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FUNCTIONAL CALCULUS ON A WIENER TYPE ALGEBRA OF ANALYTIC FUNCTIONS OF INFINITY MANY VARIABLES

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For generators of isometric strong continuous operator groups, defined on nuclear Wiener algebras of analytic complex functions on a unit Banach ball, a functional calculus is constructed. Its symbol algebra consists of Fourier-images of exponential type distributions.
Key words: Functional calculus, Wiener type algebra, Banach ball, Fourier-image of exponential type distributions.

1. C_0-group $\hat{U}_t$ over the Wiener algebra W_π

Let X, X' be complex Banach reflexive space and its dual, respectively. By $\langle X \mid X' \rangle$ we denote the corresponding duality. We use the main notations and definitions from [1]. For every $F_n' \in X_\pi'^{\odot n}$ there exists [2] a unique n-homogeneous polynomials F_n such that

$$F_n(x) := \left\langle x^{\odot n} \mid F_n' \right\rangle \text{ for all } x \in X.$$

We denote by

$$\mathcal{P}_\pi^n(X) = \{F_n : F_n' \in X_\pi'^{\odot n}\}$$

the space of so-called nuclear n-homogeneous polynomials, where the complete symmetric tensor product $X_\pi'^{\odot n}$ with the projective norm $\|\cdot\|_\pi$ endowed. It follows from it the isomertry $\mathcal{P}_\pi^n(X)$ and $X_\pi'^{\odot n}$, so on $\mathcal{P}_\pi^n(X)$ we may define the following norm

$$\| F_n \| := \| F_n' \|_\pi, F_n' \in X_\pi'^{\odot n}.$$

Definition. The ℓ_1-sum

$$W_\pi := \{F = \sum_{n \geq 0} F_n : F_n \in \mathcal{P}_\pi^n(X)\}$$

with the norm $\| F \| = \sum \| F_n \|$ is called the nuclear Wiener type algebra.

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Similar to [3], it can be shown that W_π is a Banach algebra of bounded analytic functions in $B = \{x \in X : \| x \| < 1\}$.

It is known ([4], theorem 1) that for C_0-group of linear isometric operator $\mathbb{R} \ni t \mapsto U_t \in \mathcal{L}(X)$ the C_0-group

$$\hat{U}_t F(x) = F(U_t x), (x \in B) \text{ on } W_\pi$$

is contractive.

Applying the isometric property of $\hat{U}_t$ over W_π and [5, theorem 2] we obtain that its (infinitesimal) generator $\hat{A}$ is a closable *conservative* differentiation on W_π.

We will use the fact that the C_0-group $\hat{U}_t$ over the Wiener algebra W_π is well-defined and acted as follows

$$\hat{U}_t F = \sum_{n \geq 0} \hat{U}_t^{\odot n} F_n,\ F = \sum F_n \in W_\pi,$$

where $\hat{U}_t^{\odot n}$ is defined in [4, proposition 2] by the equality

$$\hat{U}_t^{\odot n} F_n(x) = \left\langle x^{\odot n} \mid U_t'^{\otimes n} F_n' \right\rangle \text{ for all } x \in X,$$

where $U_t'^{\otimes n} = \underbrace{U_{t'} \otimes \ldots \otimes U_{t'}}_{n}$ and $\left\langle U_t x \mid y \right\rangle = \left\langle x \mid U_t' y \right\rangle$ for all $x, y \in X$, U_t' is the adjoint group of U_t.

Let $\hat{A}$ be the generator of $\hat{U}_t$ of the form ([4, proposition 3])

$$\hat{A}F(x) = \sum_{n \in \mathbb{Z}_+} \left\langle x^{\odot n} \mid \sum_{j=0}^{n} A_j' F_n' \right\rangle, A_j' := \underbrace{I' \otimes \ldots \otimes I'}_{j-1} \otimes A' \otimes \underbrace{I' \otimes \ldots \otimes I'}_{n-j+1}, x \in B,$$

defined on a norm dense subspace $\mathcal{D}(\hat{A}) = \{F = \sum F_n : F_n' \in \mathcal{D}(A')^{\odot n}\}$ in W_π, where $\mathcal{D}(A')$ is the definition's domain of adjoint generator A'.

2. Finite functions of the generator $\hat{A}$

For $\nu > 0$ we consider (see [6]) $\mathcal{E}^\nu := \{\varphi \in L_1(\mathbb{R}) : \| \varphi \|_\nu < \infty\}$ with the norm $\| \varphi \|_\nu = \sup\limits_{k \in \mathbb{Z}_+} \dfrac{\| D^k \varphi(t) \|_{L_1}}{\nu^k}$

- the subspace in $L_1(\mathbb{R})$ of entire analytic complex functions of exponential type ν on $\mathbb{C}$, whose restrictions to $\mathbb{R} \subset \mathbb{C}$ belongs to $L_1(\mathbb{R})$. Let

$$\mathcal{E} := \bigcup_{\nu>0} \mathcal{E}^\nu = \lim_{\nu\to\infty} \operatorname{ind} \mathcal{E}^\nu$$

be the inductive limit of Banach spaces $\mathcal{E}^\nu$ under the continuous embeddings $\mathcal{E}^\nu \subset \mathcal{E}^\mu$ with $\nu \le \mu$. Note that the subspace $\mathcal{E} \subset L_1(\mathbb{R})$ consists of all entire analytic complex functions on $\mathbb{C}$ of exponential type, whose restrictions to the real axis $\mathbb{R}$ belong to $L_1(\mathbb{R})$. Following [6], the functionals of the space $\mathcal{E}'$ are called *exponential type distributions* over $\mathbb{R}$. In the dual pair $\langle \mathcal{E} \mid \mathcal{E}' \rangle$ the space $\mathcal{E}$ play a role of the space test functions and the embedding $\mathcal{E} \subset \mathcal{E}'$ is dense.

It is proven in [6] that $\mathcal{E}'$ is invariant under the differentiations. Consequently, we obtain

$$\langle D^k g \mid \varphi \rangle = (-1)^k \langle g \mid D^k \varphi \rangle, k \in \mathbb{Z}_+, \tag{1}$$

for all functionals $g \in \mathcal{E}'$ and all entire functions $\varphi \in \mathcal{E}$ on $\mathbb{R}$ of exponential type. It follows from [6] that $\mathcal{E}'$ is a locally convex topological algebra with respect to the convolution

$$\mathcal{E}' \times \mathcal{E}' \ni (g,h) \mapsto g * h \in \mathcal{E}'$$

and $\mathcal{E}$ is its convolution subalgebra.

By the well-known Paley-Wiener theorem [7], the Fourier-image $\hat{\mathcal{E}}$ of the space $\mathcal{E}$, endowed with inductive topology under the Fourier transform $\mathcal{F} : \mathcal{E} \ni \varphi \to \hat{\varphi} \in \hat{\mathcal{E}}$, consist of infinitely smooth finite complex functions on $\mathbb{R}$. So,

$$\hat{\mathcal{E}} \subset \mathcal{D}(\mathbb{R}), \tag{2}$$

where $\mathcal{D}(\mathbb{R})$ means the classic Schwartz space of test functions.

Via [6] the Fourier transform $\mathcal{F}$ can be extended from the space $\mathcal{E}$ onto the strong dual space

$$\mathcal{F}^\# : \mathcal{E}' \ni g \to \hat{g} \in \hat{\mathcal{E}}',$$

where $\hat{\mathcal{E}}'$ denotes its image, i.e. $\mathcal{F}^\# |_{\mathcal{E}} = \mathcal{F}$. This extended Fourier transform $\mathcal{F}^\#$ has the property

$$\mathcal{F}^\#(g * h) = \hat{g} \cdot \hat{h}, \qquad g,h \in \mathcal{E}'.$$

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So, the extended Fourier-image $\widehat{\mathcal{E}}'$ is a topological algebra with pointwise multiplication and $\hat{\mathcal{E}}$ is its multiplication subalgebra.

Since the embedding (2) is dense, the dense embedding

$$\mathcal{D}'(\mathbb{R}) \subset \hat{\mathcal{E}}'$$

holds, where the dual spaces $\mathcal{D}'(\mathbb{R})$ and $\hat{\mathcal{E}}'$ are endowed with the strong (or weak) topologies under the dualities $\langle \mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}) \rangle$ and $\langle \hat{\mathcal{E}}, \hat{\mathcal{E}}' \rangle$, respectively. Hence, $\langle \hat{\mathcal{E}} | \hat{\mathcal{E}}' \rangle$ forms a new dual pair, which is a Fourier-image of the dual pair $\langle \mathcal{E} | \mathcal{E}' \rangle$.

The following theorem is a generalization of [7, theorem 3] and may be proven in analogical way.

Theorem 1. For every $\varphi \in \mathcal{E}$ the operator

$$\hat{\varphi}(\hat{A})F(x) = \sum_{n \in \mathbb{Z}_+} \left\langle x^{\odot n} \mid \sum_{j=0}^{n} \left[\hat{\varphi}(A_j)\right]' F_n' \right\rangle, \qquad x \in B, \tag{3}$$

belongs to the Banach algebra $\mathcal{L}(W_\pi)$ of all bounded linear operators over W_π, where the operators

$$\hat{\varphi}(A) = \int_{\mathbb{R}} U_t \varphi(t) dt, \left[\hat{\varphi}(A_j)\right]' := \underbrace{I' \otimes \ldots \otimes I'}_{j-1} \otimes \left[\hat{\varphi}(A)\right]' \otimes \underbrace{I' \otimes \ldots \otimes I'}_{n-j+1}$$

(here $\left[\hat{\varphi}(A)\right]$ is adjoint to $\hat{\varphi}(A) \in \mathcal{L}(X)$) are bounded over X and $X_\pi'^{\odot n}$, respectively. Moreover, the differential property

$$\widehat{(D\varphi)}(A) = \hat{A} \circ \hat{\varphi}(A), \qquad \varphi \in \mathcal{E}$$

holds and the mapping

$$\hat{E} \ni \hat{\varphi} \mapsto \hat{\varphi}(A) \in \mathcal{L}(W_\pi)$$

is an algebraic homomorphism, particularly

$$\widehat{(\varphi * \psi)}(A) = \hat{\varphi}(A) \cdot \hat{\psi}(A) \qquad \text{for all } \varphi, \psi \in \mathcal{E}.$$

3. Functional calculus for the generator $\hat{A}$ in the symbol algebra $\hat{\mathcal{E}}'$

Following [6] we define the completions $\mathcal{E}(W_\pi) := \mathcal{E} \otimes_\pi W_\pi$ and $\mathcal{E}''(W_\pi) := \mathcal{E}'' \otimes_\pi W_\pi$ of the tensor product $\mathcal{E} \otimes W_\pi$ and $\mathcal{E}'' \otimes W_\pi$ under the

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corresponding projective tensor topologies. Then

$$\mathcal{E}(W_\pi) = \bigcup_{\nu>0} \mathcal{E}^\nu(W_\pi) = \liminf_{\nu\to\infty} \mathcal{E}^\nu(W_\pi) = \left(\liminf_{\nu\to\infty} \mathcal{E}^\nu\right) \otimes_\pi W_\pi.$$

Each element $F \in \mathcal{E}(W_\pi)$ is a W_π-valued exponential type entire function

$$\mathbb{R} \ni t \mapsto F(x,t) = \sum_{n\in\mathbb{Z}_+} F_n(x,t) \in W_\pi,$$

which also is an analytic complex function of $x \in B$ for any fixed t.

It follows from the known Grotendieck theorem [8] that for every $F \in \mathcal{E}(W_\pi)$ there exists $\nu > 0$ such that

$$F = \sum_{j\in\mathbb{N}} F_j \otimes \varphi_j \quad \text{with} \quad F_j \in W_\pi, \quad \varphi_j \in \mathcal{E}^\nu \tag{4}$$

is absolutely convergent in $\mathcal{E}^\nu(W_\pi)$. Hence, we can well-define the elements

$$\hat{F} := \sum_{j\in Z^+} \hat{\varphi}_j(\hat{A})F_j,$$

where $\hat{\varphi}_j(\hat{A})$ is defined by (3). For any $\nu > 0$ the subspace

$$\hat{\mathcal{E}}^\nu(W_\pi) := \left\{\hat{F} : F \in \mathcal{E}^\nu(X)\right\}$$

is complete under the norm induced by the mapping $\mathcal{E}^\nu(W_\pi) \ni F \mapsto \hat{F} \in \hat{\mathcal{E}}^\nu(W_\pi)$ (see [5, lemma 5]).

We define the convolution of an exponential type distribution $g \in \mathcal{E}'$ and W_π-valued exponential type entire function $F \in \mathcal{E}(W_\pi)$, representing by a series (4), as follows

$$(F * g)(x,t) = \sum_{j\in N} F_j(x) \otimes (g * \varphi_j)(t), \qquad x \in B, \quad t \in \mathbb{R}.$$

Denote

$$(F * g)(x,t) := (I \otimes K_g)F(x,t), \quad x \in B, \quad t \in \mathbb{R},$$

where I is identity operator on W_π and the convolution $g * \varphi$ of exponential type distribution $g \in \mathcal{E}'$ and exponential type entire complex function $\varphi \in \mathcal{E}$ is defined in [5].

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For any $\nu > 0$ the subspace $\hat{\mathcal{E}}(W_\pi)$ is invariant under each operator $I \otimes K_g$ with $g \in \mathcal{E}'$ (see [5], lemma 6). If we define the inductive limit

$$\hat{\mathcal{E}}(W_\pi) := \bigcup_{\nu>0} \hat{\mathcal{E}}^\nu(W_\pi) = \lim_{\nu\to\infty} \operatorname{ind} \hat{\mathcal{E}}^\nu(W_\pi)$$

then $\hat{\mathcal{E}}(W_\pi)$ also is invariant under each operator $I \otimes K_g$ with $g \in \mathcal{E}'$.

The following theorem is a generalization of [6, theorem 6]. We denote by $\mathcal{L}\left[\hat{\mathcal{E}}(W_\pi)\right]$ the algebra of all bounded linear operators over the space $\hat{\mathcal{E}}(W_\pi)$ endowed with the strong operator topology.

Theorem 2. The mapping $\hat{\mathcal{E}}' \ni \hat{g} \to \hat{g}(\hat{A}) \in \mathcal{L}\left[\hat{\mathcal{E}}(W_\pi)\right]$, where the linear operator $\hat{g}(\hat{A})$ is defined by

$$\hat{g}(\hat{A}) : \hat{\mathcal{E}}(W_\pi) \ni \hat{F} = \sum_{j\in Z^+} \hat{\varphi}_j(\hat{A})F_j \to \hat{g}(\hat{A})\hat{F} := \sum_{j\in Z^+} (g \hat{*} \varphi_j)(\hat{A})F_j \in \hat{\mathcal{E}}(W_\pi),$$

is a continuous homomorphism from the symbol algebra $\hat{\mathcal{E}}'$ into $\mathcal{L}\left[\breve{\mathcal{E}}(W_\pi)\right]$. Moreover,

$$(\hat{D}g)(\hat{A}) = \hat{A} \circ \hat{g}(\hat{A}), \qquad g \in \mathcal{E},$$

where the generalized derivative D is defined by (1).

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ФУНКЦІОНАЛЬНЕ ЧИСЛЕННЯ НА АЛГЕБРІ ТИПУ ВІНЕРА АНАЛІТИЧНИХ ФУНКЦІЙ НЕСКІНЧЕННОЇ КІЛЬКОСТІ ЗМІННИХ

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Для генераторів ізометричних сильно неперервних операторних груп на ядерних алгебрах Вінера аналітичних комплексних функцій на одиничній банаховій кулі побудовано функціональне числення. Його алгебра символів складається з Фур'є-образів розподілів експоненціального типу.

Ключові слова: функціональне числення, алгебра Вінера, банахова куля, Фур'є-образ розподілів експоненціального типу.

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Для генераторов изометрических сильно непрерывных операторных групп на ядерных алгебрах Винера аналитических комплексных функций на единичном банаховом шаре построено функциональное исчисление. Его алгебра символов состоит из Фурье-образов распределений экспоненциального типа.

Ключевые слова: функциональное исчисление, алгебра Винера, банахов шар, Фурье-образ распределений экспоненциального типа.

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