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LAWSON MONADS AND PROJECTIVITY

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We show that power monads are not Lawson. The power monads are the most natural examples of projective monads. However there exist projective monads which are Lawson.
Key words: Lawson monad, projective monad.

0. The algebraic aspect of the theory of functors in categories of topological spaces and continuous maps was inswestigated rather recently. It is based, mainly, on the existence of monad (or triple) structure in the sense of S.Eilenberg and J.Moore [1].

Many classical constructions lead to monads: hyperspaces, spaces of probability measures, superextensions etc. There were many investigations of monads in categories of topological spaces and continuous maps(see for example the survey [2]). But it seems that the main difficulty to obtain general results in the theory of monads is the different nature of functors.

Some functional representations of the hyperspace functor were found in [3] and [4]. There was introduced a class of Lawson monads in [5] which contains sufficiently wide class of monads. Lawson monads have a functional representation, i.e., their functorial part FX can be naturally imbedded in $\mathbb{R}^{CX}$. In this paper we investigate connection between the classes of Lawson and projective monads.

The paper is arranged in the following manner. In 1 we prove that the power monads are not Lawson. In 2 we introduce an example of Lawson monad which is projective.

1. By *Comp* we denote the category of compact Haussdorff spaces (compacta) and continuous maps.

We need some definitions concerning monads and algebras. A *monad* $\mathbb{T} = (T, \eta, \mu)$ in a category $\mathcal{E}$ consists of an endofunctor $T : \mathcal{E} \to \mathcal{E}$ and natural transformations $\eta : \mathrm{Id}_{\mathcal{E}} \to T$ (unity), $\mu : T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$.

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in a category $\mathcal{E}$. The pair (X, ξ) is called a $\mathbb{T}$-*algebra* if $\xi \circ \eta X = id_X$ and $\xi \circ \mu X = \xi \circ T\xi$. Let (X, ξ), (Y, ξ') be two $\mathbb{T}$-algebras. A map $f : X \to Y$ is called a $\mathbb{T}$-algebras morphism if $\xi' \circ Tf = f \circ \xi$.

For any real $t \geqslant 0$,we denote by I_t the segment $[-t, t]$. If t_1, t_2 are real numbers with $0 \leqslant t_1 \leqslant t_2$, by $j_{t_1}^{t_2}$ we denote the natural embedding $j_{t_1}^{t_2} : I_{t_1} \to I_{t_2}$.

Let $\mathbb{F} = (F, \eta, \mu)$ be a monad in the category *Comp*. A family of $\mathbb{F}$-algebras $\{\xi_t : FI_t \to I_t \mid t \geqslant 0\}$ we call *coherente* iff for each $t_1, t_2 \in \mathbb{R}$ with $0 \leqslant t_1 \leqslant t_2$ the embedding $j_{t_1}^{t_2}$ is an $\mathbb{F}$-algebras morphism. A monad $\mathbb{F} = (F, \eta, \mu)$ is called *Lawson* if there exists a coherente family of $\mathbb{F}$-algebras $\{\xi_t : FI_t \to I_t \mid t \geqslant 0\}$ such that for each $X \in Comp$ there exists a point-separating family of $\mathbb{F}$-algebras morphisms $\{f_\alpha : (FX, \mu X) \to (I_{t(\alpha)}, \xi_{t(\alpha)}) \mid \alpha \in A\}$[5]. We will need the following lemma from [6].

Lemma A[6]. *Let $\mathbb{F} = (F, \eta, \mu)$ be a monad in a category $\mathbb{S}$ and X is an object of $\mathbb{S}$. Let $f, g : (FX, \mu) \to (Y, \xi)$ be $\mathbb{F}$-algebras morphism with $f \circ \eta X = g \circ \eta X = h$. Then $f = g = \xi \circ Fh$.*

For $X \in Comp$ and $n \in \mathbb{N}$ by $D_n X$ we denote the compactum X^n. For a map $f : X \to Y$ we define the map $D_n f : D_n X \to D_n Y$ by the rule $D_n f(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$. One can check that D_n is a covariant functor on $Comp$.

For $X \in Comp$ define the maps $\gamma X : X \to D_n X$ and $\mu X : D_n^2 X \to D_n X$ by the formulas $\gamma X(x) = (x, \dots, x)$ for $x \in X$ and $\mu X((x_1^1, \dots, x_n^1), \dots, (x_1^n, \dots, x_n^n)) = (x_1^1, \dots, x_n^n)$ for $((x_1^1, \dots, x_n^1), \dots, (x_1^n, \dots, x_n^n)) \in D_n^2 X = D_n(D_n X)$.

It is known that the triple $\mathbb{D}_n = (D_n, \eta, \mu)$ is a monad in the category $Comp$, where $\eta = \{\eta X\} : \mathrm{Id}_{Comp} \to D_n$ and $\mu = \{\mu X\} : D_n^2 \to D_n$ are the natural transformations defined above [2]. This monad is called the power monad.

Lemma 1. *Let $\{(I_t, \xi_t) \mid t \geqslant 0\}$ be a coherent family of the power monad $\mathbb{D}_n$. Then there exists $k \in \{1, \dots, n\}$ such that $\xi_t = p_k : I_t^n \to I_t$ for each $t \geqslant 0$.*

Proof. Consider any $t > 0$. Let us remark that the following equalities follow from the definition of natural transformations η, μ and from definition of $\mathbb{D}$-algebras: $\xi_t(l, \dots, l) = l$ and $\xi_t(\xi_t(x_1^1, \dots, x_1^n), \dots, \xi_t(x_n^1, \dots, x_n^n)) = \xi_t(x_1^1, \dots, x_n^n)$. Define $a_i \in I_t^n$ by the rule $a_i^i = t$ and $a_i^j = 0$ for each $j \neq i$. Let us show that there exists $k \in \{1, \dots, n\}$ such that $\xi_t(a_k) \in \{-t, t\}$. Assume the contrary. Then we can choose $q < t$ such that $\xi_t(a_i) \in I_q$ for each $k \in \{1, \dots, n\}$. Since the embedding $j_q^t : I_q \to I_t$ is a $\mathbb{D}_n$-algebras morfism, we have that $\xi_t(\xi_t(a_1), \dots, \xi_t(a_n)) = \xi_q(\xi_t(a_1), \dots, \xi_t(a_n))$. On the other hand $\xi_t(\xi_t(a_1), \dots, \xi_t(a_n)) = \xi_t(t, \dots, t) = t \notin I_q$. Hence we obtain the contradiction.

Assume that $\xi_t(a_k) = -t$. For each $l \in [-t, t]$ consider the point $a_k(l) \in I_t^n$ defined by $a_k^k(l) = t$ and $a_k^i(l) = l$ for $i \neq k$. The function $f : [0, t] \to [-t, t]$ defined by the formula $f(l) = \xi_t(a_k(l))$ is continuos. We have that $f(0) = -t$ and $f(t) = t$. There exists $s \in [0, t]$ such that $\xi_t(a_k(s)) = 0$. Then we have $a_k^k = \xi_t(t, \dots, t) = t$ and $a_k^i = \xi_t(a_k(s)) = 0$ for $i \neq k$. Thus $\xi_t(a_k) = \xi_t(a_k(s)) = 0$ and we obtain a contradiction. Hence $\xi_t(a_k) = t$.

It is also easy to check that $\xi_t(a_k(l)) > 0$ for each $l \in [-t, t]$ using the above reasoning. Assume that there exists $l \in [-t, t]$ with $\xi_t(a_k(l)) = s < t$. The above remark implies that $s > 0$. We have $\xi_t(a_k(s)) = s$. For each $l \in [-t, t]$ consider the point $b_k(l) \in I_t^n$ defined by $b_k^k(l) = l$ and $b_k^i(l) = t$ for $i \neq k$. Since $b_k^i(s) = \xi_t(t, \dots, t) = t$ for $i \neq k$ and $b_k^k(s) = \xi_t(a_k(s))$, we have that $\xi_t(b_k(s)) = t$.

Put $s_1 = \inf\{l \in [-t, s] \mid \xi_t(a_k(p)) < t$ for each $p \in [l, s]\}$. Then we have $s_1 \in [0, s)$, $\xi_t(a_k(s_1)) = t$ and $\xi_t(a_k(l)) < t$ for each $l \in (s_1, s]$. Consider the point $c(l) \in I_t^n$ defined by $c^k(l) = \xi_t(a_k(l))$ and $c^i(l) = \xi_t(b_k(l))$ for $i \neq k$. Since $\xi_t(c(l)) = t$ and $\xi_t(a_k(l)) < t$, we have that $\xi_t(b_k(l)) \in \{-t, t\}$ for each $l \in (s_1, s]$. Since $(s_1, s]$ is connected and $\xi_t(b_k(s)) = t$, we have that $\xi_t(b_k(l)) = t$ for each $l \in (s_1, s]$. But $\xi_t(b_k(s_1)) = s_1 \neq t$ and we obtain a contradiction with continuity of the function $g : [-t, t] \to [-t, t]$ defined by $g(l) = \xi_t(b_k(l))$. Hence $\xi_t(a_k(l)) = t$ for each $l \in [-t, t]$. We also have that $\xi_t(b_k(l)) = l$ for each $l \in [-t, t]$.

Consider any point $x = (x_1, \dots, x_n) \in I_t^n$. Since $\xi_t(b_k(l)) = l$ for each $l \in [-t, t]$, we have that $\xi_t(x) = \xi_t(\xi_t(b_k(x^1)), \dots, \xi_t(b_k(x^n)) = \xi_t(b_k(x^k)) = x^k$. We have shown that $\xi_t = p_n : I_t^n \to I_t$. Since $j_{t_1}^{t_2} : I_{t_1} \to I_{t_2}$ is a $\mathbb{D}_n$-algebras morfism, we have that $\xi_t = p_k$ for each $t \geqslant 0$. The lemma is proved.

Teorema 1. *The monad $\mathbb{D}_n$ is not a Lawson monad for each $n > 1$.*

Proof. Consider any $n > 1$ and any coherent family $\{(I_t, \xi_t) \mid t \geqslant 0\}$ of $\mathbb{D}_n$-algebras. Then, by Lemma 1, there exists $k \in \{1, \dots, n\}$ such that $\xi_t = p_k$ for each $t \geqslant 0$. Let X be any compactum with $|X| \geqslant 2$ and $x_1, x_2 \in X$ with $x_1 \neq x_2$. Consider any $l \in \{1, \dots, n\}$ such that $l \neq k$. Consider $y_1, y_2 \in X^n$ such that $y_1^k = y_2^k$ and $y_1^l = x_1 \neq x_2 = y_2^l$. By Lemma A, each $\mathbb{D}_n$-algebras morfism $f : (D_n X, \mu X) \to (I_t, \xi_t)$ can be represented as follows $f = \xi_t \circ D_n(f \circ \eta X)$. Then we have $f(y_1) = p_k \circ (f \circ \eta X)^n(y_1) = f(y_1^k, \dots, y_1^k) = f(y_2^k, \dots, y_2^k) = p_k \circ (f \circ \eta X)^n(y_2) = f(y_2)$. Hence we can not separate distinct points $y_1, y_2 \in X^n$ by a $\mathbb{D}_n$-algebras morfism. The theorem is proved.

2. In this section we introduce an example of a Lawson monad which is projective.

For $X \in \mathit{Comp}$ we put $SX = X \times \{0, 1\}$. If $f : X \to Y$ is a continuous map, define the map $Sf : SX \to SY$ by the formula $Sf(x, l) = (f(x), l)$. One can check that S is a covariant functor in the category Comp. By S^2X we denote the iteration $S(SX)$of the functor S.

For $X \in \mathit{Comp}$ define maps $\eta X : X \to SX$ and $\mu X : S^2X \to SX$ by the formulas $\eta X(x) = (x, 0)$ and $\mu X(x, l, m) = (x, \max\{l, m\})$. It is easy to check that ηX and μX are the components of natural transformations $\eta : \mathrm{Id}_{\mathcal{E}} \to S$ and $\mu : S^2 \to S$.

Proposition 1. *The triple $\mathbb{S} = (S, \eta, \mu)$ forms a monad in the category Comp.*

Proof. Let $X \in \mathit{Comp}$ and $(x, l) \in X \times \{0, 1\}$. Then we have $\mu X \circ \eta SX(x, l) = \mu X(x, l, 0) = (x, l) = \mu X(x, 0, l) = \mu X \circ S\eta X(x, l)$.

For each $(x, l, m, n) \in X \times \{0, 1\}^3 = S^3X$ we have $\mu X \circ \mu SX(x, l, m, n) = \mu X(x, l, \max\{m, n\}) = (x, \max\{l, m, n\}) = \mu X(x, \max\{l, m\}, n) = \mu X \circ S\mu X(x, l, m, n)$. The proposition is proved.

Theorem 2. *The monad $\mathbb{S}$ is projective and Lawson.*

Proof. For $X \in \mathit{Comp}$ define the map $\pi X : SX \to X$ by the formula $\pi X(x, s) = x$ where $(x, s) \in SX$. For each $(x, i, j) \in S^2X$ we have $\pi X \circ \mu X(x, i, j) = x = \pi X \circ \pi FX(x, i, j) = \pi X \circ F\pi X(x, i, j)$. Evidently, πX are components of the natural transformation $\pi : S \to \mathrm{Id}_{Comp}$. Hence the monad $\mathbb{S}$ is projective.

Let us show that $\mathbb{S}$ is a Lawson monad. For $t \geqslant 0$ define the map $r_t : I_t \to I_t$ by the formula $r_t(s) = \min\{1, s\}$ if $s \geqslant 0$ and $r_t(s) = \max\{-1, s\}$ if $s \leqslant 0$. Now define the map $\xi_t : SI_t \to I_t$ by the formula $\xi_t(x, 0) = x$ and $\xi_t(x, 1) = r_t(x)$. We have $r_t \circ r_t = r_t$. For $x \in I_t$ we have $\xi_t \circ \eta I_t(x) = x$. Now consider $(x, l, m) \in S^2X$. If $l = m = 0$, we have $\xi_t \circ \mu I_t(x, 0, 0) = \xi_t \circ S\xi_t(x, 0, 0)$. In the case, when $l + m = 1$, we have $\xi_t \circ \mu I_t(x, l, m) = \xi_t(x, 1) = r_t(x) = \xi_t \circ S\xi_t(x, l, m)$. Finally, if $l = m = 1$, we have $\xi_t \circ \mu I_t(x, 1, 1) = \xi_t(x, 1) = r_t(x) = r_t \circ r_t(x) = \xi_t \circ S\xi_t(x, 1, 1)$. Hence (I_t, ξ_t) is an S-algebra for each $t \geqslant 0$.

Consider any $0 \leqslant t_1 \leqslant t_2$ and the natural embedding $j_{t_1}^{t_2} : I_{t_1} \to I_{t_2}$. Then we have that $j_{t_1}^{t_2} \circ \xi_{t_1}(x,0) = x = \xi_{t_2}(x,0) = \xi_{t_2} \circ Sj_{t_1}^{t_2}(x,0)$ and $j_{t_1}^{t_2} \circ \xi_{t_1}(x,1) = r_{t_1}(x) = r_{t_2}(x) = \xi_{t_2}(x,1) = \xi_{t_2} \circ Sj_{t_1}^{t_2}(x,1)$. Hence the family $\{(I_t, \xi_t) \mid t \geqslant 0\}$ is coherent.

Now consider (x,l), $(y,s) \in SX$ with $(x,l) \neq (y,s)$. If $x \neq y$, consider the map $f : X \to [-1,1]$ such that $f(x) \neq f(y)$. Then we have $\xi_1 \circ Sf(x,l) = f(x) \neq f(y) = \xi_1 \circ Sf(y,s)$. If $x = y$, then $\{l,s\} = \{0,1\}$. We can assume that $l = 0$ and $s = 1$. Consider the map $g : X \to [-2,2]$ such that $f(X) \subset \{2\}$. Then $\xi_2 \circ S(g)(x,0) = 2$ and $\xi_2 \circ S(g)(y,1) = 1$. Since $\xi_t \circ S(\varphi)$ is an $\mathbb{S}$-algebras morphism for each $\varphi \in C(X, I_t)$, we have proved that $\mathbb{S}$ is a Lawson monad.

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МОНАДИ ЛОУСОНА ТА ПРОЕКТИВНІСТЬ

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Показано, що степінь монад не є монадодою Лоусона. Степінь монад є найбільш природним прикладом проективних монад. Проте існують проективні монади, які є монадами Лоусона.

Ключові слова: монада Лоусона, проективна монада.

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