y is the maximal metric that agrees with the initial metric on X and the metric $d_{n,k}$ on every $V(G_{n,k})$. It easily follows from the construction that Y is a proper metric space, i.e. an object of the category $\mathcal{A}$.

Let $f: Y \to P(X)$ be the map that sends $x \in X$ to $\delta_x \in P(X)$. The map f is an isometric embedding and we are going to show that there is no asymptotically Lipschitz extension of f onto Y. Assume the contrary and let $\bar{f}: Y \to P(X)$ be such an extension. Then there are $\lambda > 0$ and $\varepsilon > 0$ such that

$$d(\bar{f}(x), \bar{f}(x')) \leqslant \lambda d(x,x') + \varepsilon$$

for all $x, x' \in Y$.

Let $n > \lambda^4$. Since the maps $P(p_n)$ and $b: P(\mathbb{R}^n) \to \mathbb{R}^n$ are short, we conclude that the map $b \circ P(p_n) \circ \bar{f}|X_n$ is a (λ, ε)-Lipschitz retraction from X_n onto $\mathbb{R}^n$, which gives us a contradiction.

4. Cone and join. The cone construction is of importance in the asymptotic topology as it allows to apply asymptotic methods for investigation of topological and metric properties of spaces. The open cone construction of compact metric spaces is considered in [R].

In the case of noncompact spaces, the following construction of cone is proposed by A. Dranishnikov [D]. Let X be a proper metric space with base point x_0.

Denote by CX the quotient space of the subspace

$$\{(x,t) \in X \times \mathbb{R} : |t| \leqslant d(x,x_0)\} \subset X \times \mathbb{R}$$

with respect to the following equivalence relation $\sim$: $(x_1,t_1) \sim (x_2,t_2)$ if and only if either $(x_1,t_1) = (x_2,t_2)$ or $t_1 = -d(x_1,x_0) = -d(x_2,x_0) = t_2$. Denote by $[x,t] \in CX$ the equivalence class of X that contains (x,t). We endow CX with the quotient metric ϱ,

$$\begin{aligned}\varrho([x_1,t_1],[x_2,t_2]) = \inf\{\sum_{i=0}^{k} d((y_{2i}, s_{2i}), (y_{2i+1}, s_{2i+1})) :\\ (x_1,t_1) = (y_0,s_0),\ (x_2,t_2) = (y_{2k+1}, s_{2k+1})\\ \text{and } (y_{2k-1}, s_{2k-1}) \sim (y_{2k}, s_{2k}),\ i = 1,\ldots,k\}.\end{aligned}$$

The obtained metric space (CX, ϱ) is called the *cone* of X.

Given two proper metric spaces X, Y with base points x_0, y_0 respectively we define their wedge $X \vee Y$ as the quotient space $(X \sqcup Y)/\{x_0, y_0\}$ endowed with the maximal metric that makes the natural embeddings $X \to X \vee Y$, $Y \to X \vee Y$ to be isometric embeddings. The subspace

$$X * Y = \{t\delta_x + (1-t)\delta_y : d(x, x_0) = d(y, y_0),\ t \in [0,1]\}$$

of $P(X \vee Y)$ is called the *join* of X and Y.

Proposition. *The space $X * \mathbb{R}_+$, up to asymptotically Lipschitz equivalence, does not depend on the choice of base point.*

Proof. Let $x_1, x_2 \in X$ be base point. Hereafter, $X \vee_i \mathbb{R}_+$ and $X *_i \mathbb{R}_+$ denote respectively the wedge and the join with respect to the base point x_i, $i = 1, 2$. Denote by φ_i the distance function to the point x_i, $i = 1, 2$. Define a map $f : X *_1 \mathbb{R}_+ \to X *_2 \mathbb{R}_+$ by the formula

$$f(t\delta_x + (1-t)\delta_{\varphi_1(x)} = t\delta_x + (1-t)\delta_{\varphi_2(x)}.$$

Obviously, f is a bijective map and it is sufficient to show that the maps f and f^{-1} are asymptotically Lipschitz. Because of similarity, we prove this only for f.

Suppose that $x, y \in X$, $t\delta_x + (1-t)\delta_{\varphi_1(x)}$, $s\delta_y + (1-s)\delta_{\varphi_1(y)} \in X *_1 \mathbb{R}_+$ and

$$d(t\delta_x + (1-t)\delta_{\varphi_1(x)},\ s\delta_y + (1-s)\delta_{\varphi_1(y)}) = K.$$

There exists a short function $\alpha_1 : X \vee \mathbb{R}_+ \to \mathbb{R}$ such that

$$|t\alpha_1(x) + (1-t)\alpha_1(\varphi_1(x)) - s\alpha_1(y) - (1-s)\alpha_1(\varphi_1(y))| = K.$$

Define a function $\alpha_2 : X \vee \mathbb{R}_+ \to \mathbb{R}$ by the conditions $\alpha_2|X = \alpha_1$ and $\alpha_2(r) = \alpha_1(r) - \alpha_1(x_1) + \alpha_1(x_2)$ for $r \in \mathbb{R}_+$ (we identify X and $\mathbb{R}_+$ with the subspaces of $X \vee_i \mathbb{R}_+$ along the natural embeddings). Then α_2 is a short function and we obtain

$$\begin{aligned}
&d(f(t\delta_x + (1-t)\delta_{\varphi_1(x)}), f(s\delta_y + (1-s)\delta_{\varphi_1(y)}))\\
&\leqslant |t\alpha_2(x) + (1-t)\alpha_2(\varphi_2(x)) - s\alpha_2(y) - (1-s)\alpha_2(\varphi_2(y))|\\
&\leqslant |t\alpha_1(x) + (1-t)\alpha_1(\varphi_1(x)) - s\alpha_1(y) - (1-s)\alpha_1(\varphi_1(y))|\\
&+|s-t||\alpha_1(x_2) - \alpha_1(x_1)| + (1-t)|\alpha_1(\varphi_2(x)) - \alpha_1(\varphi_1(x))|\\
&+(1-s)|\alpha_1(\varphi_2(y)) - \alpha_1(\varphi_1(y))|\\
&\leqslant K + 4d(x_2, x_1).
\end{aligned}$$

This means that f is $(1, 4d(x_2, x_1))$-Lipschitz.

A. Dranishnikov asked in [D] whether the spaces CX and $X * \mathbb{R}_+$ are asymptotically Lipschitz equivalent for every proper metric space X. The following example demonstrates that this is not the case.

Example. Let $X = \omega$ (the set of all finite ordinals) with base point 0. We endow ω with a metric d defined as follows: $d(i, j) = \max\{i, j\}$ whenever $i \neq j$. Then $CX = \{(i, t) \in \omega \times \mathbb{R} \mid |t| \leqslant i\}$. Let $A_i = \{i\} \times [-i, i]$, because the equivalence relation $\sim$ in this case is trivial. Note that $d(A_i, C\omega \setminus A_i) \geqslant i$.

We are going to show that there is no proper asymptotically Lipschitz map from $\omega * \mathbb{R}_+$ to $C\omega$. Note first that the space $\omega * \mathbb{R}_+$ is obviously 1-connected. Suppose that there exists a proper (λ, ε)-Lipschitz map $f : \omega * \mathbb{R}_+ \to C\omega$, where $\lambda, \varepsilon > 0$. Then the image $f(\omega * \mathbb{R}_+)$ is a $\lambda + \varepsilon + 1$-connected set. Suppose that $f(\omega * \mathbb{R}_+) \cap A_i \neq \emptyset$

for some $i \in \omega$, then $f(\omega * \mathbb{R}_+) \cap A_j = \emptyset$ for all $j > \max\{\lambda + s, i\}$. Therefore, the set $f(\omega * \mathbb{R}_+)$ is compact, which contradicts to the properness of f.

Nevertheless, in some cases the cone CX and the space $X * \mathbb{R}_+$ are isomorphic as objects of the category $\mathcal{A}$. The following proposition is a counterpart of Lemma 2.4 from [D].

Proposition. *The space* $\mathbb{R}^n * \mathbb{R}_+$ *is isomorphic to* $\mathbb{R}^{n+1}_+ = \{(x_1, \dots, x_{n+1} \mid x_{n+1} \geqslant 0\}$.

Proof. Let $i: \mathbb{R}^n \vee \mathbb{R}_+ \to \mathbb{R}^{n+1}_+$ be the map acting by the formula: $i(x) = (x, 0)$, $i(t) = (0, t)$, where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. Denote by $f: \mathbb{R}^n * \mathbb{R}_+ \to \mathbb{R}^{n+1}_+$ the restriction of the composition

$$P(\mathbb{R}^n \vee \mathbb{R}_+) \xrightarrow{P(i)} P(\mathbb{R}^{n+1}_+) \xrightarrow{b} \mathbb{R}^{n+1}_+$$

onto the subspace $\mathbb{R}^n * \mathbb{R}_+$. Obviously, F is a short bijective map and therefore it is sufficient to prove that the map $g = f^{-1}$ is asymptotically Lipschitz. The explicit formula for g is

$$g(x,t) = \frac{\|x\|}{\|x\| + t}\delta_{\bar{x}} + \frac{t}{\|x\| + t}\delta_{\|x\|+t},$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, $\bar{x} = \frac{x}{\|x\|}(\|x\| + t)$; if $\|x\| = 0$, then $g(x,t) = \delta_t$. Note that it is easy to verify that g is a continuous function.

Given $(x,t), (y,s) \in \mathbb{R}^{n+1}_+$ with $\|(x,t) - (y,s)\| \leqslant 1$, we suppose, without loss of generality, that $0 < \|x\| + t \leqslant \|y\| + s$. Let $h: \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}_+$ be the homothety map centered at 0 and with coefficient $(\|y\| + s)/(\|x\| + t)$. Obviously, $d(h|\mathrm{supp}(g(x,t)), \mathrm{id}) \leqslant 1$ and therefore $d(P_2(h)(g(x,t)), g(x,t)) = d(g(h(x,t)), g(x,t)) \leqslant 1$.

Put $(x_1, t_1) = h(x,t)$; we have $\|x_1\| + t_1 = \|y\| + s$.

We are going to estimate the distance between $g(x_1, t_1)$ and $g(y, s)$. Note that $\|(x_1, t_1) - (y, s)\| \leqslant 2$. Without loss of generality we may assume that $\|x_1\| \leqslant \|y\|$. Denote by (y_1, t_1) the unique point that satisfies the conditions: $\|y_1\| = \|x_1\|$, $s_1 = t_1$, y and y_1 are collinear. Note that $|\ \|y\| - \|y_1\|\ | \leqslant |\ \|y\| - \|x_1\|\ | \leqslant 2$ and therefore $\|(y_1, s_1) - (x_1, t_1))\| \leqslant 2$. Then

$$d(g(y_1, s_1), g(x_1, t_1)) \leqslant \left| \frac{\|y_1\|}{\|y_1\| + s_1} \|x_1 - y_1\| \right| \leqslant 2.$$

Let us estimate $d(g(y_1, s_1), g(y, s))$. Put

$$C = \sup\{d(g(x,t), g(y,s)) \mid \|(x,t) - (y,s)\| \leqslant 2, \|(y,s)\| \leqslant 1.$$

Since g is continuous, we see that $C < \infty$. Then

$$\begin{aligned}
&d(g(y_1, s_1), g(y, s)) \\
&\quad = d\left(\frac{s_1}{\|y_1\| + s_1}\delta_{y_1} + \frac{\|y_1\|}{\|y_1\| + s_1}\delta_{\|y_1\|+s_1}, \frac{s}{\|y\| + s}\delta_y + \frac{\|y\|}{\|y\| + s}\delta_{\|y\|+s} \right) \\
&\quad = \sup\left\{ \left| \frac{s_1\alpha(y_1) - s\alpha(y)}{\|y\| + s} \right| \mid \alpha \in \mathrm{Lip}(\mathbb{R}^n \vee \mathbb{R}_+),\ \alpha(\|y\| + s) = 0 \right\} \\
&\quad \leqslant \left| \frac{2(\|y_1\| + s}{\|y\| + s} \right| \leqslant 3 + C.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}d(g(x,t),(y,s)) \leqslant & d(g(x,t),g(x_1,t_1)) + d(g(x_1,t_1),g(y_1,s_1) + d(g(y_1,s_1),g(y,s)) \\ \leqslant & 1+2+3+C = 6+C.\end{aligned}$$

Since $\mathbb{R}^{n+1}_+$ is a geodesic metric space, it follows from Proposition 1.4 from [D] that the map g is asymptotically Lipschitz.

5. Remarks. The example from Section 4 demonstrates that the notion of a cone needs a slight modification in order to be more closely related to the "join with $\mathbb{R}_+$" construction. Namely, given a metric space (X,d), define its *modified cone* $\tilde{C}X$ as follows. As a set, $\tilde{C}X$ coincides with CX. The metric on $\tilde{C}X$ is the maximal metric $\varrho' \leqslant \varrho$ satisfying the following condition:

$$\varrho'([x_1,-d(x_1,x_0)],[x_2,-d(x_2,x_0)]) = |d(x_1,x_0) - d(x_2,x_0)|$$

for all $x_1, x_2 \in X$.

In a forthcoming publication we are going to consider some relations between $\tilde{C}X$ and CX.

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АСИМПТОТИЧНА КАТЕГОРІЯ І ПРОСТОРИ ЙМОВІРНІСНИХ МІР

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Наведено приклад власного метричного простору, простір ймовірнісних мір якого наділений метрикою Канторовича, не є абсолютним екстензором для асимптотичної категорії в сенсі Дранішникова.

Ключові слова: асимптотична категорія, ймовірнісні міри.

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ГРУПИ, БАГАТІ $\mathfrak{X}$-ПІДГРУПАМИ

[1]Орест **АРТЕМОВИЧ**, [2]Леонід **КУРДАЧЕНКО**

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Присвячується пам'яті видатного алгебриста
Д. І. Зайцева, якому в 2002 р. виповнилося б 60 років

Наведено огляд результатів про групи, багаті нормальними (відповідно субнормальними, нільпотентними) підгрупами та близькі до них групи, отримані за останні десятиріччя.

Ключові слова: мінімальне не $\mathfrak{X}$-група, умова мінімальності для $\mathfrak{X}$-підгруп, умова максимальності для $\mathfrak{X}$-підгруп.

0. Нехай ν - деяка властивість, яку можуть мати підгрупи. Ця властивість може бути внутрішньою (наприклад, $\nu=$ бути нормальною (norm), субнормальною (sn), майже нормальною (an), переставною (perm) підгрупою або підгрупою, що має доповнення (comp) і т. і.), і зовнішньою, тобто у цьому випадку ν означає бути підгрупою, що належить до деякого класу груп $\mathfrak{X}$ (скорочено бути $\mathfrak{X}$-групою). Найважливішими з цих класів є клас $\mathfrak{A}$ всіх абельових груп ($\nu = \mathrm{ab}$), всіх груп, які мають скінченний комутант ($\nu = \mathrm{BFC}$), всіх FC-груп ($\nu = \mathrm{FC}$), $\mathfrak{N}_k$ всіх нільпотентних груп ступеня k ($\nu = \mathrm{nil}(k)$), $\mathfrak{N}$ всіх нільпотентних груп ($\nu = \mathrm{nil}$), всіх гіперцентральних груп ($\nu = \mathrm{hyp}$), $\mathfrak{S}_d$ всіх розв'язних груп класу d ($\nu = \mathrm{sol(d)}$), $\mathfrak{S}$ всіх розв'язних груп ($\nu = \mathrm{sol}$), $\mathfrak{F}$ всіх скінченних груп ($\nu = \mathrm{fin}$), всіх скінченно породжених груп ($\nu = \mathrm{fg}$). Якщо G – група, то через $\mathbb{L}_{\text{non-}\nu}(G)$ (відповідно $\mathbb{L}_\nu(G)$) позначимо систему всіх тих підгруп із G, які не мають властивості ν (відповідно мають властивість ν). Одна з перших задач теорії груп, яка зберігає своє значення і до цього часу, полягає у вивченні впливу на будову групи систем $\mathbb{L}_\nu(G)$ та $\mathbb{L}_{\text{non-}\nu}(G)$ для найважливіших природних впастивостей ν. Першим кроком у цьому напрямі стала класична стаття Р. Дедекінда [25], в якій вивчено скінченні групи, всі підгрупи яких нормальні, тобто групи, в яких система $\mathbb{L}_{\text{norm}}(G)$, збігається з системою усіх підгруп або, що рівносильно, система $\mathbb{L}_{\text{non-norm}}(G)$ порожня. Далі були праці Г. Міллера та Х. Морено [56], в якій вивчали скінченні групи, всі власні підгрупи яких абельові, тобто система $\mathbb{L}_{\text{non-ab}}(G)$ складається з усіх власних підгруп або, що рівносильно, $\mathbb{L}_{\text{non-ab}}(G) = \{G\}$. Важливою була праця О. Ю. Шмідта [139], в якій вивчали скінченні групи, всі власні підгрупи

яких нільпотентні, тобто система $\mathbb{L}_{\text{nil}}(G)$ складається з усіх власних підгруп або, що рівносильно, $\mathbb{L}_{\text{nil}}(G) = \{G\}$. Після цих праць почали вивчати групи, скінченні і нескінченні, всі власні підгрупи яких мають деяку важливу властивість ν. Важливу роль для розвитку теорії нескінченних груп відіграла проблема Шмідта про нескінченні групи, всі власні підгрупи яких скінченні (див., наприклад, [134] та [100]). Він почав вивчення скінченних груп, у яких система підгруп $\mathbb{L}_{\text{non-}\nu}(G)$ є "досить малою" (або "досить великою") у деякому сенсі. У праці [140] він отримав опис скінченних груп, у яких всі підгрупи з множини $\mathbb{L}_{\text{non-norm}}(G)$ спряжені, а в [141] описав скінченні групи, в яких множина $\mathbb{L}_{\text{non-norm}}(G)$ складається з двох класів спряжених підгруп. Що означає "бути досить малим" для нескінченних груп? Отримуємо досить великий простір для різних підходів. Один з таких підходів започаткував С. М. Черніков, який запропонував розглядати групи, в яких система $\mathbb{L}_{\text{non-}\nu}(G)$ задовольняє деяку умову скінченності; зокрема такі класичні умови скінченності, як умови мінімальності та максимальності. У [131] розглянуто групи, в яких множина $\mathbb{L}_{\text{non-ab}}(G)$ задовольняє умову мінімальності, а у [132] розглянуто групи, у яких множина $\mathbb{L}_{\text{non-norm}}(G)$ складається тільки зі скінченних підгруп. Цей підхід виявився дуже цікавим та результативним. Мета статті – зробити огляд результатів, які одержано у цій галузі за останні десятиріччя. Ми не можемо навести всі результати, тому що їх досить багато. Ми обмежимось важливими властивостями, наприклад, нормальність, субнормальність, майже нормальність, абельовість та їх узагальнення.

Всі терміни, які ми використовували, можна знайти в [69].

1. Групи з "малими" системами ненормальних підгруп

Ми вже зазначали, що скінченні групи, всі підгрупи яких нормальні, описав ще Р. Дедекінд [25]. Його результати поширив пізніше Р. Бер на довільні групи [13]. Такі групи тому й отримали назву дедекіндових. Вони мають досить просту будову: дедекіндова група або абельова або має вигляд $A \times B \times Q$, де A – абельова періодична $2'$-група, B – елементарна абельова 2-група, Q – група кватерніонів. С. М. Черніков [132] розглянув групи, всі нескінченні підгрупи яких нормальні, тобто система $\mathbb{L}_{\text{non-norm}}(G)$ всіх ненормальних підгруп складається тільки зі скінченних підгруп. Таких груп дещо більше.

Нехай G – нескінченна група, всі нескінченні підгрупи якої нормальні. Якщо G неабельова, то вона періодична; якщо G локально скінченна, то вона або дедекіндова, або містить таку нормальну квазіциклічну підгрупу K, що G/K – скінченна дедекіндова група. (С. М. Черніков [132]).

Останній результат отримав С. М. Черніков за умови існування нескінченної абельової підгрупи; пізніше отримав теорему В. П. Шунков [142], яка забезпечила існування такої підгрупи. Умова локальної скінченності не є зайвою, оскільки існують нескінченні періодичні групи, всі власні підгрупи яких скінченні. Приклади таких груп побудував О. Ю. Ольшанський [121, §28]. Групи, в яких система $\mathbb{L}_{\text{non-norm}}(G)$ задовольняє умову мінімальності (групи з умовою Min-(non-norm)), також розглядав С. М. Черніков. З результатів його праці [136] випливають такі твердження.

Нехай G – нескінченна група з умовою Min-(non-norm). Якщо G неперіодична, то вона абельова; якщо ж G локально скінчена, то вона або дедекіндова, або

черніковська.
Зауважимо, що у працях [132, 136] розглядали узагальнення наведених тут випадків. Ми не будемо детально розглядати їх, оскільки вони відображені в інших оглядових статтях (див., наприклад, [133, 134, 100, 128]).

Дуальною до умови мінімальності є умова максимальності. Групи, в яких система $\mathbb{L}_{\text{non-norm}}(G)$ задовольняє умову максимальності (групи з умовою Max-(non-norm)) розглядалися в працях Л. А. Курдаченка, М. Ф. Кузенного, М. М. Семка [109] та Дж. Кутоло [22]. Якщо клас локально ступінчатих груп з умовою Min-(non-norm) є простим об'єднанням класу дедекіндових груп та класу груп із звичайною умовою мінімальності, то для груп з умовою Max-(non-norm) ситуація інша. Основні результати цих праць можна сформулювати у такому вигляді.

Нехай G – локально ступінчата група, яка задовольняє умову Max-(non-norm). Тоді G – група одного з наступних типів:

1) G – майже поліциклічна група;

2) G – дедекіндова група;

3) $\zeta(G)$ має таку квазіциклічну p-підгрупу P, що G/P – скінченно породжена дедекіндова група;

4) $G = H \times L$, де $H \cong \mathbb{Q}_2$, L – скінченна неабельова дедекіндова група.

Якщо в групі всі скінченно породжені підгрупи нормальні, то і всі її підгрупи нормальні. Природно виникає запитання про протилежну ситуацію: що можна сказати про групи, всі нескінченно породжені підгрупи (підгрупи, що не мають скінченої системи породжуючих елементів) яких нормальні, тобто система $\mathbb{L}_{\text{non-norm}}(G)$ складається тільки зі скінченно породжених підгруп. Ці групи розглядали у працях Л. А. Курдаченко та В. В. Пилаєв [111], Дж. Кутоло [22], Дж. Кутоло та Л. А. Курдаченко [23]. У цих працях є такі результати.

Нехай група G має зростаючий ряд підгруп, кожен фактор якого – локально майже розв'язна група. Кожна нескінченно породжена підгрупа групи G тоді і тільки тоді нормальна, коли G – група одного з наступних типів:

1) G дедекіндова;

2) G має таку нормальну квазіциклічну підгрупу K, що G/K – скінченно породжена дедекіндова група;

3) група G задовольняє такі умови:

3а) центр $\zeta(G)$ має квазіциклічну p-підгруппу K, що G/K – мінімаксна абельова група зі скінченною періодичною частиною;

3б) $Sp(G/K) = \{p\}$;

3в) $G/FC(G)$ – група без скруту;

3г) якщо A – абельова підгрупа G, то $A/(A \cap K)$ скінченно породжена;

4) $G = T \times A$, де $A \cong \mathbb{Q}_2$, T – скінченна дедекіндова група;

5) група G задовольняє такі умови:

5а) $G = (A \times T) \rtimes \langle g \rangle$, де $A \cong \mathbb{Q}_p$ для деякого простого числа p, T - скінченна дедекіндова підгрупа;

5б) якщо T неабельова, то $p = 2$;

5в) елемент g індукує на силовській p-підгрупі T_p підгрупи T ступеневий автоморфізм;

5г) існує таке число $r \in \mathbb{N}$, що $a^g = a^c$, де $c = p^r$ або $c = -p^r$ для кожного $a \in AT_{p'}$, $T_{p'}$ – силовська p'-підгрупа T.

У працях [14] та [98] введено дуже цікаві нові умови скінченності – слабкі умови максимальності та мінімальності для різних типів підгруп. Умови та пов'язані з ними результати зацікавили науковців, тому невдовзі було написано багато праць, присвячених цим умовам для різноманітних типів підгруп. Наша мета – розглядати результати цих досліджень, тому що їх згадували в оглядовій статті [102]. Наведемо тільки ті результати, які стосуються тематики нашої праці.

Нехай $\mathfrak{M}$ – деяка система підгруп групи G. Говоритимемо, що $\mathfrak{M}$ задовольняє слабку умову мінімальності (відповідно максимальності) або група G задовольняє слабку умову мінімальності для $\mathfrak{M}$-підгруп чи скорочено Min-∞-$\mathfrak{M}$ (відповідно максимальності для $\mathfrak{M}$-підгруп чи скорочено Max-∞-$\mathfrak{M}$), якщо G не має таких нескінченних збігаючих (відповідно зростаючих) рядів $\{H_n | n \in \mathbb{N}\}$ підгруп із системи $\mathfrak{M}$, що індекси $|H_n : H_{n+1}|$ (відповідно $|H_{n+1} : H_n|$) нескінченні для кожного $n \in \mathbb{N}$. Якщо $\mathfrak{M} = \mathbb{L}_{\text{non-norm}}(G)$, то отримуємо групи зі слабкою умовою мінімальності (відповідно максимальності) для ненормальних підгруп або скорочено з умовою Min-∞-(non-norm) (відповідно Max-∞-(non-norm)). Ці групи вивчали у працях Л. А. Курдаченко та В. Е. Горецький [45], де виявлено, що *локально майже розв'язна група G тоді і тільки тоді задовольняє умову Min-∞-(non-norm) (відповідно Max-∞-(non-norm)), коли вона або дедекіндова, або мінімаксна.* Зазначимо також низку інших результатів про групи з обмеженнями на систему $\mathbb{L}_{\text{non-norm}}(G)$. Якщо в групі всі циклічні підгрупи нормальні, то і всі її підгрупи нормальні. Тому виникає питання про будову груп, всі нециклічні підгрупи яких нормальні, тобто система $\mathbb{L}_{\text{non-norm}}(G)$ складається тільки з циклічних підгруп. Це питання сформулював С. М. Черніков [133]. Розв'язанню його та розгляду деяких його узагальнень присвячено праці Ф. М. Лимана [113-117]. Також у працях [118, 103-108, 119, 122, 123, 125, 126] досліджували метагамільтонові групи – групи, в яких система $\mathbb{L}_{\text{non-norm}}(G)$ складається тільки з абельових підгруп.

2. Групи з "малими" системами підгруп, що не є майже нормальними

Підгрупа H групи G називається майже нормальною в G, коли $\mathbf{cl}_G(H)=\{H^g | g \in G\}$ – клас усіх спряжених з H підгруп – скінченна множина. Якщо підгрупа H нормальна в G, то $\mathbf{cl}_G(H)=\{H\}$, так що майже нормальні підгрупи – це природне узагальнення нормальних підгруп. Підгрупа H тоді і тільки тоді майже нормальна в G, коли її нормалізатор $N_G(H)$ має скінченний індекс в G (звідси і походить назва таких підгруп). Очевидно, перетин двох майже нормальних підгруп та підгрупа, породжена двома майже нормальними підгрупами, будуть майже нормальними підгрупами. Інакше кажучи, множина $\mathbb{L}_{\text{an}}(G)$ всіх майже нормальних підгруп групи буде граткою. На відміну від гратки всіх нормальних підгруп, ця гратка не буде повною. Групи, в яких $\mathbb{L}_{\text{an}}(G)$ буде повною граткою, розглядали Л. А. Курдаченко та С. Рінауро [42]. Результати цієї праці засвідчують, що досить багато таких груп мають центр скінченного індексу, тобто є *скінченними над центром.* Скінченні над центром групи відіграють тут роль, подібну до тієї, яку відіграють дедекіндові групи при вивченні груп з малою множиною $\mathbb{L}_{\text{non-norm}}(G)$. Про це свідчать, зокрема, такі два результати, що вже стали класичними. *Група G, кожна підгрупа якої майже нормальна (тобто $\mathbb{L}_{\text{non-an}}(G) = \emptyset$), скінченна над центром* (Б. Нейман [63]), і *група G, кожна* *абельова підгрупа якої майже нормальна (тобто $\mathbb{L}_{\text{non-an}}(G)$ складається з неабельових підгруп), також скінченна над центром* (І. І. Еремін [94]). І. І. Еремін почав розглядати групи, в яких множина $\mathbb{L}_{\text{non-an}}(G)$ складається зі скінченних підгруп. Він отримав деякі умови, за яких такі групи скінченні над центром [95]. Опис такого роду груп за умови їх локальної майже розв'язності одержали Л. А. Курдаченко, С. С. Левіщенко та М. М. Семко [124].

Нехай G – неперіодична локально майже розв'язна група.

I Якщо G неперіодична, то кожна нескінченна підгрупа G тоді і тільки тоді майже нормальна, коли G – група одного з таких типів:

Iа G має центр скінченного індексу;

Iб $G = A\langle b\rangle$, $|b| = p$ – просте число, $A = C_G(A)$ – вільна абельова підгрупа 0-рангу $p-1$, b індукує на A раціонально незвідний автоморфізм (тобто кожна неодинична $\langle b\rangle$-інваріантна підгрупа A має скінченний індекс);

Iв G має таку скінченну підгрупу F, що G/F – група типу (2).

II Якщо G періодична, то кожна нескінченна підгрупа G тоді і тільки тоді майже нормальна, коли G – група одного з таких типів:

IIа G має центр скінченного індексу;

IIб $G = D\langle g\rangle$, $D = C_G(D)$ – подільна абельова підгрупа спеціального рангу $p-1$, p – просте число, $g^p \in D$, кожна власна $\langle g\rangle$-інваріантна підгрупа D скінченна;

IIв $G = D\langle g\rangle$, $D = C_G(D)$ – подільна абельова p-підгрупа спеціального рангу $\leqslant (q-1)$, де q – найменше просте число з множини $\Pi(\langle g\rangle)$, $|g|$ – p'-число, для кожного елементу $1 \neq y \in \langle g\rangle$ кожна власна $\langle y\rangle$-інваріантна підгрупа D скінченна;

IIг G має таку скінченну підгрупу F, що G/F – група типу (2) або (3).

Ці результати узагальнено у працях С. Франціозі, Ф. де Жіованні та Л. А. Курдаченка [30], де розглядали групи, в яких множина $\mathbb{L}_{\text{non-an}}(G)$ складається зі скінченно породжених підгруп. Виявилось таке: *якщо G – група, яка має зростаючий ряд підгруп, кожен фактор якого – локально нільпотентна або локально скінченна група, і кожна нескінченно породжена підгрупа якої майже нормальна, то або $G/\zeta(G)$ скінченна, або G – майже розв'язна $\mathbb{A}_3$-група.* Майже розв'язні $\mathbb{A}_3$-групи, в яких множина $\mathbb{L}_{\text{non-an}}(G)$ складається зі скінченно породжених підгруп, розпадаються на багато типів, які досить детально вивчено у [30]. Зауважимо, що у цій праці розглядали поставлене М. С. Черніковим [133] запитання про будову груп, в яких множина $\mathbb{L}_{\text{non-an}}(G)$ складається з нециклічних підгруп. Групи, в яких множина $\mathbb{L}_{\text{non-an}}(G)$ задовольняє умову мінімальності – групи з умовою Min-(non-an) – розглядали Л.А.Курдаченко та В.В.Пилаєв [110], які отримали такий результат.

I Неперіодична група G тоді і тільки тоді задовольняє умову Min-(non-an), коли вона має центр скінченного індексу або містить таку скінченну нормальну підгрупу F, що $H = G/F$ – група одного з таких типів.

1 $H = A \rtimes \langle b\rangle$, $|b| = p$ – просте число, $A = C_H(A)$ – вільна абельова підгрупа 0-рангу $p-1$, b індукує на A раціонально незвідний автоморфізм.

2 $H = K \times L$, K – подільна черніковська підгрупа, L – група типу (1).

II Локально скінченна група G тоді і тільки тоді задовольняє умову Min-(non-an), коли вона черніковська або має центр скінченного індексу.

Групи, в яких множина $\mathbf{L}_{\text{non-an}}(G)$ задовольняє умову максимальності – групи з умовою Max-(non-an) – розглядали Л. А. Курдаченко, М. Ф. Кузенний та М. М. Семко [109]. Вони мають більш просту будову, а саме *локально майже розв'язна група G тоді і тільки тоді задовольняє умову Max-(non-an), коли вона майже поліциклічна або має центр скінченного індексу.* Групи, в яких множина $\mathbf{L}_{\text{non-an}}(G)$ задовольняє слабку умову мінімальності (відповідно максимальності) – групи з умовою Min-∞-(non-an) (відповідно Max-∞-(non-an)) – розглядали у працях Дж. Кутоло та Л. А. Курдаченко [24]. *Якщо група G, яка має зростаючий ряд підгруп, кожен фактор якого – локально майже розв'язна група, задовольняє умову Min-∞-(non-an) (відповідно Max-∞-(non-an)), то або $G/\zeta(G)$ скінченна, або G – майже розв'язна $\mathbb{A}_3$-група.*

Зазначимо, що у працях С. Франціозі, Ф. де Жіованні та Л. А. Курдаченко [31] розглядали групи, в яких множина $L_{\text{non-an}}(G)$ складається з субнормальних підгруп.

3. Групи з "малими" системами підгруп, що не є субнормальними

Відомо, що скінченна група, всі підгрупи якої субнормальні (тобто множина $\mathbf{L}_{\text{non-sn}}(G)$ порожня), нільпотентна. Щодо нескінченних груп ситуація зовсім інша. Існують локально нільпотентні розв'язні групи без центру, всі власні підгрупи яких субнормальні. Такі приклади побудували Г. Хайнекен та І. Мохамед [37-39], Б. Хартлі [33], Ф. Менегаццо [55]. Ми не будемо детально розглядати питання про будову груп, всі підгрупи яких субнормальні, оскільки його детально описано в літературі, зокрема у книзі Дж. Леннокса та С. Стоунхевера [53]. Зазначимо лише декілька важливих результатів, одержаних останнім часом.

Нехай G – група, всі підгрупи якої субнормальні. Тоді G розв'язна (В. Мьорес [57]).

Нехай G – група, всі підгрупи якої субнормальні. G буде нільпотентною в кожному з таких випадків: 1) якщо G періодична та гіперцентральна (В. Мьорес [58]); *2) якщо G періодична і резидуально скінченна* (Х. Сміт [75]); *3) якщо G має таку нормальну нільпотентну підгрупу A, що G/A обмежена* (Х. Сміт [76]); *4) якщо G періодична і резидуально нільпотентна* (Х. Сміт [77]); *5) якщо G – група без скруту* (Х. Сміт [78]).

Деякі умови нільпотентності груп, всі підгрупи яких субнормальні, пов'язані з властивостями нормальних замкнень елементів, досліджували Л. А. Курдаченко та Х. Сміт [43]. Групи, в яких множина $\mathbf{L}_{\text{non-sn}}(G)$ задовольняє умову мінімальності – групи з умовою Min-(non-sn) – розглядали С. Франціозі та Ф. де Жіованні [30]. При досить звичайних обмеженнях такі групи вичерпуються черніковськими та групами, всі підгрупи яких субнормальні. Вивчення груп, у яких множина $\mathbf{L}_{\text{non-sn}}(G)$ задовольняє умову максимальності – груп з умовою Max-(non-sn) – виявилось результативнішим. Ці групи розглядали Л. А. Курдаченко та Х. Сміт [43]. Доведено, що *локально нільпотентна група тоді і тільки тоді задовольняє умову Max-(non-sn), коли кожна її підгрупа субнормальна. Локально майже розв'язна група G тоді і тільки тоді задовольняє умову Max-(non-sn), коли вона є групою одного з таких типів:*

1) G – майже поліциклічна група;

2) кожна підгрупа G субнормальна;

3) $G \neq B(G)$, $G/B(G)$ – скінченно породжена, майже абельова та не має скруту, $B(G)$ нільпотентна, для будь-якого елемента $g \notin B(G)$ кожен G-інваріантний абельовий фактор підгрупи $B(G)$ скінченно породжений як $\mathbb{Z}\langle g\rangle$-модуль. Через $B(G)$ позначено радикал Бера групи G, тобто підгрупу, породжену всіма субнормальними циклічними підгрупами з G.

Дуже близькими до груп з умовою Max-(non-sn) виявились групи, в яких множина $\mathbf{L}_{\text{non-sn}}(G)$ складається зі скінченно породжених підгруп. Їх розглядали Г. Хайнекен та Л. А. Курдаченко [36]. Групи, в яких множина $\mathbf{L}_{\text{non-sn}}(G)$ задовольняє слабку умову мінімальності – групи з умовою Min-∞-(non-sn) – розглядали Л. А. Курдаченко та Х. Сміт [44]. Тут ситуація подібна до звичайної умови мінімальності, як показує основний результат цієї роботи.

Нехай група G має зростаючий ряд підгруп, кожен фактор якого - локально нільпотентна або локально скінченна група. Якщо G задовольняє умову Min-∞-(non-sn), то або кожна підгрупа в G субнормальна, або G майже розв'язна і мінімаксна.
Групи, в яких множина $\mathbf{L}_{\text{non-sn}}(G)$ задовольняє слабку умову максимальності – групи з умовою Max-∞-(non-sn) – розглядали Л. А. Курдаченко та Х. Сміт [45]. Ситуація значно складніша. Зокрема, *якщо локально скінченна група G задовольняє умову Max-∞-(non-sn), то або кожна підгрупа з G субнормальна, або G – черніковська група; якщо ж група Бера задовольняє умову Max-∞-(non-sn), то кожна підгрупа з G субнормальна.*

4. Мінімальні не $\mathfrak{X}$-групи та близькі до них групи

До недавнього часу небагато було відомо про нескінченні групи без власної факторизації. Першим нескінченну абельову групу без власної факторизації (тобто квазіциклічну p-групу $\mathbb{C}_{p^\infty}$) побудував Г. Прюфер [67] у своїй дисертації (1921). Перш ніж була побудована перша неабельова нескінченна група без власної факторизації пройшло майже половина століття.

Першими приклад розв'язної групи без власної факторизації побудували Г. Хайнекен та І. Мохамед [37]. Побудована група виявилась ненільпотентною, тоді всі її власні підгрупи нільпотентні та субнормальні. Це спричинило подальше вивчення груп з нільпотентними власними підгрупами (детальніше див. [112, 70, 38, 54, 33, 137, 11, 55, 48-50, 34, 10, 3, 52]) і груп із субнормальними власними підгрупами (див. [15, 71, 20, 72, 73, 21, 57-61]).

Ненільпотентні групи, всі власні підгрупи яких нільпотентні та субнормальні, в честь першовідкривачів прийнято називати *групами типу Хайнекена-Мохамеда.* Одна властивість груп типу Хайнекена-Мохамеда фактично спричинила виділення такого означення. Група G називається *нерозкладною*, якщо будь-які дві її власні підгрупи знову породжують власну підгрупу в G; і називається *розкладною* – в іншому випадку. Групи типу Хайнекена-Мохамеда є прикладами нерозкладних груп. Даючи відповідь на запитання 1 із книги Б. Амберга, С. Франчіозі і Ф. Жіованні [1], один із авторів виявив, що *недосконала група нерозкладна тоді і тільки тоді, коли вона не має власної факторизації.* Нерозкладні розв'язні групи вивчав О. Д. Артемович [83, 84]. Якщо абельова група нерозкладна, то вона є p-групою для деякого простого числа p і ізоморфна або циклічній p-групі $\mathbb{C}_{p^n}$, або квазіциклічній p-групі $\mathbb{C}_{p^\infty}$. Будь-яка група типу Хайнекена-Мохамеда є нерозкладною неабельовою групою.

Саме дослідження в цьому напрямі спонукали до задачі *охарактеризувати мінімальні не $\mathfrak{X}$-групи (якщо вони існують), де $\mathfrak{X}$ – деякий клас груп.* Нагадаємо, що група G, яка не є $\mathfrak{X}$-групою, тоді як всі її власні підгрупи є $\mathfrak{X}$-групами, називається *мінімальною не $\mathfrak{X}$-групою.* Зростаюча зацікавленість дослідників до груп типу Хайнекена-Мохамеда, які є також мінімальними ненільпотентними групами, значною мірою стимулювала вивчення мінімальних не $\mathfrak{X}$-груп. Необхідність у вивченні таких груп природно виникає і при дослідженні різних задач мінімальності. Мінімальні не $\mathfrak{X}$-групи – це групи з "малою" множиною не $\mathfrak{X}$-підгруп. Напевно, нереально повністю охопити всі дослідження в цьому напрямі. Опишемо найцікавіші з них. Історично перший результат належить Г. Міллеру та Х. Морено [56], які, як вже неодноразово зазначалось, досліджували скінченні мінімальні неабельові групи. Пізніше О. Ю. Шмідт [139] і Б. Хупперт [40] ввели в розгляд скінченні мінімальні ненільпотентні і відповідно скінченні мінімальні ненадрозв'язні групи. Л. Редеі [68] отримав опис скінченних мінімальних ненільпотентних груп. Учні Л. О. Шеметкова [138, частина VI] досліджували мінімальні групи, які не належать цій формації. Нескінченні мінімальні ненільпотентні групи почали вивчати Н. Ньюмен і Дж. Вайголд. Сформулюємо деякі результати щодо нескінченних груп.

Нехай $k \in \mathbb{N}$, G – група, в якої $\mathbf{L}_{\text{non-nil(k)}}(G) = \{G\}$. Тоді G може бути породжена щонайбільше $k+1$ елементами.

Нехай $k \in \mathbb{N}$, G – група, в якої $\mathbf{L}_{\text{non-nil(k)}}(G) = \{G\}$. Тоді $G/Fratt(G)$ – неабельова проста група.

Нехай $k \in \mathbb{N}$, G – проста група, в якої $\mathbf{L}_{\text{non-nil(k)}}(G) = \{G\}$ (відповідно $\mathbf{L}_{\text{non-nil)}}(G) = \{G\}$). Тоді (а) кожна пара максимальних підгруп G має одиничний перетин; (б) якщо $1 \neq x \in G$, то знайдеться такий елемент g, що $\langle g^{-1}xg, x\rangle = G$; (в) G не має елементів порядку 2. (М. Ньюмен, Дж. Уайголд [64]).

Як вже згадувалось, приклади таких груп побудував О. Ю. Ольшанський [121, §28]. *Якщо G – група, всі власні підгрупи якох нільпотентні, то або G скінченно породжена, або G локально нільпотентна.* Приклади, побудовані О. Ю. Ольшанським, показують, що опис таких скінченно породжених груп практично неможливий. Щодо нескінченно породжених груп, всі власні підгрупи яких нільпотентні, то, як зазначалось, приклади груп, побудовані Г. Хайнекеном та І. Мохамедом, мають цю властивість. Опис таких груп також ще не отримано. Такі результати довів Х. Сміт [74].

Нехай G – розв'язна ненільпотентна група, всі власні підгрупи якої нільпотентні. Якщо G не має максимальних підгруп, то виконуються такі умови: а) G – зчисленна p-група для деякого простого числа p; б) $G/[G,G]$ – квазіциклічна p-група; в) кожна підгрупа групи G субнормальна; г) $[G,G]^p \neq [G,G]$ і кожна гіперцентральна фактор-група групи G є абельовою (зокрема, $[G,G] = \gamma_n(G)$ для всіх $n \geqslant 2$); д) $\zeta(G)$ містить у собі кожну подільну підгрупу; е) $C_G([G,G])$ – абельова підгрупа і $[G,G]$ – несуттєва підгрупа (тобто з рівності $H[G,G] = G$ завжди випливає $H = G$) (зокрема, G не має в собі власних підгруп скінченного індексу); є) якщо H – скінченна підгрупа $[G,G]$, то $H^G \neq [G,G]$; ж) гіперцентр групи G збігається з її центром.

Нехай G – нерозв'язна локально нільпотентна група, всі власні підгрупи якої нільпотентні. Тоді виконуються такі умови: а) G – p-група для деякого простого числа p; б) G – група Фіттінга; в) G задовольняє нормалізаторну умову; г) $\zeta(G)$ містить у собі кожну власну подільну підгрупу; д) існує така нільпотентна підгрупа H, що $G = H^G$; е) гіперцентр групи G збігається з її центром.

Р. Брандл, С. Франчіозі та Ф. де Жіованні [9] вивчали групи зі скінченними мінімальними ненільпотентними групами автоморфізмів.

Систематичне вивчення мінімальних не $\mathfrak{X}$-груп почали з праць В. В. Беляєва. Він дослідив мінімальні не FC-групи [93, 90, 91]. З його результатів і праці М. Кузуцуоглу та Р. Філліпса [51], зокрема випливає, що *локально скінченна мінімальна не FC-група є p-групою.* В. В. Беляєв і М. Ф. Сесекін [93, 90] досліджували мінімальні не BFC-групи (або, що еквівалентно, групи типу Міллера-Морено); В. В. Беляєв показав, що *досконала локально скінченна мінімальна не FC-група або проста, або p-група для деякого простого числа p.* М. Кузуцуоглу та Р. Філліпс [51] доповнили цей результат, показавши, що *не існує простих локально скінченних мінімальних не FC-груп.* Як вже зазначалось, А. Азар [3] та Ф. Ляйнен [52] також розглядали мінімальні не FC-групи. Ж. Женг та К. Шум [81, 82] визначили, що *недосконала група, яка не є скінченним розширенням свого центру, але будь-яка власна підгрупа якої скінченна над центром, є мінімальною не FC-групою.* Вивчаючи локально скінченні групи з майже абельовими власними підгрупами, В. В. Беляєв [92] виявив, що *локально скінченна мінімальна не майже абельова група є або групою Чаріна* (див. [127, 92]), *або нерозкладною метабельовою групою.* Незалежно мінімальні не майже абельові групи вивчала Б. Бруно [16-18]. О. Д. Артемович [83] з'ясував, що нерозкладні метабельові групи в деякому розумінні дуже близькі до груп типу Хайнекена-Мохамеда, і *розв'язні нерозкладні групи є p-групами.* Б. Бруно і Р. Філліпс [19, 12] також досліджували недосконалі мінімальні не майже нільпотентні групи. Б. Бруно і Р. Філліпс [12] визначили, що *недосконала мінімальна не майже абельова група (відповідно мінімальна не майже нільпотентна група періодична.* З результатів В. В. Беляєва [92] та Б. Бруно [16-18] випливає, що *розглядувані групи або розкладні (і тоді вони близькі до груп Чаріна), або нерозкладні.* За результатами А. Азара [6] вони недосконалі.

Х. Отал, Х. Пена, Б. Хартлі [65, 35], А. Азар, А. Арікан [8] вивчали мінімальні не CC-групи (про які досі все ще мало відомо). В [65] виявили, що *локально ступінчата мінімальна не CC-група G є F-досконалою зліченною досконалою локально скінченною p-групою.* М. Ху [79] охарактеризував групи, всі власні підгрупи яких є групами Бера, з максимальною підгрупою; показав, що *мінімальні неберові групи не містяться в класі груп типу Хайнекена-Мохамеда.* Пізніше М. Ху [80] і незалежно М. Діксон, М. Еванс, Х. Сміт [28, 29] вивчали групи, всі власні підгрупи яких є нільпотентними розширеннями скінченних груп і відповідно груп скінченного рангу. Близьким до зазначених досліджень є цикл робіт А. Азара з учнями [3, 4, 5, 7]. Х. Отал та Х. Пена [66], Ф. Наполітані та Е. Пегораро [62] шукали мінімальні групи, які не є розширеннями нільпотентних груп за допомогою черніковських груп. О. Д. Артемович [85] охарактеризував мінімальні не майже гіперцентральні групи. Задачі про мінімальні не $\mathfrak{X}$-групи залишаються актуальними і публікації, присвяченні їм, продовжують періодично виходити з друку.

5. Групи з "малими" системами неабельових і ненільпотентних підгруп та деякі близькі до них групи

Як ми вже зазначали, опис скінченних неабельових груп, всі власні підгрупи яких абельові (тобто $\mathbf{L}_{\text{non-sn}}(G) = \{G\}$), зроблено у праці Г. Міллера та Х. Морено [56], є одним з перших важливих результатів абстрактної теорії груп. Щодо нескінченних груп з цією властивістю, то їх існування виявив досить недавно О. Ю. Ольшанський (див. книгу [121, §28]). Водночас результати О. Ю. Ольшанського засвідчують, що говорити про опис цих груп зараз практично неможливо.

Групи, в яких множина $\mathbf{L}_{\text{non-ab}}(G)$ задовольняє умову мінімальності – групи з умовою Min-(non-ab) – почав розглядати М. С. Черніков [131]. З його результатів випливає, що неабельова локально розв'язна група з умовою Min-(non-ab)-черніковська. В. П. Шунков [143] розширив цей результат на локально скінченні групи. Групи, в яких множина $\mathbf{L}_{\text{non-ab}}(G)$ задовольняє умову максимальності – групи з умовою Max-(non-ab) – не вичерпуються абельовими та групами, що задовольняють Max. Простий приклад групи, що є вінцевим добутком групи простого порядку та нескінченої циклічної групи, засвідчує це. Групи з умовою Max-(non-ab) розглянули значно пізніше Д. І. Зайцев та Л. А. Курдаченко [101], з'ясували, що *локально майже розв'язна група G, яка не є майже поліциклічною тоді і тільки тоді G задовольняє умову Max-(non-ab), коли вона містить у собі нормальну абельову підгрупу A з такими властивостями: a) $A = C_G(A)$, b) G/A скінченно породжена, майже абельова і не має скруту, c) $\mathbb{Z}\langle g\rangle$-модуль A скінченно породжений для кожного елемента $g \notin A$.*

Природний наступний етап досліджень – вивчення груп, у яких множина $\mathbf{L}_{\text{non-ab}}(G)$ задовольняє слабку умову мінімальності (груп з умовою Min-∞-(non-ab)). Ці групи розглядав Д. І. Зайцев [99]. Тут ситуація подібна до звичайної умови мінімальності, оскільки *неабельова майже розв'язна група G тоді і тільки тоді задовольняє умову Min-∞-(non-ab), коли вона майже розв'язна і мінімаксна.* Групи, в яких множина $\mathbb{L}_{\text{non-sn}}(G)$ задовольняє слабку умову максимальності – групи з умовою Max-∞-(non-ab) – розглядали Л. С. Казарін, Л. А. Курдаченко, І. Я. Суботін [41]. Ситуація тут складніша, ніж для груп з умовою Max-(non-ab), лише неабельові локально скінченні групи з умовою Max-∞-(non-ab) будуть мінімаксними (тобто черніковськими), опис інших класів потребує спеціальних термінів і ми не будемо його наводити.

Наслідуючи Д. І. Зайцева [96], називатимемо нільпотентну групу G класу нільпотентності k *стало нільпотентною*, якщо кожна нескінченна підгрупа з G, що має клас нільпотентності k (зокрема, сама група), має власну нескінченну підгрупу класу нільпотентності k. *Кожна нескінченна нільпотентна група G класу нільпотентності k має власну нескінченну підгрупу класу нільпотентності k. Якщо G – нільпотентна група без скруту, то кожна її неодинична підгрупа є стало нільпотентною. Нехай G – нільпотентна група, в якої $\mathbf{L}_{\text{nil(k)}}(G) = \{G\}$. Тоді G скінченна.* (Д. І. Зайцев [96]). *Локально нільпотентна*

група G, що має нільпотентну підгрупу класу нільпотентності k, тоді і тільки тоді містить у собі стало нільпотентну підгрупу класу нільпотентності k, коли вона не є черніковською. (Д. І. Зайцев [97]).
Питання про існування у періодичних груп нільпотентних підгруп класу $\leqslant k$ розглядав А. Н. Остиловський [120], який отримав такі результати.

Нехай G – бінарно скінченна група, яка не задовольняє Min. Якщо кожна нескінченна підгрупа, що має нескінченний індекс, задовольняє Min або є нільпотентною класу $\leqslant k$, то G – нільпотентна група класу $\leqslant k$.

Нехай G – бінарно скінченна група, що не є нільпотентною класу $\leqslant k$. Якщо множина $\mathbf{L}_{\text{nil(k)}}(G) = \{G\}$ задовольняє слабку умову мінімальності, то G – черніковська група.

О. Д. Артемович [86] вивчав локально ступінчаті групи з умовою мінімальності для не майже гіперцентральних підгруп. Х. Сміт [74] почав розглядати групи G, у яких множина $\mathbf{L}_{\text{non-nil}}(G)$ задовольняє умови мінімальності і максимальності та слабкі умови мінімальності і максимальності. Зокрема, локально нільпотентна група без скруту з цими умовами нільпотентна. Групи, в яких $\mathbf{L}_{\text{non-nil}}(G) = \{G\}$ задовольняє умову максимальності, вивчали М. Діксон та Л. А. Курдаченко [26, 27]. Перша праця містить локально нільпотентні групи з цією властивістю, друга – розв'язні. Наведемо основні результати першої праці.

Нехай G – локально нільпотентна група з умовою Max-(non-nil), T – її періодична частина, R – скінченний резидуал G. Якщо G ненільпотентна та G/R нескінченно породжена, то виконуються такі умови: а) $R \leqslant T$ та T/R скінченна; б) G/T – нільпотентна мінімаксна група і $Sp(G/T) = \{p\}$ для деякого простого числа p; в) R – p-підгрупа; г) G має таку нільпотентну нормальну підгрупу U, що G/U квазіциклічна p-група; д) якщо S – ненільпотентна підгрупа з G, то $G = US$.

Нехай G – локально нільпотентна група з умовою Max-(non-nil), R – скінченний резидуал G. Якщо G ненільпотентна, G/R скінченно породжена, а R нільпотентна, то виконуються такі умови: а) R – подільна черніковська підгрупа; б) кожна власна G-інваріантна підгрупа R скінченна; в) $[G, R] = R$.

Нехай G – локально нільпотентна група з умовою Max-(non-nil), T – її періодична частина, R – скінченний резидуал G. Якщо G ненільпотентна та немінімаксна, а G/R скінченно породжена, то виконуються такі умови: R періодична, не має власних підгруп скінченного індексу, ненільпотентна, але кожна її власна підгрупа нільпотентна, G розв'язна і, крім того, R є розширенням нільпотентної групи за допомогою черніковської; зокрема, R – p-підгрупа для деякого простого числа p, що має зростаючий ряд нільпотентних G-інваріантних підгруп $\langle 1 \rangle = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n \leqslant \cdots \cup_{n \in \mathbb{N}} A_n = R$.

Праця М. Діксона та Л. А. Курдаченка [27] містить опис розв'язних груп з умовою Max-(non-nil). О. Д. Артемович [87-89] досліджував локально нільпотентні групи з умовою максимальності для негіперцентральних підгруп. Ми не будемо детально розглядати їхню будову. Групи, в яких множина $\mathbf{L}_{\text{non-nil}}(G) = \{G\}$ задовольняє слабку умову максимальності, вивчали Л. А. Курдаченко, П. Шумяцький та І. Я. Суботін [47]. Зокрема, локально скінченні групи з цією умовою є або локально нільпотентними, або черніковськими. С. Франчіозі, Ф. де Жіованні, Я. П. Сисак [32] охарактеризували деякі класи груп з умовою мінімальності для не FC-підгруп. М. С. Черніков [129, 130] вивчав групи з різними умовами (слабкої) π-максимальності та (слабкої) π-мінімальності. О. Д. Артемович [2] досліджував розв'язні групи з умовою мінімальності та відповідно мінімальності для підгруп, які не є черніковськими над нільпотентними.

З цього випливає, що мінімальні не $\mathfrak{X}$-групи особливо пов'язані з групами з умовами мінімальності та максимальності для не $\mathfrak{X}$-підгруп. Це спостереження і та особлива роль, яку відіграє поняття "нільпотентність" (і такі його можливі узагальнення як "майже нільпотентність", "гіперцентральність", "майже гіперцентральність") у теорії груп, а також (на прикладі груп типу Хайнекена-Мохамеда) зв'язок наявності "малої" родини власних ненільпотентних підгруп з відсутністю власних факторизацій у групі підтверджують необхідність досліджувати досить мало вивчені групи, близькі до нерозкладних, тобто групи з "малими" множинами ненільпотентних (відповідно негіперцентральних) підгруп. На цьому шляху виникає багато перспективних, цікавих і важливих задач, розв'язки яких потребують нових підходів і напрацювання нових методів. Значною мірою застосовують методи теорії кілець та модулів, демонструючи глибокі взаємозв'язки теорії груп і теорії кілець.

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GROUPS WITH MANY $\mathfrak{X}$-SUBGROUPS

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We give survey of results on groups with many normal (respectively subnormal, nilpotent) subgroups and related topics.

Key words: minimal non-$\mathfrak{X}$-group, minimal condition on $\mathfrak{X}$-subgroups, maximal condition on $\mathfrak{X}$-subgroups.

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ПРО ДИФЕРЕНЦІАЛЬНІ КІЛЬЦЯ, НАД ЯКИМИ ВСІ ДИФЕРЕНЦІАЛЬНІ СКРУТИ ТРИВІАЛЬНІ

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Запропоновано деякі способи побудови диференціальних радикальних фільтрів у неодинарних диференціальних кільцях. З'ясовано, що кожному такому фільтру відповідає диференціальний скрут у категорії лівих диференціальних модулів над базовим диференціальним кільцем. Описано один тип диференціальних кілець, над якими всі диференціальні скрути тривіальні.

Ключові слова: диференціювання, диференціальні кільця, диференціально прості кільця, диференціальні модулі, диференціальні скрути, диференціальні радикальні фільтри, тривіальні диференціальні скрути.

1. Зауважимо, що теорія скрутів у категорії модулів над асоціативним кільцем досить добре розвинена. Їй присвячено багато монографій. Найфундаментальнішою з цієї галузі є праця Дж. Голана [1]. В категорії одинарних диференціальних модулів над одинарним диференціальним кільцем диференціальні скрути і диференціальні радикальні фільтри вперше розглядали в [2]. Загальним питанням теорії радикалів у кільцях, модулях і, навіть, в довільних абелевих і Гротендикових категоріях, також присвячено низку монографій і статей, серед яких зазначимо [3, 4, 5]. Проте радикалам і скрутам у категорії диференціальних модулів приділено мало уваги. Причиною є те, що ця категорія ізоморфна до категорії звичайних модулів над кільцем диференціальних операторів основного диференціального кільця. Проте зазначений ізоморфізм вирішує тільки загальнотеоретичні проблеми і зовсім не вирішує конкретних питань про локалізації диференціальних кілець та про радикальні фільтри в диференціальних кільцях. Це, зокрема, пояснюється складністю механізму зв'язку між ідеалами (односторонніми чи двосторонніми) кільця коефіцієнтів та ідеалами кільця диференціальних операторів (див. [6]). Тому виникає потреба (досліджуючи локалізації диференціальних кілець) мати змогу користуватись адекватною мовою диференціальних понять, а не їх трансформаціями в кільці диференціальних операторів.

За аналогією з [2] з'ясовуємо, що деякі найпростіші факти про звичайні радикальні фільтри без ускладнень переносяться на випадок кілець з багатьма диференціюваннями. Спочатку узагальнюються такі поняття, як диференціальний скрут, диференціальний радикальний фільтр тощо на випадок кілець з багатьма диференціюваннями та формулюються і доводяться найпростіші їхні властивості. Скрути, визначені цими радикальними фільтрами, тісно пов'язані. Зокрема, кільце дробів, побудоване за допомогою отриманого диференціального радикального фільтра, при певних обмеженнях на скрут можна перетворити в диференціальне кільце стосовно диференціювання, яке продовжує диференціювання основного кільця. Цей факт використовують потім для вивчення одинарних комутативних диференціальних кілець, всі диференціальні скрути над якими тривіальні.

2. Попередні відомості та факти. Всі розглядувані кільця припускатимемо асоціативними з одиницею, а всі модулі лівими й унітарними. Потрібні основні поняття зі звичайної теорії кілець можна знайти в [7], а матеріал з диференціальної алгебри можна використати з [8] або [9].

Нехай R – кільце. Тоді відображення $\delta\colon R \to R$ називається його *диференціюванням*, якщо для будь-яких елементів a і b з R правильні рівності $\delta(a+b) = \delta(a) + \delta(b)$ і $\delta(ab) = \delta(a)b + a\delta(b)$. Елемент $a \in R$, для якого $\delta(a) = 0$, називається *константою щодо* δ. Диференціювання, стосовно якого всі елементи кільця є константами, називається *тривіальним*. Кільце з заданим на ньому диференціюванням називається *одинарним диференціальним кільцем*. На практиці часто доводиться розглядати більше, ніж одне диференціювання. Тому надалі кільце R називатимемо *диференціальним*, якщо на ньому задано скінченну множину попарно комутуючих диференціювань $\delta_1, \dots, \delta_n$. *Константами* довільного диференціального кільця називають елементи, які є константами щодо всіх структурних диференціювань цього кільця. Якщо R – диференціальне кільце з диференціюваннями $\delta_i, i = 1, \dots, n$, то можна збудувати нове кільце D_R, яке називається кільцем лінійних диференціальних операторів кільця R.

Нагадаємо, що лінійний диференціальний оператор від диференціювань $\delta_1, \dots, \delta_n$ з коефіцієнтами з кільця R зображається у вигляді

$$\sum_{i_1=0, i_2=0, \dots, i_n=0}^{k_1, k_2, \dots, k_n} a_{i_1 i_2 \dots i_n} {d_1}^{i_1} {d_2}^{i_2} \dots {d_n}^{i_n},$$

де $a_{i_1 i_2 \dots i_n} \in R$, d_i – диференціальні невідомі.

Дію такого оператора на елемент кільця задають за правилом

$$\left(\sum_{i_1=0, i_2=0, \dots, i_n=0}^{k_1, k_2, \dots, k_n} a_{i_1 i_2 \dots i_n} {d_1}^{i_1} {d_2}^{i_2} \dots {d_n}^{i_n} \right)(a) =$$

$$= \sum_{i_1=0, i_2=0, \dots, i_n=0}^{k_1, k_2, \dots, k_n} a_{i_1 i_2 \dots i_n} \left({\delta_1}^{i_1} \circ {\delta_2}^{i_2} \circ \dots \circ {\delta_n}^{i_n} \right)(a)$$

для кожного $a \in R$. Сукупність усіх таких лінійних операторів можна перетворити в кільце. Як операцію додавання беремо звичайне додавання лівих поліномів від некомутативних невідомих $d_1, d_2, \dots, d_n$, а як множення – операцію, яка індукується співвідношеннями $\delta_i \cdot a = a\delta_i + \delta_i(a)$, $a \in R$, з врахуванням дистрибутивності та асоціативності. Це кільце називається *кільцем лінійних диференціальних* *операторів* від диференціювань $d_1, d_2, \ldots, d_n$ з коефіцієнтами з диференціального кільця R і позначають через $\mathcal{D}_R$ (див., наприклад, [8]).

Аналогічно визначають ліві модулі з багатьма диференціюваннями. Точніше, нехай M – лівий R-модуль. Відображення $\partial\colon M \to M$ називається *диференціюванням модуля* M, якщо для всіх $m, m_1, m_2 \in M$ і кожного $a \in R$ виконуються рівності:

1) $\partial(m_1 + m_2) = \partial(m_1) + \partial(m_2)$;

2) $\partial(am) = \delta(a)m + a\partial(m)$.

Модуль M, на якому задано скінченну кількість попарно комутуючих диференціювань $\partial_1, \partial_2, \ldots, \partial_n$, називається ***лівим диференціальним R-модулем.*** Якщо M_1 і M_2 – ліві диференціальні R-модулі з диференціюваннями $\partial'_1, \partial'_2, \ldots, \partial'_n$ і $\partial''_1, \partial''_2, \ldots, \partial''_n$ відповідно, то R-модульний гомоморфізм $f\colon M_1 \to M_2$ називається *диференціальним гомоморфізмом*, якщо для будь-якого i, $1 \leqslant i \leqslant n$, виконуються рівності $f(\partial'_i(m)) = \partial''_i(f(m_1))$ для будь-якого $m \in M_1$. Категорію всіх лівих диференціальних R-модулів і всіх диференціальних гомоморфізмів позначатимемо через $R - D\mathrm{mod}$. При $n = 1$ цю категорію називатимемо *одинарною* категорією лівих диференціальних модулів. Добре відомо, що категорія $R - D\mathrm{mod}$ ізоморфна до категорії лівих модулів над кільцем диференціальних операторів кільця R. Цей ізоморфізм визначаємо так: якщо M – лівий диференціальний R-модуль, то $\mathcal{D}_R$-модульна структура на M задається за допомогою співвідношення

$$\left(\sum_{i_1=0, i_2=0, \ldots, i_n=0}^{k_1, k_2, \ldots, k_n} a_{i_1 i_2 \ldots i_n} {d_1}^{i_1} {d_2}^{i_2} \ldots {d_n}^{i_n}\right)(m) =$$

$$= \sum_{i_1=0, i_2=0, \ldots, i_n=0}^{k_1, k_2, \ldots, k_n} a_{i_1 i_2 \ldots i_n} \left({\partial_1}^{i_1} \circ {\partial_2}^{i_2} \circ \cdots \circ {\partial_n}^{i_n}\right)(m),$$

де $\sum\limits_{i_1=0, i_2=0, \ldots, i_n=0}^{k_1, k_2, \ldots, k_n} a_{i_1 i_2 \ldots i_n} {d_1}^{i_1} {d_2}^{i_2} \ldots {d_n}^{i_n} \in \mathcal{D}_R, m \in M$. Навпаки, якщо M – лівий $\mathcal{D}_R$-модуль, то відображення $\partial_i\colon M \to M$, які задають за законом $\partial_i(m) = d_i \cdot m$, $1 \leqslant i \leqslant n$, є диференціюваннями модуля M. Отже, кожний $\mathcal{D}_R$-модуль можна розглядати як диференціальний R-модуль. Легко перевірити, що кожний $\mathcal{D}_R$-модульний гомоморфізм автоматично є диференціальним R-гомоморфізмом.

Аналогічно до того, як це робиться в одинарному випадку, визначаємо поняття диференціального скруту в категорії лівих диференціальних модулів з багатьма диференціюваннями. Говоритимемо, що в категорії $R - Dmod$ задано *диференціальний скрут* σ, якщо кожному лівому диференціальному R-модулю M зіставляється деякий його диференціальний підмодуль $\sigma(M)$ і правильні такі умови.

ДС1. Для кожного диференціального гомоморфізму $f\colon M \to N$

$$f(\sigma(M)) \subseteq \sigma(N).$$

ДС2. Якщо $N \subseteq \sigma(M)$, де $M, N \in R - Dmod$, то $\sigma(N) = N$.

ДС3. $\sigma(M/\sigma(M)) = 0$ для кожного $M \in R - Dmod$.

диференціальний підмодуль $\sigma(M) \subseteq M$ називається *σ-періодичною частиною* диференціального модуля M. Якщо $\sigma(M) = M$ (відповідно, $\sigma(M) = 0$), то M називається *σ-періодичним* (*σ-напівпростим,* або *модулем без σ-скруту*). Якщо одна з цих умов виконується для всіх диференціальних модулів M, то назвемо σ *тривіальним скрутом*. Обидва тривіальні скрути є диференціальними автоматично.

Нехай R – диференціальне кільце з диференціюваннями $\delta_1, \delta_2, \ldots, \delta_n$. Введемо позначення $a^{(i_1,i_2,\ldots,i_n)} = \left(\delta_1{}^{i_1} \circ \delta_2{}^{i_2} \circ \cdots \circ \delta_n{}^{i_n}\right)(a)$ для будь-якого елемента a кільця R. Крім того, нехай $a^{(\infty)} = \left\{a^{(i_1,\ldots,i_n)} \mid i_1, \ldots, i_n = 0, 1, \ldots\right\}$.

Нагадаємо, що лівий ідеал I кільця R називається *диференціальним,* якщо для кожного a з I маємо $\delta_i(a) \in I$ для всіх $i = \overline{1,n}$.

Нехай R – диференціальне кільце з диференціюванням δ. Система $\mathcal{F}$ лівих диференціальних ідеалів кільця R називається *диференціальним радикальним фільтром* кільця R, якщо виконуються такі умови.

ДФ1. Якщо $I \in \mathcal{F}$ і $I \subseteq J$, де J – диференціальний ідеал кільця R, то $J \in \mathcal{F}$.

ДФ2. Якщо $I \in \mathcal{F}$ і $a \in R$, то $(I : a^{(\infty)}) \in \mathcal{F}$.

ДФ3. Якщо $I \subset J$, де $J \in \mathcal{F}$ – такий, що $(I : a^{(\infty)}) \in \mathcal{F}$ для кожного $a \in J$, то $I \in \mathcal{F}$.

Як і у випадку звичайних радикальних фільтрів, легко перевірити, що $I \bigcap J \in \mathcal{F}$. Зауважимо таке: коли R розглядати з тривіальним диференціюванням, то диференціальний радикальний фільтр зводиться до звичайного радикального фільтра. В цьому випадку умови ДФ1, ДФ2, ДФ3 позначатимемо через Ф1, Ф2, Ф3 відповідно.

Система $\mathcal{B}$ лівих диференціальних ідеалів, що належать до диференціального радикального фільтра $\mathcal{F}$, називається *базою* для $\mathcal{F}$, якщо кожний диференціальний ідеал, який належить до $\mathcal{F}$, містить деякий диференціальний ідеал з $\mathcal{B}$. Для спрощення термінології лівий ідеал, породжений множинами $a_1^{(\infty)}, a_2^{(\infty)}, \ldots, a_n^{(\infty)}$, називатимемо ідеалом, *диференціально породженим* $a_1, a_2, \ldots, a_n$. Такі ідеали інколи називаються *диференціально скінченно породженими.*

3. Зв'язок між звичайними і диференціальними радикальними фільтрами. Наступні три леми доводяться аналогічно до відповідних лем для звичайних радикальних фільтрів. Зауважимо, що перші дві з них не вимагають існування одиниці в кільці R.

Лема 1. *Нехай R – диференціальне кільце і $\mathcal{B}$ – система скінченно породжених (як ліві диференціальні ідеали) диференціальних двосторонніх ідеалів кільця R, замкнена стосовно множення ідеалів. Тоді система лівих диференціальних ідеалів $\mathcal{F}_{\mathcal{B}} = \{T \mid T -$ лівий диференціальний ідеал, $T \supseteq B, B \in \mathcal{B}\}$ є диференціальним радикальним фільтром кільця R.*

Лема 2. *Нехай R – диференціальне кільце, I– ідемпотентний двосторонній ідеал кільця R. Тоді система лівих диференціальних ідеалів кільця I, які містять ідеал I, є диференціальним радикальним фільтром.*

Зауважимо, що в лемі 2 диференціальність ідеала I не вимагається, оскільки кожний ідемпотентний ідеал диференціального кільця є диференціальним.

Лема 3. *Нехай S – двосторонній диференціальний ідеал кільця R і в кільці R кожний лівий диференціальний ідеал є двостороннім. Тоді система лівих диференціальних ідеалів $\mathcal{F}_S = \{T \,|\, T -$ лівий диференціальний ідеал, $S + T = R\}$ є диференціальним радикальним фільтром кільця R.*

Наведені леми засвідчують, що диференціальні радикальні фільтри існують. Для доведення того факту, що кожному диференціальному радикальному фільтру відповідає скрут в категорії $R - Dmod$, нам треба довести такі дві леми.

Лема 4. *Нехай R – диференціальне кільце і $\mathcal{F}$ – диференціальний радикальний фільтр. Тоді система $\overline{\mathcal{F}}$ таких лівих ідеалів кільця R, що відповідає таким $B \in \mathcal{F}$, що $B \subseteq T$, є радикальним фільтром кільця R (звичайним).*

Доведення. Умову Ф1 для системи $\overline{\mathcal{F}}$ перевіряємо тривіально. Перевіримо властивість Ф2 радикального фільтра. Нехай $\mathcal{K} \in \overline{\mathcal{F}}$ і $a \in R$. Тоді існує $I \in \mathcal{F}$ такий, що $I \subseteq \mathcal{K}$. Отримуємо

$$(\mathcal{K} : a) \supseteq (I : a) \supseteq (I : a^{(\infty)}) \in \mathcal{F}.$$

З Ф1 випливає, що $(\mathcal{K} : a) \in \overline{\mathcal{F}}$.

Для перевірки умови Ф3 нехай $\mathcal{K} \supset T, \mathcal{K} \in \overline{\mathcal{F}}$ і $(T : \lambda) \in \overline{\mathcal{F}}$ при будь-якому $\lambda \in \mathcal{K}$. Тоді існує $I \in \mathcal{F}$ такий, що $I \subseteq \mathcal{K}$, й існують $I_\lambda \in \mathcal{F}$ такі, що $(T : \lambda) \supseteq I_\lambda$ для кожного $\lambda \in I$. Позначимо через S новий диференціальний ідеал $\sum\limits_{\lambda \in I} I_\lambda \lambda^{(\infty)}$. Тоді $I \bigcap S$ – диференціальний лівий ідеал кільця R і $I \bigcap S \subset I \in \mathcal{F}$. Крім того, $I \bigcap S : \lambda^{(\infty)} \supseteq I_\lambda$ при будь-якому $\lambda \in I$. Отже, за властивістю ДФ3 $I \bigcap S \in \overline{\mathcal{F}}$. Ми одержали, що $T \supseteq S \supseteq I \bigcap S \in \overline{\mathcal{F}}$, звідки випливає, що $T \in \overline{\mathcal{F}}$.

Лема доведена.

Лема 5. *Нехай R – диференціальне кільце з кільцем лінійних диференціальних операторів $\mathcal{D}_R$. Якщо $\mathcal{F}$ – диференціальний радикальний фільтр кільця R, то система $\underline{\mathcal{F}}$ лівих ідеалів кільця $\mathcal{D}_R$, кожний з яких містить лівий ідеал вигляду $\mathcal{D}_R I$, де $I \in \mathcal{F}$ – радикальний фільтр кільця $\mathcal{D}_R$.*

Доведення. Легко перевірити, що умова Ф1 виконується. Перевіримо умову Ф2. Нехай $T \in \underline{\mathcal{F}}$ і $\lambda = \sum\limits_{i_1=0, i_2=0, \dots, i_n=0}^{k_1, k_2, \dots, k_n} a_{i_1 i_2 \dots i_n} {d_1}^{i_1} {d_2}^{i_2} \dots {d_n}^{i_n}$ є довільним елементом кільця $\mathcal{D}_R$. Тоді $\mathcal{K}_\lambda = \bigcap\limits_{i_1=0, i_2=0, \dots, i_n=0}^{k_1, k_2, \dots, k_n} (I : a^{(\infty)}_{i_1 i_2 \dots i_n}) \in \mathcal{F}$, де $T \supseteq \mathcal{D}_R I, I \in \mathcal{F}$. Отже, $\mathcal{K}_\lambda \subseteq \mathcal{D}_R I$, тобто $(T : \lambda) \supseteq \mathcal{D}_R \mathcal{K}_\lambda$. Доведемо тепер, що $\underline{\mathcal{F}}$ володіє властивістю Ф3. Нехай $\mathcal{K} \supseteq T, \mathcal{K} \in \underline{\mathcal{F}}$ і $(T : \lambda) \in \underline{\mathcal{F}}$ при будь-якому $\lambda \in \mathcal{K}$. Тоді існує $I \in \mathcal{F}$ такий, що $\mathcal{K} \supseteq \mathcal{D}_R I$ і для кожного $\lambda \in \mathcal{K}$ існує $I_\lambda \in \mathcal{F}$ такий, що $(T : \lambda) \subseteq I_\lambda$. Зокрема, $I_\lambda \lambda^{(\infty)} \supseteq I \bigcap T$ при будь-якому $\lambda \in I$. Позначимо через S лівий диференціальний ідеал $\sum\limits_{\lambda \in I} I_\lambda \lambda^{(\infty)}$. Тоді $S \bigcap I \subset I$ і $S \bigcap I : \lambda^{(\infty)} \supseteq I_\lambda$ при будь-якому $\lambda \in I$. Отже, за властивістю ДФ3 одержуємо, що $S \bigcap I \in \mathcal{F}$, звідки $T \supseteq I \bigcap T \supseteq S \bigcap I \Rightarrow T \supseteq \mathcal{D}_R(S \bigcap I)$, що і треба було довести.

Теорема 1. *Якщо $\mathcal{F}$ – диференціальний радикальний фільтр диференціального кільця R, то функція $\tau(M) = \{x \,|\, x \in M, Ix = 0, I \in \mathcal{F}\}$, де $M \in R - Dmod$, є диференціальним скрутом в категорії $R - Dmod$.*

Доведення. Оскільки за лемою 5 $\underline{\mathcal{F}}$ є радикальним диференціальним фільтром кільця $\mathcal{D}_R$, то за теоремою Габріеля-Маранди ([11]) йому відповідає скрут $\underline{\tau}$ у категорії $\mathcal{D}_R - mod$. Але категорія $\mathcal{D}_R - mod$ ізоморфна категорії $R - Dmod$, і при цьому ізоморфізмі скрут $\underline{\tau}$ переходить у деякий скрут τ_1 в категорії $R - Dmod$. Легко переконатись в тому, що $\tau_1 = \tau$. Теорему доведено.

У наступному прикладі використовуємо термінологію теорії узагальнених функцій, з якою можна ознайомитись в [10].

Приклад. Нехай R – кільце нескінченно-диференційовних фінітних дійснозначних функцій, яке розглядається як одинарне диференціальне кільце стосовно звичайного диференціювання. Тоді сукупність усіх узагальнених функцій з класом фінітних основних функцій R є диференціальним R-модулем щодо звичайного диференціювання узагальнених функцій. Зауважимо, що це природний приклад диференціального модуля, який не має структури диференціального кільця. Побудуємо диференціальний радикальний фільтр кільця R. Для цього візьмемо ненульову фінітну нескінченно-диференційовну функцію $f(x)$, яка в нулі разом зі всіма своїми похідними дорівнює нулю. Розглянемо диференціально головний ідеал I, породжений функцією f, тобто

$$I = (f, f', f'', \dots, f^{(n)}, \dots).$$

Приймемо $\mathcal{F} = \{I, I^2, \dots, I^n, \dots\}$. За лемою 1 $\mathcal{F}$ – диференціальний радикальний фільтр, а за теоремою 1 йому відповідає скрут τ у категорії $R - Dmod$. Виявляється, що диференціальна періодична частина модуля узагальнених функцій збігається з множиною всіх регулярних у нулі узагальнених функцій, тобто це множина усіх узагальнених функцій типу Дірака.

Як бачимо, з кожним диференціальним радикальним фільтром $\mathcal{F}$ пов'язані два звичайні радикальні фільтри $\underline{\mathcal{F}}$ і $\overline{\mathcal{F}}$ кільця $\mathcal{D}_R$ і кільця R відповідно. Позначимо через $Q_{\overline{\mathcal{F}}}$ кільце дробів кільця R стосовно радикального фільтра $\overline{\mathcal{F}}$, а через $Q_{\underline{\mathcal{F}}}$ - кільце дробів кільця $\mathcal{D}_R$ за радикальним фільтром $\underline{\mathcal{F}}$.

Лема 6. Якщо R - диференціальне кільце і $\mathcal{F}$ – диференціальний радикальний фільтр кільця R, якому відповідає точний скрут у категорії $R - Dmod$, то диференціювання δ кільця R можна продовжити до диференціювання кільця $Q_{\overline{\mathcal{F}}}$.

Доведення (це наслідок з основної теореми Голана з [11]).

Теорема 2. *Нехай R – диференціальне кільце і $\mathcal{F}$ – диференціальний радикальний фільтр кільця R. Тоді*

$$Q_{\underline{\mathcal{F}}} \cong \mathcal{D}_{Q_{\underline{\mathcal{F}}}}.$$

Перейдемо до застосування техніки диференціальних радикальних фільтрів під час розв'язання завдання про те, коли в категорії диференціальних модулів над комутативним кільцем всі диференціальні скрути тривіальні. Для цього нам потрібні деякі результати, які належать до диференціальної алгебри.

До кінця статті розглядатимемо лише одинарні диференціальні кільця.

Нагадаємо, що комутативне диференціальне кільце називається *диференціально простим,* якщо в ньому немає нетривіальних диференціальних ідеалів. Зауважимо, що кожне таке кільце має одиницю та характеристику (див. [12]).

Теорема 3. *Нехай R – комутативне диференціально просте кільце характеристики $p > 0$ з диференціюванням δ. Якщо $a_0 + a_1\delta + \ldots + a_n\delta^n = 0$, де $a_0, a_1, \ldots, a_n \in R, a_n \neq 0$ при деякому натуральному n, то*

$$R \cong K[x]/(x^p),$$

де K – деяке поле характеристики $p > 0$.

Доведення. Оскільки $\delta(x^p) = 0$ для будь-якого $x \in R$, то x^pR – диференціальний ідеал кільця R при довільному $x \in R$. За умов диференціальної простоти кільця R маємо, що або $x^pR = 0$, або $x^pR = R$. Це означає, що x або зворотний, або нільпотентний, тобто, що R – локальне кільце з максимальним ідеалом N, який є нільідеалом.

Виберемо тепер оператор

$$b_0 + b_1\delta + \ldots + b_k\delta^k = 0 \tag{1}$$

з найменшим можливим k таким, що $b_k \neq 0$. Тоді коефіцієнти b_k, які трапляються в операторах вигляду (1), утворюють диференціальний ідеал, відмінний від нульового. На підставі диференціальної простоти кільця R отримаємо, що цей ідеал збігається з кільцем R. Це означає, що

$$c_0 + c_1\delta + \ldots + c_{k-1}\delta^{k-1} + \delta^k = 0 \tag{2}$$

при деяких $c_0, c_1, \ldots, c_{k-1} \in R$.

Враховуючи (2), отримаємо, що N – нільпотентний ідеал кільця R, тобто що $(N)^k = 0$. Справді, нехай $z = y_1y_2\ldots y_k$ – добуток k довільних елементів ідеалу N. Очевидно, $\delta(z), \delta^2(z), \ldots, \delta^{k-1}(z)$ знову будуть елементами ідеалу N, а завдяки (2) і $\delta^k(z) \in N$. Диференціюючи (2), застосоване до елемента z, бачимо, що $\delta^n(z) \in N$ при будь-якому натуральному n. Диференціальний ідеал $\{z\}$, породжений елементом z, є диференціальним ідеалом, який міститься в ідеалі N. На підставі диференціальної простоти кільця R маємо, що $\{z\} = 0$, а отже, і $z = 0$. Оскільки z – довільний елемент з N^k, то $N^k = 0$.

Для завершення доведення достатньо застосувати теорему 4.1 з праці Блокка [12], яка стверджує таке: якщо в комутативному диференціально простому кільці R характеристики $p > 0$ існує такий елемент $x \neq 0$, що $xN = 0$, то це кільце ізоморфне $K[x]/(x^p)$, де K – поле характеристики p.

Теорема доведена.

Нехай R – довільне кільце (без диференціювання). Розглянемо кільце $R[x]$ поліномів з коефіцієнтами з R від комутуючої змінної x. Припустимо, що кільце R має ненульову характеристику $p > 0$. Тоді $R[x]$ є диференціальним кільцем стосовно звичайного диференціювання, а ідеал (x^p) є диференціальним ідеалом, тому фактор-кільце $R_p = R[x]/(x^p)$ є диференціальним кільцем, яке називається *кільцем зрізаних поліномів з коефіцієнтами з R.*

Правильне таке твердження.

Твердження 1. *Нехай K – кільце характеристики $p > 0$. Тоді кільце лінійних диференціальних операторів кільця зрізаних поліномів з коефіцієнтами з K ізоморфне до кільця $p \times p$-матриць над кільцем K.*

Доведення. Нехай $R = K[x]/(x^p)$ і $\mathcal{D}_R$ – кільце лінійних диференціальних операторів кільця R. Тоді кожний елемент з $\mathcal{D}_R$ має вигляд $\sum\limits_{i,j=0}^{p-1} a_{ij}x^i\delta^j$. Побудуємо відображення з $\mathcal{D}_R$ у кільце матриць порядку p над K. Нехай

$$X = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p-1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

тоді приймемо

$$\kappa\left(\sum_{i,j=0}^{p-1} a_{ij}x^i\delta^j\right) = \sum_{i,j=0}^{p-1} a_{ij}X^iD^j.$$

Для перевірки того, що κ – кільцевий гомоморфізм, достатньо перевірити, що $DX = XD + E$, де E – одинична матриця, але це перевіряється безпосереднім обчисленням.

Покажемо, що κ – сюр'єктивний гомоморфізм. Для цього обчислимо добутки X^iD^j і D^iX^j. Отримаємо

$$X^iD^j = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{i!}{1!} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{(i+1)!}{2!} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \frac{(p-1)!}{(p-i-1)!} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & \frac{i!}{1!} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots & 0 & \frac{(i+1)!}{2!} & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & \dots & * \end{pmatrix}$$

й аналогічно

$$D^iX^j = \begin{pmatrix} * & 0 & \dots & \dots & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & * & 0 & \dots & 0 & \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & * & 0 & \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & * & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & & & \vdots & \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & \dots & 0 \end{pmatrix},$$

де $*$ позначає деякий елемент кільця K, причому зворотний, а кожний із верхніх блоків матриць має i рядків.

З вигляду матриць X^iD^j і D^iX^j легко зрозуміти, що за допомогою них можна отримати всі матричні одиниці кільця $Mat(K,p)$. Отже, κ – сюр'єктивний гомоморфізм.

Ін'єктивність відображення κ випливає з того, що кожна матриця виражається через матричні одиниці однозначно (це можна перевірити безпосередньо).

Твердження доведено.

Твердження 2. *Нехай R – таке комутативне диференціально просте кільце характеристики $p > 0$, що кільце $\mathcal{D}_R$ ізоморфне до кільця матриць над деяким тілом K. Тоді кільце R ізоморфне до кільця зрізаних поліномів над деяким полем характеристики $p > 0$.*

Доведення. Позаяк $\mathcal{D}_R$ лівий векторний простір скінченної вимірності над тілом K, то при деякому n

$$k_0 + k_1\delta + \ldots + k_n\delta^n = 0. \tag{3}$$

Розглянувши елементи k_i як оператори з $\mathcal{D}_R$ і звівши (3) до вигляду $a_0 + a_1\overline{\delta} + \ldots + k_m\overline{\delta}^m = 0$, одержимо залежність, яка є в умовах теореми 3, звідки і випливає наше твердження.

Теорема доведена.

Теорема 4. *Нехай R – комутативне диференціально просте кільце. Якщо в категорії $R - Dmod$ всі диференціальні скрути тривіальні, то R ізоморфне до кільця зрізаних поліномів над полем характеристики $p > 0$.*

Доведення. Розглянемо спочатку випадок характеристики 0. Познер [13] довів, що комутативне диференціально просте кільце R характеристики 0 є областю цілісності, а отже, і кільце $\mathcal{D}_R$ є областю цілісності. Проте над областю цілісності всі скрути тривіальні лише в тому випадку, коли ця область цілісності є полем. Позаяк в $\mathcal{D}_R$ елемент δ завжди незворотний, то звідси випливає, що над диференціально простим кільцем характеристики 0 завжди існують нетривіальні диференціальні скрути.

Нехай тепер char$R = p > 0$. Тоді за результатом В. А. Андрунакієвича та Ю. М. Рябухіна [14] кільце $\mathcal{D}_R$ є кільцем матриць над локальним досконалим кільцем.

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ON DIFFERENTIAL RINGS FOR WHICH ALL DIFFERENTIAL TORSION THEORIES ARE TRIVIAL

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Some methods of constructing differential radical filters are proposed. In addition every differential radical filter is in correspondence with some differential torsion theory in the category of left differential modules over the basic differential ring. One kind of differential rings for which all differential torsion theories are trivial is described.

Key words: derivation, differential rings, differentially simple rings, differential modules, differential torsion theories, differential radical filters, trivial differential torsion theories.

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СПЕКТРАЛЬНА ЗАДАЧА ОСЕСИМЕТРИЧНОЇ ТЕОРІЇ ПРУЖНОСТІ

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Отримано зображення напружено-деформованого стану (НДС) при довільному осесиметричному навантаженні на торцях товстостінного циліндра у вигляді ряду за власними функціями. Доведено існування зліченої кількості власних значень. Коефіцієнти ряду знаходять з умови мінімуму інтеграла квадрата відхилення розв'язку від заданих граничних умов на торці. Запропоновано два чисельні методи знаходження коефіцієнтів ряду. Знайдено розв'язок крайової задачі для бігармонічного рівняння в циліндричній системі координат.

Ключові слова: бігармонічне рівняння, власні значення, власні функції, проблема моментів, цілі функції, осесиметрична теорія пружності, напружено-деформований стан, циліндр.

У статті розглянуто НДС товстостінного циліндра: внутрішній радіус – R_1, зовнішній – R_2, нескінченної або скінченної довжини H, вісь x збігається з віссю симетрії циліндра. Циліндр навантажений довільними осесиметричними торцевими зусиллями при вільних від навантаження бокових поверхнях. Запропоноване формулювання задачі може описувати пружний півпростір, куля, куля або півпростір з вирізом, суцільний циліндр і т.п. Для того щоб знайти НДС товстостінного циліндра, потрібно проінтегрувати рівняння Ляме [1-3] при заданих граничних умовах. Осесиметричні задачі розглядали в [1-7]. Огляд літератури наведено в [2-3]. У працях [5,6] запропоновано чисельний розв'язок рівнянь Ляме, в [4] наведений наближений розв'язок. В [7] подано розрахунок тонкостінного циліндра у випадку, коли його можна моделювати циліндричною оболонкою. Проте проблема побудови наближеного з заданою точністю аналітичного розв'язку для довільного навантаження на торцях залишилася невирішеною.

1. Знаходження розв'язку методом розкладання в ряд за власними функціями. Ляв [1], виразивши переміщення і напруження у вигляді

$$w = -\frac{1}{2G}\frac{\partial^2}{\partial r \partial x}L, \quad u = \frac{1}{2G}\left[2(1-\nu)\nabla^2 L - \frac{\partial^2}{\partial x^2}L\right], \ \sigma_r = \frac{\partial}{\partial x}\left(\nu\nabla^2 L - \frac{\partial^2}{\partial r^2}L\right),$$

$$\sigma_x = \frac{\partial}{\partial x}\left((2-\nu)\nabla^2 L - \frac{\partial^2}{\partial x^2}L\right),\ \tau_{rx} = \tau = \frac{\partial}{\partial r}\left((1-\nu)\nabla^2 L - \frac{\partial^2}{\partial x^2}L\right), \quad (1)$$

звів розв'язання рівнянь Ляме до знаходження бігармонічної функції $L = L(x,r)$, що задовольняє відповідні граничні умови та рівняння

$$\nabla^2\nabla^2 L(x,r) = 0,\ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}, \quad (2)$$

де ν - коефіцієнт Пуассона, G - модуль зсуву.

Сформульовану задачу розрахунку НДС товстостінного циліндра зведемо до знаходження такого розв'язку бігармонічного рівняння (2), який задовольняє нульові граничні умови на бокових поверхнях

$$\sigma_r(x,R_j) = 0,\ \tau(x,R_j) = 0,\quad j = 1,2 \quad (3)$$

і задані граничні умови в напруженнях на торцях циліндра [див. 1, 2, 3]. Цей розв'язок подамо методом розділення змінних у вигляді ряду

$$L(x,r) = \sum_{k=1}^{\infty} Re\{\left(Z_{0,k}^2(\beta_k r) + rZ_{1,k}^1(\beta_k r)\right)\exp(-\beta_k x) +$$
$$+\left(Z_{0,k}^4(\beta_k r) + rZ_{1,k}^3(\beta_k r)\right)\exp(\beta_k x)\}, \quad (4)$$

де $\beta_k \in C$ – шукані спектральні параметри, для яких окремі розв'язки в зображенні (4) задовольняють граничні умови (3). Введені функції $Z_{m,k}^j(\beta_k r) = g_{k,j}J_m(\beta_k r) + c_{k,j}N_m(\beta_k r)$, $j = \overline{1,4}$ є циліндричними, де $g_{k,j}, c_{k,j}$ - невідомі загалом комплексні коефіцієнти, $J_m(\beta_k r)$ - функції Бесселя першого роду, $N_m(\beta_k r)$ - функції Бесселя другого роду або функції Неймана [8].

Спочатку припустимо, що циліндр за змінною x в сторону додатних значень є півбезмежним. З фізичних міркувань випливає, що в цьому випадку окремий частковий розв'язок бігармонічного рівняння (2) у зображенні (4) повинен мати вигляд

$$L(x,\beta r) = Re\{\left(Z_0^2(\beta r) + rZ_1^1(\beta r)\right)\exp(-\beta x)\}, \quad (5)$$

де $Re(\beta) > 0$, $Z_0^1(\beta r) = gJ_0(\beta r) + cN_0(\beta r)$, $Z_0^2(\beta r) = bJ_0(\beta r) + aN_0(\beta r)$, a, g, c, b, - невідомі комплексні коефіцієнти. Якщо функція Лява $L(x,\beta r)$ задовольняє граничні умови (3), то її називатимемо власною функцією. Підставивши функцію Лява (5) у зображення напружень (1) і використавши властивості Беселевих функцій, виразимо нормальні та дотичні напруження через введені циліндричні функції:

$$\sigma_r = Re\{\beta\left((1-2\nu)\beta Z_0^1(\beta r) + \left[\frac{\beta}{r}Z_1^2(\beta r) - \beta^2 Z_0^2(\beta r) - \beta^2 r Z_1^1(\beta r)\right]\right)\exp(-\beta x)\},$$
$$\sigma_x = -Re\{\beta\left[2(2-\nu)\beta Z_0^1(\beta r) - \beta^2\left(Z_0^2(\beta r) + rZ_1^1(\beta r)\right)\right]\exp(-\beta x)\}, \quad (6)$$
$$\tau = Re\{\beta^2\left[\beta Z_1^2(\beta r) - 2(1-\nu)Z_1^1(\beta r) - \beta r Z_0^1(\beta r)\right]\exp(-\beta x)\}.$$

Знайдемо рівняння, яке повинно задовольняти спектральний параметр β, щоб функція $L(x,\beta r)$ стала власною функцією. З зображення (6) випливає, що граничні умови (3), з яких можна визначити власні значення β, набувають вигляду

$$\left[\frac{1}{R_j} Z_1^2(\beta R_j) + (1-2\nu) Z_0^1(\beta R_j) - \beta Z_0^2(\beta R_j) - \beta R_j Z_1^1(\beta R_j)\right] = 0,$$

$$\left[\beta Z_1^2(\beta R_j) - 2(1-\nu) Z_1^1(\beta R_j) - \beta R_j Z_0^1(\beta R_j)\right] = 0, \ j = 1, 2. \qquad (7)$$

Використавши вираз власних функцій (5), зведемо граничні умови (7) до чотирьох лінійних рівнянь з невідомими комплексними коефіцієнтами a, g, c, b

$$\beta\left(bJ_1(\beta R_j) + aN_1(\beta R_j)\right) - 2(1-\nu)\left(gJ_1(\beta R_j) + cN_1(\beta R_j)\right) - \beta R_j\big(gJ_0(\beta R_j) +$$

$$+ cN_0(\beta R_j)\big) = 0, \ \ \left(bJ_1(\beta R_j) + aN_1(\beta R_j)\right) + (1-2\nu) R_j\left[gJ_0(\beta R_j) + cN_0(\beta R_j)\right] -$$

$$-\beta R_j^2\left(gJ_1(\beta R_j) + cN_1(\beta R_j)\right) - \beta R_j\left(bJ_0(\beta R_j) + aN_0(\beta R_j)\right) = 0, \ j = 1, 2. \quad (8)$$

Як відомо, відмінний від нуля розв'язок системи (8) можливий тільки за умови рівності нулю визначника цієї системи. Якщо розкрити визначник, то отримаємо характеристичне рівняння, яке можна розв'язати чисельно. Отже, потрібно знайти набір спектральних параметрів β_k і власних функцій (5), які б у вигляді ряду задовольнили відповідні граничні умови в напруженнях на торцях циліндра. Врахувавши зображення (6), (7), легко визначити таке.

Теорема 1. *Якщо функція Лява (5) задовольняє другу граничну умову (7), то нормальні торцеві напруження, які їй відповідають, є самозрівноваженими.*

Доведення. За означенням, нормальні торцеві напруження є самозрівноваженими тоді і тільки тоді, коли сумарне інтегральне нормальне зусилля T_x, яке діє на торці циліндра, дорівнює нулю. Теорему доведемо безпосереднім обчисленням. Врахувавши співвідношення (6), знайдемо зусилля T_x, яке діє на торці циліндра

$$T_x = 2\pi \int_{R_1}^{R_2} r\sigma_x(0, r)\, dr = 2\pi\beta^3 \left[\beta Z_1^2(\beta r) - 2(1-\nu) Z_1^1(\beta r) - \beta r Z_0^1(\beta r)\right]_{R_1}^{R_2}.$$

Використавши другу граничну умову (7), отримаємо твердження теореми. Теорема 1 правильна також для окремих розв'язків (4), а отже, і для будь-якого набору розв'язків у вигляді (4), (5). Звідси випливає, що одних розв'язків вигляду (4), (5) у випадку, коли на торці циліндра діє несамозрівноважене сумарне зусилля $T_x \neq 0$, недостатньо. Спочатку треба розв'язати відому задачу [1,2] для рівномірно розподілених на торці циліндра нормальних напружень σ_x і долучити в подання функції Лява (4) доданок $L_0(x, r) = \frac{G}{E}\sigma_0 x\left(\nu r^2 + \frac{1-2\nu}{3}x^2\right)$, де $\sigma_0 = \frac{T_x}{S}$, S - площа торцевої поверхні, E - модуль Юнга. Функція $L_0(x, r)$ відповідає нульовому значенню спектрального параметра $\beta_0 = 0$. Легко перевірити, що вона є власною функцією.

2. Обчислення НДС суцільного циліндра. Розглянемо суцільний циліндр $R_1 = 0$, $R_2 = R$,. Введемо безрозмірну радіальну змінну γ, $0 \leqslant \gamma \leqslant 1$, $r = R\gamma$.

У цьому випадку коефіцієнти a, c в зображенні (5) повинні дорівнювати нулю і характеристичне рівняння системи (8) спрощується до вигляду

$$z^2\left[J_0^2(z)+J_1^2(z)\right]=\delta J_1^2(z), \tag{9}$$

де $\delta=2(1-\nu)$, $z=\beta R$ - безрозмірний спектральний параметр. Легко зауважити, що дійсні розв'язки характеристичного рівняння (9) можливі тільки за $z^2<\frac{\delta}{2}$. З поведінки функцій Бесселя при $z^2<\frac{\delta}{2}$ випливає, що характеристичне рівняння (9) має тільки один дійсний корінь $z_0=0$, якому відповідає власна функція $L_0(x,r)$. Інші корені характеристичного рівняння будуть комплексними і мають вигляд $z_k=(\pm\mu_k\pm i\alpha_k)R$. Ми повинні вибрати корені з додатними дійсними частинами $z_k=\mu_k+i\alpha_k$, $\overline{z_k}=\mu_k-i\alpha_k$, де $\mu_k>0$, $k=1,2,3...$

Розв'язування рівняння (9) зведемо до задачі пошуку нулів функції

$$F(z)=z^2\left[J_0^2(z)+J_1^2(z)\right]-\delta J_1^2(z). \tag{10}$$

Використовуючи означення порядку і типу функції [див. 9, 10], можна показати, що функція $F(z)$ є цілою порядку 1 типу 2.

Теорема 2. *Функція $F(z)$ має зліченну кількість нулів $z_k=R\beta_k$.*

Доведення. Приймемо $z=\sqrt{\zeta}$. Функція $F\left(\sqrt{\zeta}\right)$ стосовно комплексної змінної ζ є функцією порядку $\frac{1}{2}$. З теореми 5 [10, с.268] випливає доведення.

Теорема 3. *Система функцій $\{J_0(z_k\gamma), J_1(z_k\gamma)\}$, $k=0,1,2,...$ є повною в комплексному крузі $|\gamma|\leqslant 1$.*

Доведення. Оскільки $J_0(z), J_1(z)$ - цілі функції порядку $\rho=1$ типу $\sigma=1$, а для послідовності нулів $z_k=R\beta_k$ виконується умова $\lim\limits_{k\to\infty}\frac{k}{|z_k|}<e$, то доведення теореми 3 випливає з твердження теореми 18 [див. 9, с. 283].

Наслідок. *Система функцій $\{J_0(z_k\gamma)\}$, $k=0,1,2,...$ є повною на одиничному відрізку $\gamma\in[0,1]$ в класі аналітичних функцій.*

З граничних умов (8) (у випадку суцільного циліндра) випливає, що невідомі коефіцієнти g_k, b_k комплексні та лінійно залежні між собою

$$b_k=\zeta_k g_k,\ \text{де}\quad \zeta_k=\frac{m_k+\delta}{\beta_k},\ m_k=\sqrt{\delta-(z_k)^2}. \tag{11}$$

Розмістимо корені z_k, а отже і β_k, в порядку зростання їх дійсних частин. Кожному кореню z_k відповідає окремий розв'язок (5), (11), в який входить довільний комплексний коефіцієнт g_k.

2.1. Знаходження НДС півбезмежного циліндра методом моментів. У цьому випадку функція Лява, яка визначає НДС суцільного циліндра, набуває вигляду

$$L(x,R\gamma)=L_0(x,R\gamma)+\sum_{k=1}^{\infty}Re\{g_k\varphi_k(z_k\gamma)\exp(-\beta_k x)\}, \tag{12}$$

де $\gamma=\frac{r}{R}$, $\varphi_k(z_k\gamma)=\varsigma_k J_0(z_k\gamma)+R\gamma J_1(z_k\gamma)$. Комплексні коефіцієнти g_k знайдемо з відомих граничних умов. Припустимо, що на торці циліндра задано нормальні та дотичні напруження, які можна подати у вигляді

$$\sigma(\gamma) = \sigma_0 + f_1(\gamma) = \sigma_0 + \sum_{j=0}^{\infty} a_j \gamma^{2j}, \ \tau_g(\gamma) = f_2(\gamma) = \sum_{j=0}^{\infty} b_j \gamma^{2j+1}, \ 0 \leqslant \gamma \leqslant 1, \quad (13)$$

де $\sigma_0 \neq 0$ - задає рівномірний стиск-розтяг циліндра. Функції $f_1(\gamma)$, $f_2(\gamma)$ повинні задовольняти коректні фізичні умови

$$\int_0^1 \gamma f_1(\gamma)\, d\gamma = 0, \ f_2(1) = 0, \ |f_1(\gamma)| < \infty, \ |f_2(\gamma)| < \infty, \ \gamma \in [0,1]. \quad (14)$$

Зображення (13) не накладає обмежень на значення прикладених зовнішніх дотичних і нормальних зусиль, оскільки довільну функцію інтегровну на проміжку [0, 1] можна апроксимувати з заданою точністю парними та непарними поліномами.

Для знаходження невідомих коефіцієнтів g_k використаємо метод моментів [11] разом з методом найменших квадратів. Врахувавши співвідношення (11), (12), (13), подамо граничні умови на торці циліндра у вигляді

$$f_1(\gamma) = \sum_{j=0}^{\infty} a_j \gamma^{2j} = \sum_{k=1}^{\infty} Re\,[c_k \chi_k(\gamma)],$$

$$f_2(\gamma) = \sum_{j=0}^{\infty} b_j \gamma^{2j+1} = \sum_{k=1}^{\infty} Re\,[c_k \psi_k(\gamma)], \quad (15)$$

де $\chi_k(\gamma) = (m_k - 2) J_0(z_k\gamma) + z_k\gamma J_1(z_k\gamma)$, $\psi_k(\gamma) = m_k J_1(z_k\gamma) - z_k\gamma J_0(z_k\gamma)$, $c_k = x_k + iy_k = \frac{z_k^2}{R^2} g_k$, x_k, y_k - визначають дійсну й уявну частину невідомого комплексного коефіцієнта c_k, i - уявна одиниця. Виділимо у функцій $\chi_k(\gamma)$, $\psi_k(\gamma)$ дійсну й уявну частину $\chi_k(\gamma) = \chi_{rk}(\gamma) + i\chi_{yk}(\gamma)$, $\varphi_k(\gamma) = \psi_{rk}(\gamma) + i\psi_{yk}(\gamma)$.

З наслідку теореми 3 випливає, що система функцій $\{\chi_k(\gamma)\}$, $k = 0, 1, 2, ...$ є повною на відрізку $\gamma \in [0,1]$, а $\{\psi_k(\gamma)\}$, $k = 1, 2, 3, ...$ - повною в класі аналітичних функцій, які дорівнюють нулю при $\gamma = 0$, $\gamma = 1$. Це відповідає обмеженням (14) на граничні навантаження.

Розглянемо інтеграл квадратичного відхилення шуканого розв'язку від заданих зовнішніх граничних навантажень на торці циліндра

$$\Phi\{x_1, ...x_N, y_1, ...y_N\} = \int_0^1 \gamma \left\{ \left\{ \sum_{k=1}^{N} [x_k \chi_{rk}(\gamma) - y_k \chi_{yk}(\gamma)] - f_1(\gamma) \right\}^2 + \right.$$

$$\left. + \left\{ \sum_{k=1}^{N} [x_k \psi_{rk}(\gamma) - y_k \psi_{yk}(\gamma)] - f_2(\gamma) \right\}^2 \right\} d\gamma, \quad (16)$$

де N задає кількість членів ряду (15). Комплексні коефіцієнти c_k, визначимо з умови мінімуму функціонала (16). Для його мінімізації знайдемо частинні похідні $\frac{\partial \Phi\{x_1, ..., x_N, y_1, ..., y_N\}}{\partial x_j}$, $\frac{\partial \Phi\{x_1, ..., x_N, y_1, ..., y_N\}}{\partial y_j}$, $j = \overline{1, N}$ і прирівняємо їх до нуля.

Після громіздких обчислень інтегралів одержимо систему $2N$ лінійних рівнянь для визначення $2N$ невідомих x_k, y_k, $k=\overline{1,N}$ у вигляді

$$\sum_{k=1}^{N}\left\{x_k B_{k,j}^1 - y_k B_{k,j}^2\right\} = \int_0^1 \gamma\left[f_1(\gamma)\,\chi_{rj}(\gamma) + f_2(\gamma)\,\psi_{rj}(\gamma)\right],$$

$$\sum_{k=1}^{N}\left\{x_k B_{k,j}^3 - y_k B_{k,j}^4\right\} = \int_0^1 \gamma\left[f_1(\gamma)\,\chi_{yj}(\gamma) + f_2(\gamma)\,\psi_{yj}(\gamma)\right],\ j=\overline{1,N}, \tag{17}$$

де $B_{k,j}^1 = Re\,(N_{k,j})$, $B_{k,j}^2 = Im\,(N_{k,j})$, $B_{k,j}^3 = -Im\,(A_{k,j})$, $B_{k,j}^4 = Re\,(A_{k,j})$,

$$2N_{k,j} = D_{k,j}(z_k, z_j) + D_{k,j}(z_k, \overline{z}_j) + M_{k,j}(z_k, z_j) + M_{k,j}(z_k, \overline{z}_j),$$

$$2A_{k,j} = -D_{k,j}(z_k, z_j) + D_{k,j}(z_k, \overline{z}_j) - M_{k,j}(z_k, z_j) + M_{k,j}(z_k, \overline{z}_j).$$

Коефіцієнти $M_{k,j}(z_k, z_j)$, $D_{k,j}(z_k, z_j)$, $k=\overline{1,N}$, $j=\overline{1,N}$ для значень індексів $k \neq j$ знаходять за формулами

$$M_{k,j}(z_k, z_j) = m_k m_j F_{1,1}(z_k, z_j) - m_k z_j G(z_j, z_k) - m_j z_k G(z_k, z_j) + z_k z_j F_{0,3}(z_k, z_j),$$

$$D_{k,j}(z_k, z_j) = Q_k Q_j F_{0,1}(z_k, z_j) + Q_k z_j G(z_k, z_j) + Q_j z_k G(z_j, z_k) +$$

$$+ z_k z_j F_{1,3}(z_k, z_j),\quad F_{0,1}(z_k, z_j) = W(z_k, z_j)\left[z_k J_1(z_k)\, J_0(z_j) - z_j J_1(z_j)\, J_0(z_k)\right],$$

$$G(z_k, z_j) = W(z_k, z_j)\left[z_k J_1(z_k)\, J_1(z_j) + z_j J_0(z_j)\, J_0(z_k) - 2z_j F_{0,1}(z_j, z_k)\right],$$

$$F_{1,3}(z_k, z_j) = W(z_k, z_j)\left[2z_k G(z_k, z_j) - 2z_j G(z_j, z_k) - z_k J_0(z_k)\, J_1(z_j) +\right.$$

$$\left. + z_j J_0(z_j)\, J_1(z_k)\right],\quad W(z_k, z_j) = \frac{1}{z_k^2 - z_j^2},\quad Q_k = m_k - 2,$$

$$F_{0,3}(z_k, z_j) = \frac{1}{z_k}\left[z_j F_{1,3}(z_k, z_j) + J_1(z_k)\, J_0(z_j) - 2G(z_j, z_k)\right],$$

$$F_{1,1}(z_k, z_j) = W(z_k, z_j)\left[z_k J_2(z_k)\, J_1(z_j) - z_j J_2(z_j)\, J_1(z_k)\right].$$

Коефіцієнти $D_{k,k}(z_k, z_k)$, $M_{k,k}(z_k, z_k)$, $k=\overline{1,N}$ знаходять за формулами

$$6D_{k,k}(z_k, z_k) = 3Q_k^2\left[J_1^2(z_k) + J_0^2(z_k)\right] + z_k^2\left[J_1^2(z_k) + J_2^2(z_k)\right] + 6Q_k J_1^2(z_k),$$

$$12M_{k,k}(z_k, z_k) = 6m_k^2\left[J_1^2(z_k) - J_0(z_k)\, J_2(z_k)\right] - 12m_k J_1^2(z_k) +$$

$$+ z_k^2\left[2J_1^2(z_k) + 3J_0^2(z_k) - J_2^2(z_k)\right].$$

Як бачимо, інтегральні коефіцієнти $B_{k,j}^m$, $k=\overline{1,N}$, $j=\overline{1,N}$, $m=\overline{1,4}$ у лівій частині системи рівнянь (17) знайдено в явному вигляді через функції Бесселя, а в правій частині – інтеграли від відомих функцій. Розв'яжемо цю систему $2N$ лінійних рівнянь, одержимо дійсні й уявні частини комплексних коефіцієнтів c_k, $k=\overline{1,N}$. Знайдемо шукані комплексні коефіцієнти $g_k = \frac{R^2}{z_k^2}c_k$. Далі за поданнями (1), (12) знайдемо НДС суцільного циліндра.

2.2. Знаходження НДС обмеженого суцільного циліндра. Розглянемо циліндр скінченної висоти H, навантажений з двох боків довільними нормальними та дотичними напруженнями з однаковим нормальним зусиллям $T_x = S\sigma_0$

$$\sigma_N(B_m,\gamma) = \sigma_0 + \sum_{j=0}^{N-1} a_j^m \gamma^{2j}, \quad \tau_g(B_m,\gamma) = \sum_{j=0}^{N-1} b_j^m \gamma^{2j+1}, \tag{18}$$

де індекс $m = 0$ відповідає нижньому торцю, $m = 1$ - верхньому торцю, $B_0 = 0$, $B_1 = \frac{H}{R}$, а функції $\sigma_N(B_m,\gamma)$, $\tau_g(B_m,\gamma)$, $m = 0,1$ стосовно змінної γ задовольняють умови, аналогічні (14). Функція Лява для скінченного циліндра має зображення

$$L = L_0(x, R\gamma) + \sum_{k=1}^{N} Re\{[\varsigma_k J_0(z_k\gamma) + rJ_1(z_k\gamma)][g_k e^{(-\beta_k x)} + q_k e^{(\beta_k x)}]\}. \tag{19}$$

Для знаходження невідомих g_k, q_k використаємо метод розкладення граничних умов в ряд за степенями змінної γ. Врахувавши зображення (6), (19) і зовнішні навантаження (18), після перетворень одержимо граничні умови у вигляді двох рівнянь

$$\sum_{k=1}^{N} Re\{\beta_k^2 [g_k \exp(-z_k B_m) - q_k \exp(z_k B_m)] [(m_k - 2) J_0(z_k\gamma) + z_k\gamma J_1(z_k\gamma)]\} =$$

$$= \sum_{j=0}^{N-1} a_j^m \gamma^{2j}, \qquad \sum_{k=1}^{N} Re\{\beta_k^2 [g_k \exp(-z_k B_m) +$$

$$+ q_k \exp(z_k B_m)] [m_k J_1(z_k\gamma) - z_k\gamma J_0(z_k\gamma)]\} = \sum_{j=0}^{N-1} b_j^m \gamma^{2j+1}, \tag{20}$$

де $m = 0,1$. Прирівнявши в рівняннях (20) коефіцієнти при однакових степенях змінної γ, одержимо чотири системи по N лінійних рівнянь для визначення невідомих комплексних коефіцієнтів g_k, q_k

$$\sum_{k=1}^{N} Re\left\{z_k^{2j+3}(m_k - 2 - 2j)[g_k \exp(-z_k B_m) + q_k \exp(z_k B_m)]\right\} = 2(j+1)A_j b_j^m,$$

$$\sum_{k=1}^{N} Re\left\{z_k^{2j+2}(m_k - 2 - 2j)[g_k \exp(-z_k B_m) - q_k \exp(z_k B_m)]\right\} = A_j a_j^m, \tag{21}$$

де $j = \overline{0, N-1}$, $m = 0,1$, $A_j = (-1)^j 2^{2j} j! j! R^2$.

Розв'яжемо систему лінійних рівнянь (21) і визначимо невідомі g_k, q_k. Далі за формулами (1), (19) знайдемо НДС циліндра заданої висоти H.

3. Знаходження НДС півбезмежного товстостінного циліндра. Введемо безрозмірні змінні $2R = R_1 + R_2$, $2h = R_2 - R_1$, $\varepsilon R = h$, $r = R\gamma$, $z = \beta R$, $\beta R_j = z\alpha_j$, $\alpha_j = 1 + (-1)^j \varepsilon$, $j = 1,2$. Після громіздких перетворень подамо умову рівності нулю визначника системи рівнянь (8) у вигляді трансцендентного рівняння стосовно комплексного спектрального параметра z

$$\left(\alpha_1^2 z^2-\delta\right)\left(\alpha_2^2 z^2-\delta\right) B_{11}(z)+\alpha_1^2 \alpha_2^2 z^4 B_{00}(z)+\left(\alpha_1^2 z^2-\delta\right) \alpha_2^2 z^2 B_{10}(z)+$$

$$+\left(\alpha_2^2 z^2-\delta\right) \alpha_1^2 z^2 B_{01}(z)-\frac{4 z^2}{\pi^2}\left(\alpha_1^2+\alpha_2^2\right)+\frac{8 \delta}{\pi^2}=0, \tag{22}$$

де

$$B_{kj}(z)=\left[J_k\left(\alpha_1 z\right) N_j\left(\alpha_2 z\right)-J_k\left(\alpha_2 z\right) N_j\left(\alpha_1 z\right)\right]^2, \ k=0,1, \ j=0,1.$$

Безпосереднім обчисленням можна перевірити, що для функцій $B_{kk}(z)$, $z^2 B_{01}(z)$, $z^2 B_{10}(z)$ точка $z=0$ є регулярною. Отже, характеристичне рівняння (22) визначає цілу функцію, яка має нескінченну кількість нулів z_k, $k=1,2,3\ldots$.

Підставивши зображення (5) для кожного значення z_k в граничні умови (8), (аналогічно як для суцільного циліндра), виразимо невідомі коефіцієнти a_k, c_k, b_k через g_k у вигляді $b_k=R\varsigma_{k,1}g_k$, $c_k=\varsigma_{k,2}g_k$, $a_k=R\varsigma_{k,3}g_k$. Функція Лява в цьому випадку має зображення

$$L(x,R\gamma)=L_0(x,R\gamma)+\sum_{k=1}^{\infty} Re\{Rg_k[\gamma J_1(z_k\gamma)+$$

$$+\varsigma_{k,1}J_0(z_k\gamma)+\varsigma_{k,2}\gamma N_1(z_k\gamma)+\varsigma_{k,3}N_0(z_k\gamma)]\exp\left(\frac{-z_k x}{R}\right)\}. \tag{23}$$

3.1. Розкладення граничних умов у ряди за степенями радіальної змінної γ. Нехай на торці циліндра задані відомі зовнішні нормальні та дотичні зусилля, які задовольняють відповідні для товстостінного циліндра умови (14), де враховано, що $\alpha_1\leqslant\gamma\leqslant\alpha_2$. Граничні умови на торці циліндра в загальному випадку мають вигляд $\sigma_x(0,\gamma)=\sigma_0+\sum_{j=0}^{N-1}a_j\gamma^{2j}$, $\tau(0,\gamma)=\sum_{j=0}^{N-1}b_j\gamma^{2j+1}$, де $\alpha_1\leqslant\gamma\leqslant\alpha_2$, N задає точність апроксимації граничних умов і кількість членів ряду (23). Щоб розкласти в ряд за степенями γ функцію Лява (23), потрібно апроксимувати $\ln(\gamma)$ на проміжку $\alpha_1\leqslant\gamma\leqslant\alpha_2$ поліномом стосовно γ. Задовольнимо ці граничні умови з врахуванням подань (1), (23) і прирівняємо коефіцієнти при однакових степенях змінної γ, після нескладних перетворень одержимо систему $2N$ лінійних рівнянь для визначення невідомих g_k, $k=\overline{1,N}$

$$\sum_{k=1}^{N} Re\left\{z_k^2 g_k\left\{B_j^1 z_k^{2j}\left(z_k\varsigma_{k,1}-\delta_j^0-2(j+1)\right)+z_k\varsigma_{k,2}c_{j,k}^0+\left(z_k\varsigma_{k,3}-\delta_j^0\varsigma_{k,2}\right)c_{j,k}^1\right\}\right\}=$$

$$=R^2 b_j, \qquad \sum_{k=1}^{N} Re\{z_k^2 g_k\{2(j+1)B_j^1 z_k^{2j}(z_k\varsigma_{k,1}-\alpha)+z_k^{2j}B_{j-1}^1+$$

$$+z_k\varsigma_{k,2}c_{j-1,k}^1+\left(z_k\varsigma_{k,3}-\alpha\varsigma_{k,2}\right)c_{j,k}^0\}\}=R^2 a_j, \ j=\overline{0,N-1}, \tag{24}$$

де

$$c_{j,k}^1=\pi_1 z_k^{2j-1}\left[z_k^2 B_j^1\left(f_k-d_j-j_1\right)+c_2 B_{j-1}^1-z_k^{-2}v_1\delta_j^1\right], \ \pi_1=\frac{2}{\pi}, \ j_1=(2j+2)^{-1},$$

$$c_{j,k}^{0} = \pi_1 z_k^{2j-2}\left[z_k^2 B_j^0 (C-d_j) + c_2 B_{j-1}^0\right],\ j=\overline{1,N-1},\ c_{0,k}^0 = \pi_1 C,$$
$$4c_{0,k}^{1} = \pi_1\left[z_k(2C-1) - 2\pi_1 v z_k^{-1}\right],\ v_1 = -\left(1-\varepsilon^2\right)^{-2},\ v = -2\left(1+\varepsilon^2\right)v_1,$$
$$f_k = \ln\left[(1+\varepsilon)z_k\right] + \frac{c_2}{v_1},\ c_2 = \frac{1}{4\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon},\ B_k^i = \frac{(-1)^k}{2^{2k+i}(k+i)!k!},\ d_j = \sum_{n=1}^{j}\frac{1}{n},$$

$\alpha = 2(2-\nu)$, $c_{-1,k}^1 = 0$, δ_j^1 - символ Кронекера, C - постійна Ейлера-Маскароні [8].

Розв'яжемо систему лінійних рівнянь (24) і визначимо невідомі g_k, $k=\overline{1,N}$. Далі за зображеннями (1), (23) знайдемо НДС товстостінного циліндра. Як і для суцільного циліндра скінченої висоти H можна розрахувати НДС товстостінного циліндра висотою H .

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SPECTRAL OF AXIALLY SYMMETRIC PROBLEMS OF ELASTICITY THEORY

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A presentation of the stressed-strained state (SSS) for a cylinder, end-loaded arbitrary of axially symmetrically, is obtained in the form of series in own functions. It is shown that spectral values will be complex conjugate numbers and there will be an infinite number of them. The series coefficients are defined from the condition of integrals minimum of deviation quadratic for the solution from the given boundary conditions on the end. Two numerical methods for calculation of the series coefficients are proposed. A solution of the boundary problem for a biharmonic equation is found in cylindrical coordinate system.

Key words: biharmonic equation, eigenvalues, eigenfunctions, problem of moments, entire functions, axially symmetric elasticity theory, stressed-strained state, cylinder.

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ТРИ ЗАДАЧІ ПРО СТАНОВО ЗАМКНЕНІ ГРУПИ АВТОМОРФІЗМІВ ОДНОРІДНОГО КОРЕНЕВОГО ДЕРЕВА

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Сформульовано три проблеми про такі зображення груп автоморфізмів кореневого дерева, образ яких є станово замкненою групою автоморфізмів.

Ключові слова: **кореневе дерево, група автоморфізмів, група станово замкнених автоморфізмів.**

1. Кореневим називається дерево з виділеною вершиною – його коренем. Множина вершин дерева (T, x_0), що перебувають на заданій відстані k (у природній метриці симпліціального графа) називається сферою радіуса k. Коренева вершина x_0 утворює сферу радіуса 0. Дерево (T, x_0) називається сферично однорідним, якщо степені всіх вершин кожної зі сфер однакові. Це означає, що кожна вершина k сфери $(k \geqslant 0)$ дерева (T, x_0) з'єднана з тим самим числом n_k вершин $(k+1)$-і сфери. Кожне нескінченне сферично однорідне кореневе дерево однозначно, з точністю до ізоморфізму, визначається послідовністю чисел $< n_1, n_2, \ldots >$, яка називається його сферичним типом [1]. Кореневе дерево T називається однорідним, якщо його сферичний тип має вигляд $< n, n, \ldots >$ для деякого $n \in \mathbb{N}$. Число n називається сферичним індексом однорідного дерева. Для кожного натурального n з точністю до ізоморфізму кореневих дерев існує лише одне однорідне кореневе дерево сферичного індексу n, яке позначатимемо T_n. Для довільної вершини y дерева T_n символом $T(y)$ позначимо кореневе піддерево дерева T_n з коренем y. Зрозуміло, що T_n і $T(y)$ ізоморфні як кореневі дерева.

2. Автоморфізмом кореневого дерева називається таке бієктивне перетворення множини його вершин, яке зберігає інцидентність вершин і залишає нерухомим корінь дерева. Кожен автоморфізм кореневого дерева T_n переставляє між собою лише вершини, що належать до однієї сфери, причому з'єднані з однією й тією вершиною сфери меншого радіуса. Тому автоморфізм f однозначно визначається вершино поміченим деревом D_f (Рис.), яке називається [2] портретом автоморфізму f.

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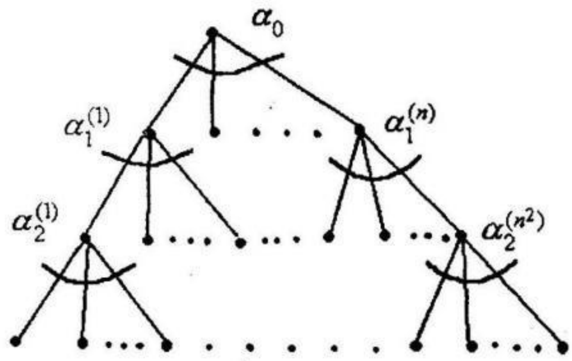
Для довільної вершини u дерева D_f помічене кореневе піддерево $D_f(u)$ з коренем u також є портретом деякого автоморфізму f_u дерева T_n. Цей автоморфізм називається [2] u-им станом автоморфізму f. Отже, кожному автоморфізму f дерева T_n відповідає множина R_f найможливіших станів цього автоморфізму.



Автоморфізм f називається скінченно становим, якщо $|R_f| < \infty$. Як відомо, автоморфізми дерева T_n можна інтерпретувати як автоматні підстановки над n-елементним алфавітом, які задаються автоматами Мілі [3]. При такій інтерпретації скінченно становим автоморфізмам дерева T_n відповідають скінченно автоматні підстановки і навпаки.

3. Нехай $AutT_n$ — група автоморфізмів дерева T_n. Підгрупа $G < AutT_n$ називається становo замкненою [2], якщо разом з кожним автоморфізмом містить всі його стани. Сама $AutT_n$, її підгрупа скінченно станових автоморфізмів, підгрупа автоморфізмів, які нетривіально діють лише на якомусь початку дерева, є прикладами становo замкнених підгруп. Серед інших відомих прикладів — групи Григорчука [3], групи Гупта-Сідкі [4],[5] та багато інших. Широкий клас груп, що занурюються в $AutT_n$, описують таким твердженням.

Теорема. *Нехай G — група зі спадним субнормальним рядом підгруп*

$$G = G_0 > G_1 > G_2 > \dots$$

таким, що $\cap_{i=0}^{\infty} = \{1\}$ і G_i/G_{i+1} ізоморфно занурюється в симетричну групу S_n для всіх $n = 0, 1, 2, \dots$. Тоді група G ізоморфно занурюється в $AutT_n$.

Доведення теореми випливає з того, що для такої групи G природно конструюється дія на ультраметричному просторі, який збігається з простором кінців однорідного дерева сферичного індекса n (див [6]).

Проте поки що зовсім незрозуміло, які групи можна занурити в $AutT_n$ так, щоб їх образи були групами скінченно станових ізоморфізмів T_n. У багатьох випадках такі занурення невідомі для конкретних груп, хоча питання про їх існування цікаве з багатьох поглядів. Тому ми сформулюємо такі задачі.

1. Чи зображається точно вінцевий добуток кратності m

$$A_m = \underbrace{\mathbb{Z} \wr \mathbb{Z} \wr \dots \wr \mathbb{Z}}_{m}$$

нескінченної циклічної групи $\mathbb{Z}$ на себе як скінченно станова група автоморфізмів дерева T_n при $n \geqslant 2$?

2. Побудувати точні станово замкнені зображення вільної групи F_2 рангу 2 автоморфізмами дерева T_n, $n \geqslant 2$. Зображення F_2 автоморфізмами дерева T_n сконструйовано для всіх $n \geqslant 2$. Ці зображення не є станово замкненими (див., напр., [7]).
3. Побудувати мінімальну за включенням станово замкнену групу автоморфізмів дерева T_n.

Ці задачі, на наш погляд, можуть пояснити саму природу скінченної становості, яка поки що залишається малозрозумілою.

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THREE PROBLEMS ON STATED CLOSED AUTOMORPHISM GROUP OF HOMOGENEOUS ROOTED TREE

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Here are formulated three problems in this paper on such representations of automorphism groups of the rooted tree that their images are the stated closed automorphism groups.

Key words: rooted tree, automorphism group, stated closed automorphism group.

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